Short note on Ardhuin and Herbers (2002), JFM, *vol.* 451, *pp.* 1-33

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GOVERNING EQUATION

The physical descriptions in this note will be mixed vectorial and index notation. As a convention we have x_1, x_2 denoting the two horizontal dimensions of a Cartesian frame of reference with its origin at the fluid free surface while z denotes the co-ordinate along the vertical pointing upward in that frame. Index notation will be applied where convenient on the horizontal dimensions where the indices $\{i, j\} \in \{1, 2\}$ and the usual conventions regarding the summation of double indices apply. With these conventions we consider the case of an incompressible, inviscid fluid and a slowly (relative to a typical wavelength scale) varying bathymetry. Applying a deep water scaling ($\tanh k_0 h \to 0(1)$) a non-dimensional set may be obtained as

$$\nabla^2\phi + \phi_{zz} = 0, \qquad \forall z \in \mathcal{D} \tag{1a}$$

$$\phi_z + \nabla h \cdot \nabla\phi = 0, \qquad z = -h(\mathbf{x}, t) \tag{1b}$$

$$\phi_{tt} + \phi_z = -\epsilon\left[\partial_t + \frac{\epsilon}{2}\nabla\phi \cdot \nabla\right]\left\{|\nabla\phi|^2 + (\phi_z)^2\right\}, \qquad z = \epsilon\eta(\mathbf{x}, t) \tag{1c}$$

$$\eta + \phi_t = -\frac{\epsilon}{2}\left(|\nabla\phi|^2 + (\phi_z)^2\right). \qquad z = \epsilon\eta(\mathbf{x}, t) \tag{1d}$$

where $\epsilon \equiv k_0 a_0$.

In anticipation of small wave steepness we introduce a Taylor expansion around the free surface as in

$$\mathcal{F}|_\eta = \sum_{r=1}^{\infty} \frac{(\epsilon\eta)^r}{r!}\frac{\partial^r}{\partial z^r}\{\mathcal{F}\}|_0. \tag{2}$$

We omit terms of higher order than $O(\epsilon)$ yielding

$$\nabla^2\phi + \phi_{zz} = 0, \qquad \forall z \in \mathcal{D} \tag{3a}$$

$$\phi_z + \nabla h \cdot \nabla\phi = 0, \qquad z = -h(\mathbf{x}, t) \tag{3b}$$

$$\begin{aligned}&\phi_{tt} + \phi_z = \\ &\epsilon\left(\phi_t\phi_{tz} - \frac{1}{2}\left(|\nabla\phi|^2 + (\phi_z)^2\right)\right)_t - \epsilon\nabla \cdot (\phi_t\nabla\phi), \qquad z = 0\end{aligned} \tag{3c}$$

$$\eta + \phi_t = \epsilon\phi_t\phi_{tz} - \frac{\epsilon}{2}\left(|\nabla\phi|^2 + (\phi_z)^2\right), \qquad z = 0 \tag{3d}$$

SOLUTION THROUGH WKB

We anticipate propagating (in the horizontal plane and time) wave-like solutions with slowly varying properties in the dimensions of propagation. The rate of variation is made explicit through the introduction of slow time and horizontal space scales $\{X_i, \tau\} = \{\beta x_i, \gamma t\}$. Here β and γ are parameters - assumed small - denoting spatial and temporal inhomogeneity. The bottom is taken to vary $O(1)$ on the slow scale: $h = H(\mathbf{X})$. A general solution of the anticipated form reads

$$\left[\begin{array}{c} \phi(x,z,t) \\ \eta(x,t) \end{array} \right] = \sum_{j=0}^{\infty} \sum_{\substack{p=-\infty \\ s}}^{\infty} \epsilon^j \left[\begin{array}{c} \phi^s_{j,p}(\mathbf{x},z,t) \\ \eta^s_{j,p}(\mathbf{x},t) \end{array} \right] = \sum_{j=0}^{\infty} \sum_{\substack{p=-\infty \\ s}}^{\infty} \epsilon^j \left[\begin{array}{c} \widetilde{\phi}^s_{j,p}(\mathbf{X},z,\tau) \\ \widetilde{\eta}^s_{j,p}(\mathbf{X},\tau) \end{array} \right] e^{-i\chi^s_p(x,t)} \tag{4}$$

where the p is a counter over the harmonic components. For comparison purposes with Ardhuin and Herbers (2002) we will let the counter p coincide with a decomposition into an orthogonal wavenumber space with equidistant wavenumber spacing in the vicinity of an origin, $X = 0$ say. A local wavenumber and frequency are defined as

$$\mathbf{k}_p = \nabla \chi^s_p (\mathbf{x}, t)\,, \qquad -s\omega_p = \partial_t \chi^s_p (\mathbf{x}, t)\,. \tag{5}$$

In the following we will use the ∇_1 operator denoting $\{\partial_{X_1}, \partial_{X_2}\}$. We will keep all small parameters explicit for ease of recognition but will assume at the outset an ordering scheme where $O(\beta) = O(\gamma) = O(\epsilon)$.

The lowest order solution

At leading order our set (3) reduces to

$$\widetilde{\Phi}^s_{0,p,zz} - k_p^2 \widetilde{\Phi}^s_{0,p} = 0, \qquad \forall z \in \mathcal{D} \tag{6a}$$

$$\widetilde{\Phi}^s_{0,p,z} = 0, \qquad z = -H(\mathbf{X}) \tag{6b}$$

$$\widetilde{\Phi}^s_{0,p,z} = \omega_p^2 \widetilde{\Phi}^s_{0,p,z}, \qquad z = 0 \tag{6c}$$

$$\widetilde{\zeta}^s_{0,p} = is\omega_p \widetilde{\Phi}^s_{0,p} \qquad z = 0 \tag{6d}$$

At this order we find for the potential function

$$\widetilde{\Phi}^s_{0,p}(\mathbf{X}, T, z) = -i \frac{\widetilde{\zeta}^s_{0,p}}{s\omega_p} \frac{\operatorname{Ch} Q_p}{\operatorname{Ch} q_p}, \tag{7}$$

where we make use of the shorthand notation

$$\operatorname{Ch} \equiv \cosh, \qquad \operatorname{Sh} \equiv \sinh \tag{8}$$

$$Q_p \equiv k_p(h+z), \qquad q_p \equiv k_p h. \tag{9}$$

and $T_p = \operatorname{Sh} q_p / \operatorname{Ch} q_p$. We will write the lowest order solution as

$$\widetilde{\Phi}^s_{0,p}(\mathbf{X}, T, z) = \varphi^s_p(\mathbf{X}, \tau) \frac{\operatorname{Ch} Q_p}{\operatorname{Ch} q_p} = \varphi^s_p(\mathbf{X}, \tau) f(z, H), \tag{10}$$

thus separating the vertical structure from the amplitude function.

The first order solution

At the following order in terms of $O(\beta,\gamma,\epsilon)$ we have

$$\widetilde{\Phi}^s_{1,p,zz} - k_p^2\widetilde{\Phi}^s_{1,p} = -i\frac{\beta}{\epsilon}\left(2\mathbf{k}_p\cdot\nabla_1\widetilde{\Phi}^s_{0,p} + \widetilde{\Phi}^s_{0,p}\nabla_1\cdot\mathbf{k}_p\right), \qquad \forall z\in\mathcal{D} \tag{11a}$$

$$\widetilde{\Phi}^s_{1,p,z} = -i\frac{\beta}{\epsilon}\frac{\varphi^s_p}{\mathrm{Ch}\,q_p}\mathbf{k}_p\cdot\nabla_1 H, \qquad z=-H(\mathbf{X}) \tag{11b}$$

$$\begin{aligned}\left(\gamma^2\partial^2_\tau + \partial_z\right)\Phi^s_{1,p} = {} & i\frac{\gamma}{\epsilon}\left(2s\omega_p\varphi^s_{p,\tau} + s\omega_{p,\tau}\varphi^s_p\right)e^{i\chi^s_p} \\ & + i\sum_{\substack{n\\ s_1,s_2}}\mathcal{D}^{s_1,s_2}_{n,p-n}\varphi^{s_1}_n\varphi^{s_2}_{p-n}e^{i\left(\chi^{s_1}_n+\chi^{s_2}_{p-n}\right)}, \qquad z=0\end{aligned} \tag{11c}$$

$$\begin{aligned}\zeta^s_{1,p} - is\omega_p\Phi^s_{1,p} = {} & -\frac{\gamma}{\epsilon}\varphi^s_{p,\tau}e^{i\chi^s_p} \\ & + \sum_{\substack{n\\ s_1,s_2}}\mathcal{R}^{s_1,s_2}_{n,p-n}\varphi^{s_1}_n\varphi^{s_2}_{p-n}e^{i\left(\chi^{s_1}_n+\chi^{s_2}_{p-n}\right)}, \qquad z=0\end{aligned} \tag{11d}$$

where

$$\begin{aligned}\mathcal{D}^{s_1,s_2}_{n,p-n} \equiv {} & \left(s_1\omega_n + s_2\omega_{p-n}\right)\left(k_nk_{p-n}T_nT_{p-n} - \mathbf{k}_n\cdot\mathbf{k}_{p-n}\right) \\ & - \frac{1}{2}\left[s_2\omega_{p-n}k_n^2\left(1-T_n^2\right) + s_1\omega_nk_{p-n}^2\left(1-T_{p-n}^2\right)\right].\end{aligned} \tag{12}$$

and

$$\begin{aligned}R^{s_1,s_2}_{n,p-n} = {} & \\ & -\frac{1}{2}\left\{s_1s_2\omega_n\omega_{p-n}\left(k_nT_n + k_{p-n}T_{p-n}\right) - \left(\mathbf{k}_n\cdot\mathbf{k}_{p-n} - k_nk_{p-n}T_nT_{p-n}\right)\right\} = \\ & -\frac{1}{2}\left\{s_1s_2\omega_n\omega_{p-n}\left(\omega_n^2+\omega_{p-n}^2\right) + \omega_n^2\omega_{p-n}^2 - \mathbf{k}_n\cdot\mathbf{k}_{p-n}\right\}\end{aligned} \tag{13}$$

Essentially at this order the set (11) describes a forcing problem in the vertical on the potential amplitude $\phi^s_{1,p}$. A general solution to such a problem of second order can be readily found. A more natural co-ordinate in the vertical than z is $Q_p = k_p(z+H)$ and written in that co-ordinate the forcing problem (governing equation) becomes

$$\widetilde{\Phi}^s_{1,p,QQ} - \widetilde{\Phi}^s_{1,p} = -\frac{i}{k_p^2\,\mathrm{Ch}\,q_p}\frac{\beta}{\epsilon}\left[\mathcal{E}^{(1),s}_p\,\mathrm{Ch}\,Q_p + \mathcal{E}^{(2),s}_p\,\mathrm{Sh}\,Q_p + \mathcal{E}^{(3),s}_pQ_p\,\mathrm{Sh}\,Q_p\right] \tag{14}$$

with

$$\mathcal{E}^{(1),s}_p \equiv 2\left(\nabla_1\varphi^s_p - \varphi^s_pT_p\nabla_1q_p\right)\cdot\mathbf{k}_p + \varphi^s_p\nabla_1\cdot\mathbf{k}_p \tag{15a}$$

$$\mathcal{E}^{(2),s}_p \equiv 2\varphi^s_pk_p\mathbf{k}_p\cdot\nabla_1H \tag{15b}$$

$$\mathcal{E}^{(3),s}_p \equiv 2\varphi^s_p\frac{\mathbf{k}_p}{k_p}\cdot\nabla_1k_p \tag{15c}$$

A general solution for (14) reads

$$\widetilde{\Phi}^s_{1,p} = -\frac{i\beta}{\epsilon k_p^2 \operatorname{Ch} q_p}\left\{\frac{e^{Q_p}}{2}\left[\int^{Q_p}\mathcal{P}^s_p e^{-Q'_p}dQ'_p + \mathcal{C}^s_{1,p}\right] - \frac{e^{-Q_p}}{2}\left[\int^{Q_p}\mathcal{P}^s_p e^{Q'_p}dQ'_p + \mathcal{C}^s_{2,p}\right]\right\} \tag{16}$$

where

$$\mathcal{P}^s_p \equiv \mathcal{E}^{(1),s}_p \operatorname{Ch} Q'_p + \mathcal{E}^{(2),s}_p \operatorname{Sh} Q'_p + \mathcal{E}^{(3),s}_p Q'_p \operatorname{Sh} Q'_p. \tag{17}$$

Working out the integrals while applying the condition on $\widetilde{\Phi}^s_{1,p,z}$ at $Q_p = 0$ yields:

$$\begin{aligned}\widetilde{\Phi}^s_{1,p} = -\frac{i\beta}{\epsilon 2k_p^2 \operatorname{Ch} q_p}\Bigg\{\left(\mathcal{E}^{(2),s}_p Q_p + \frac{1}{2}\mathcal{E}^{(3),s}_p Q_p^2 + \mathcal{C}'^s_{1,p}\right)\operatorname{Ch} Q_p \\ + \left(\mathcal{E}^{(1),s}_p - \frac{1}{2}\mathcal{E}^{(3),s}_p\right) Q_p \operatorname{Sh} Q_p\Bigg\},\end{aligned} \tag{18a}$$

$$\begin{aligned}\widetilde{\Phi}^s_{1,p,Q} = -\frac{i\beta}{\epsilon 2k_p^2 \operatorname{Ch} q_p}\Bigg\{\left(\left(\mathcal{E}^{(1),s}_p + \frac{1}{2}\mathcal{E}^{(3),s}_p\right) Q_p + \mathcal{E}^{(2),s}_p\right)\operatorname{Ch} Q_p \\ + \left(\mathcal{E}^{(1),s}_p + \mathcal{E}^{(2),s}_p Q_p + \frac{1}{2}\mathcal{E}^{(3),s}_p\left(Q_p^2 - 1\right) + \mathcal{C}'^s_{1,p}\right)\operatorname{Sh} Q_p\Bigg\}.\end{aligned} \tag{18b}$$

Insertion of the latter set in the boundary condition at $Q_p = q_p$ yields

$$\begin{aligned}\left(\gamma^2\partial_\tau^2 + \omega_p^2\right)\mathcal{C}'^s_{1,p}e^{i\chi^s_p} = -k_p\Big[\big(2\left(\nabla_1\varphi^s_p - \varphi^s_p T_p \nabla_1 q_p\right)\cdot\mathbf{k}_p \\ +\varphi^s_p\left(\nabla_1\cdot\mathbf{k}_p - \mathbf{k}_p\cdot\frac{\nabla_1 k_p}{k_p}\right)\big)\left(T_p + q_p\left(1 - T_p^2\right)\right) + 2\varphi^s_p\mathbf{k}_p\cdot\nabla_1 q_p\Big]e^{i\chi^s_p} \\ - 2k_p^2\frac{\gamma}{\beta}\left(2s\omega_p\varphi^s_{p,\tau} + s\omega_{p,\tau}\varphi^s_p\right)e^{i\chi^s_p} - 2k_p^2\frac{\epsilon}{\beta}\sum_{\substack{n\\ s1,s2}}\mathcal{D}^{s_1,s_2}_{n,p-n}\varphi^{s_1}_n\varphi^{s_2}_{p-n}e^{i\left(\chi^{s_1}_n + \chi^{s_2}_{p-n}\right)}.\end{aligned} \tag{19}$$

In the present note we will not be concerned with the nature of the non-linear terms in (19) and since we require no verification on the non-stationary terms we reduce (19) to

$$\begin{aligned}\left(\gamma^2\partial_\tau^2 + \omega_p^2\right)\mathcal{C}'^s_{1,p}e^{i\chi^s_p} = -k_p\Bigg\{\Bigg[2\left(\nabla_1\varphi^s_p - \varphi^s_p T_p\nabla_1 q_p\right)\cdot\mathbf{k}_p \\ +\varphi^s_p\left(\nabla_1\cdot\mathbf{k}_p - \mathbf{k}_p\cdot\frac{\nabla_1 k_p}{k_p}\right)\Bigg]\left(T_p + q_p\left(1 - T_p^2\right)\right) + 2\varphi^s_p\mathbf{k}_p\cdot\nabla_1 q_p\Bigg\}e^{i\chi^s_p}.\end{aligned} \tag{20}$$

to only include terms due to medium inhomogeneity.

CROSS-LINK TO EXPRESSIONS ARDHUIN AND HERBERS

For verification purposes of the terms denoting the resonant interaction of the wave field with the bottom we will link the expressions in This Note (TN) with those of Ardhuin and Herbers (2002) (denoted by AH) in an improvisatory manner. In the limit $X \to 0$ the Lagrangian definitions and quantities as used in AH will simplify and for comparison purposes we note that

$$\overbrace{\sum_{p,s} \phi_{0,p}^s = \sum_{p,s} \varphi_p^s f_p e^{i\chi_p^s}}^{\text{TN}} \equiv \overbrace{\phi_1 = \sum_{\mathbf{k},s} \frac{\operatorname{Ch}(kz + kH)}{\operatorname{Ch} kh} \Phi_{1,\mathbf{k}}^s e^{iS_{1,\mathbf{k}}}}^{\text{AH}}$$

$$\varphi_p^s \equiv \widehat{\Phi}_{1,\mathbf{k}}^s$$

and

$$\underbrace{\mathbf{k}_p}_{TN} \equiv \underbrace{\lim_{X\to 0} \mathbf{k}_r = \mathbf{k}}_{AH}$$

Also we will let $\epsilon = \beta$ and simply express our spatial derivatives in the slow spatial scales.

Comparison with (D1)

It is easy to see that (14) can be written as

$$\begin{aligned}\widetilde{\Phi}_{1,p,zz}^s - k_p^2 \widetilde{\Phi}_{1,p}^s = \\ -i\Bigg\{ \left(2\left(\nabla_1 \varphi_p^s - \varphi_p^s T_p \nabla_1 q_p\right) \cdot \mathbf{k}_p + \varphi_p^s \nabla_1 \cdot \mathbf{k}_p\right) \frac{\operatorname{Ch} Q_p}{\operatorname{Ch} q_p} \\ +2\varphi_p^s \left(k_p \mathbf{k}_p \cdot \nabla_1 H \frac{\operatorname{Sh} Q_p}{\operatorname{Ch} q_p} + \mathbf{k}_p \cdot \frac{\nabla_1 k_p}{k_p} Q_p \frac{\operatorname{Sh} Q_p}{\operatorname{Sh} q_p} \right) \Bigg\}\end{aligned} \quad (21)$$

which can be manipulated to

$$\begin{aligned}\widetilde{\Phi}_{1,p,zz}^s - k_p^2 \widetilde{\Phi}_{1,p}^s = -i \frac{1}{\operatorname{Ch} q_p} \{ (2\mathbf{k}_p \cdot \nabla_1 + \nabla_1 \cdot \mathbf{k}_p) \operatorname{Ch} Q_p \\ +2\left((z+H)\operatorname{Sh} Q_p - HT_p \operatorname{Ch} Q_p\right) \mathbf{k}_p \cdot \nabla_1 k_p \\ +2\left(k_p \operatorname{Sh} Q_p - k_p T_p \operatorname{Ch} Q_p\right) \mathbf{k}_p \cdot \nabla_1 H \} \varphi_p^s\end{aligned} \quad (22)$$

which - within the limitations stated above - is in correspondence with (D1) as can be seen upon inspection.

Comparison with (D2,3,4,5,6)

The expression (18a) can be restated through minor manipulations as

$$\begin{aligned}\widetilde{\Phi}_{1,p}^s = -i \frac{\operatorname{Ch} Q_p}{\operatorname{Ch} q_p} \left[(z+H)\mathbf{k}_p \cdot \nabla_1 H \varphi_p^s + (z+H)^2 \mathbf{k}_p \cdot \frac{\nabla_1 k_p}{2k_p} \varphi_p^s + \frac{1}{2k_p^2} C_{1,p}^{\prime s} \right] \\ -i \frac{\operatorname{Sh} Q_p}{\operatorname{Ch} q_p} (z+H) \left[\frac{1}{2k_p} \left(2\mathbf{k}_p \cdot \nabla_1 \varphi_p^s + \varphi_p^s \nabla_1 \cdot \mathbf{k}_p\right) - \varphi_p^s \frac{\mathbf{k}}{k_p} \cdot \left(\left(HT_p + \frac{1}{2k_p} \right) \nabla_1 k_p + k_p T_p \nabla_1 H \right) \right]\end{aligned} \quad (23)$$

and is thus in agreement with (D2,3,4,5,and 6) apart from a different definition of the undetermined $C'_{1,p}$ versus $\Phi^s_{3,\mathbf{k}}$ in AH which is merely a matter of convenience and (naturally) of no consequence to any of the following steps nor the final solution.

Comparison with (D7)

At the free surface we find (20), which through minor manipulations and retaining only terms due to bottom inhomogeneity can be written as

$$\left(\gamma^2\partial_\tau^2+\omega_p^2\right)\mathcal{C}''^s_{1,p}e^{i\chi_p^s}=i\left[\frac{\left(T_p+q_p\left(1-T_p^2\right)\right)}{2k_p}\left(2\left(\nabla_1\varphi_p^s-\varphi_p^sT_p\nabla_1 q_p\right)\cdot\mathbf{k}_p\right.\right.$$
$$\left.\left.+\varphi_p^s\left(\nabla_1\cdot\mathbf{k}_p-\mathbf{k}_p\cdot\frac{\nabla_1 k_p}{k_p}\right)\right)+\varphi_p^s\frac{\mathbf{k}_p}{k_p}\cdot\nabla q_p\right]e^{i\chi_p^s}. \quad (24)$$

where $\mathcal{C}''^s_{1,p}=-i(2k_p^2)^{-1}\mathcal{C}'^s_{1,p}$. Or alternatively

$$\left(\gamma^2\partial_\tau^2+\omega_p^2\right)\mathcal{C}''^s_{1,p}e^{i\chi_p^s}=i\left[\frac{\left(T_p+q_p\left(1-T_p^2\right)\right)}{k_p}\left(\mathbf{k}_p\cdot\nabla_1+\underbrace{\frac{\nabla_1\cdot\mathbf{k}_p}{2}}_{\text{factor }2}\right.\right.$$
$$\left.\left.-\underbrace{\mathbf{k}_p}_{\text{factor }2}\cdot\left(\left(T_pH+\frac{1}{2k_p}\right)\nabla_1 k_p+T_pk_p\nabla_1 H\right)\right)+\frac{\mathbf{k}_p}{k_p}\cdot\nabla_1 q_p\right]\varphi_p^se^{i\chi_p^s}. \quad (25)$$

which is essentially in the form of (D7). The (minor) differences between (25) and (D7) are under-braced in the former.

VERIFICATION

In this part we will rewrite the secular terms in a more recognizable form as they should sustain the conservation of energy. Essentially we seek the equation governing the slow variation of the amplitude by equating the secular terms to zero.

1DH energy conservation of a linear and stationary wave field

In 1DH (replacing X_i by X) neglecting the effects of non-stationarity and non-linearity we essentially have

$$\left(T_p + q_p(1-T_p^2)\right)\left[\varphi^s_{p,X} - q_{p,X}T_p\varphi^s_p\right] + q_{p,X}\varphi^s_p = 0 \tag{26}$$

yielding

$$\frac{\varphi^s_{p,X}}{\varphi^s_p} = q_{p,X}\left(1-T_p^2\right)\left[\frac{q_pT_p - 1}{T_p + q_p\left(1-T_p^2\right)}\right] \tag{27}$$

we can use the identity (for derivation: see Appendix to note)

$$q_{p,X} = H_X k_p \frac{T_p}{\left(T_p + q_p\left(1-T_p^2\right)\right)s}$$

so that we may have

$$\frac{\varphi^s_{p,X}}{\varphi^s_p} = q_{p,X}\left(1-T_p^2\right)\left[\frac{q_pT_p - 1}{T_p + q_p\left(1-T_p^2\right)}\right] = -\frac{C_{g,p,X}}{2C_{g,p}} \tag{28}$$

or alternatively and equivalently

$$\frac{\varphi^s_{p,X}}{\varphi^s_p} = -H_X k_p\left[\frac{2\,\mathrm{Sh}\,2q_p + 2q_p\left(1-\mathrm{Ch}\,2q_p\right)}{\left(\mathrm{Sh}\,2q_p + 2q_p\right)^2}\right] \tag{29}$$

Both (28) and (29) represent amplitude evolution in correspondence with energy conservation; clearly they are equivalents to the form

$$\frac{d}{dX}\left(C_{g,p}|\varphi^s_p|^2\right) = 0 \tag{30}$$

2DH energy conservation of a linear and stationary wave field

In two horizontal dimensions the amplitude variation is governed by

$$\left(2\left(\nabla_1\varphi^s_p - \varphi^s_pT_p\nabla_1 q_p\right)\cdot\mathbf{k}_p + \varphi^s_p\left(\nabla_1\cdot\mathbf{k}_p - \mathbf{k}_p\cdot\frac{\nabla_1 k_p}{k_p}\right)\right)\left(T_p + q_p\left(1-T_p^2\right)\right) + 2\varphi^s_p\mathbf{k}_p\cdot\nabla_1 q_p = 0 \tag{31}$$

which can be recast into

$$\left[2\nabla_1\varphi^s_p\cdot\mathbf{k}_p + \varphi^s_p\nabla_1\cdot\mathbf{k}_p\right]\left(T_p + q_p\left(1-T_p^2\right)\right) - \mathbf{k}_p\cdot\frac{\nabla_1 k_p}{k_p}\left[3T_p + q_p\left(1-3T_p^2\right)\right]\varphi^s_p = 0 \tag{32}$$

Note that we can have the identity (for $\partial_\tau \mathbf{k} = 0$)

$$\nabla_1 \cdot \mathbf{C}_{g,p} = \frac{1}{2k_p^{3/2} T_p^{1/2}} \left[\left(T_p + q_p \left(1 - T_p^2 \right) \right) \nabla_1 \cdot \mathbf{k}_p \right. \\ \left. - \mathbf{k}_p \cdot \frac{\nabla_1 k_p}{k_p} \left(3T_p + q_p \left(1 - 3T_p^2 \right) \right) \right] \quad (33)$$

It is then easy to see that (32) reduces to

$$2\nabla_1 \varphi_p^s \cdot \mathbf{C}_{g,p} + \varphi_p^s \nabla_1 \cdot \mathbf{C}_{g,p} = 0 \quad (34)$$

And can be readily transformed in the conservative form

$$\nabla_1 \cdot \left(\mathbf{C}_{g,p} |\varphi_p^s|^2 \right) = 0 \quad (35)$$

which concludes our argument.

CONSEQUENCES

Our verification supports believe that (25) is indeed the correct expression for the resonant bottom-induced (slow variations only) forcing problem, which would result in minor corrections in (D7) of AH. Consequences for the other expressions could be significant but are very small in fact and the final result, (D12), remains unaffected (This is obvious since our verification states a similar conservation law). Alternatively, if we were to recast (25) in a form similar to (D8) we would have

$$F_{3,1,p}^{\Phi,\text{he}} = -t \frac{C_{g,p}}{2k_p} \nabla_1 \cdot \left(\mathbf{k}_p F_{1,1,p}^{\Phi} \right) \\ + t \frac{C_{g,p}}{k_p} \mathbf{k}_p \cdot \left[T_p \nabla_1 q_p + \frac{\nabla_1 k_p}{2k_p} \right] F_{1,1,p}^{\Phi} \\ - \underbrace{t \frac{\omega_p}{2T_p k_p^2} \mathbf{k}_p \cdot \nabla_1 q_p F_{1,1,p}^{\Phi}}_{\text{factor 2}} \quad (36)$$

which is identical to (D8) except for a factor 2 on the last term in the RHS. It can be seen that the last two terms essentially read

$$tF_{1,1,p}^{\Phi} \mathbf{k}_p \cdot \left\{ \frac{C_{g,p}}{k_p} \left[T_p \nabla_1 q_p + \frac{\nabla_1 k_p}{2k_p} \right] - \frac{\omega_p}{2T_p k_p^2} \nabla_1 q_p \right\} = \\ tF_{1,1,p}^{\Phi} \mathbf{k}_p \cdot \left\{ \frac{1}{4\omega_p k_p} \left[3T_p + q_p \left(1 - 3T_p^2 \right) \right] \frac{\nabla_1 k_p}{k_p} \right\} = - \frac{tF_{1,1,p}^{\Phi} \mathbf{k}_p}{2} \cdot \nabla_1 \frac{C_{g,p}}{k_p} \quad (37)$$

Note that

$$\nabla_1 \left(\frac{C_{g,p}}{k_p} \right) = \frac{\nabla_1 \left(C_{g,p} T_p \right)}{k_p T_p} \quad (38)$$

and thus the identity (37) essentially is an equivalent to (D11). This yields for (36)

$$F_{3,1,p}^{\Phi,\text{he}} = -\frac{t}{2} \left\{ \frac{C_{g,p}}{k_p} \nabla_1 \cdot \left(\mathbf{k}_p F_{1,1,p}^{\Phi} \right) + \mathbf{k}_p \cdot \nabla_1 \frac{C_{g,p}}{k_p} \right\} \\ = -\frac{t}{2} \left\{ F_{1,1,p}^{\Phi} \nabla_1 \cdot \mathbf{C}_{g,p} + \mathbf{C}_{g,p} \cdot \nabla_1 F_{1,1,p}^{\Phi} \right\} \\ = -\frac{t}{2} \nabla_1 \cdot \left(\mathbf{C}_{g,p} F_{1,1,p}^{\Phi} \right) \quad (39)$$

Restating (39) in terms of the energy spectrum will yield an equivalent to (D12).

CONCLUSIONS

The following can be concluded

- through comparison with AH we see that all terms found here are identical except a minor typo in (D7) and - perhaps consequently but not consistent - in (D8) of AH.
- through verification with (simplified) analytical expressions the derivation in this note is supported

References

Ardhuin, F. and Herbers, T. H. C. (2002). “Bragg scattering of random surface gravity waves by irregular sea bed topography.” *J. Fluid Mech.*, 451, 1–33.