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Weakly nonlinear non-symmetric gravity waves on water of finite depth

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A weakly nonlinear Hamiltonian model for two-dimensional irrotational waves on water of finite depth is developed. The truncated model is used to study families of periodic travelling waves of permanent form. It is shown that non-symmetric periodic waves exist, which appear *via* spontaneous symmetry-breaking bifurcations from symmetric waves.

1. Introduction

Frequently in physics the equations that describe a phenomena are invariant under the action of some symmetry group. In general the solutions have the same symmetry group. In some cases, however, it has been observed that bifurcations can lead to solutions that have a smaller symmetry group. These are examples of spontaneous symmetry breaking. The symmetry group of the equations remains unchanged, but solutions spontaneously break symmetry in the absence of any external perturbation.

A very clear example of this situation is Hopf bifurcation where temporal symmetry is broken. Another physical example of symmetry breaking is the appearance of hexagonal convection cells in Bénard convection.

Symmetry breaking plays an important role in several physical disciplines, including pattern formation in reaction diffusion problems, convective flows in geophysical phenomena, neurobiology, statistical physics, physical chemistry. For a good analysis of bifurcations in the presence of symmetries using group representation theory see Sattinger (1980, 1983).

Since Stokes (1849) discussed the problem of two-dimensional progressive symmetric gravity waves, a great deal of work has been done to study all properties of travelling waves of permanent form on water of finite and infinite depth. But to date all the families of waves that have been found correspond to symmetric waves. To define what symmetry means, consider a travelling wave of permanent form that is given by a function $f(x)$, which describes the shape. A wave is said to be symmetric if the origin of the wave can be chosen such that $f(x) = f(-x)$. This relation defines a symmetry group.

Several attempts have been carried out trying to find non-symmetric water waves. For example in the case of deep water Chen & Saffman (1980) found that gravity waves of finite amplitude are not unique. They found bifurcations to families of waves in which all crests are not equal. They did in fact find bifurcations into apparently non-symmetric solutions, but these solutions proved to be just shifted symmetric waves. They searched for symmetry-breaking bifurcations in the regular Stokes family and in two families that bifurcate from this one, that they called Class 2 and Class 3 irregular waves. They were not able to find any genuine symmetry-breaking bifurcation. Irregular waves of Class 2 on water of finite depth have been computed by Vanden-Broeck (1983).

Recently Longuet-Higgins (1985) carried out calculations using a method based on a Fourier expansion of the shape of the wave in the potential plane. He did not find any bifurcation to non-symmetric waves from the regular branch. Later Zufiria & Saffman (unpublished work) using the same method looked for symmetry-breaking bifurcations in the Class 3 waves. Some were found, but in regions where the convergence of the method was bad, and the bifurcations moved or disappeared as the number of modes used in the expansion was varied.

During the last decade a lot of work has been done trying to understand the local asymptotic expansion near the crest for the Stokes limiting wave of 120° (Grant 1973; Norman 1974; Longuet-Higgins & Fox 1977, 1978; Olfe & Rottman 1980). Again all the work has considered only symmetric expansions. Work in progress shows that even though the crests are always symmetric in first approximation, non-symmetric solutions are possible when the influence of neighbouring crests is considered.

During the last year the Hamiltonian theory of water waves has proved to be very successful in answering some important questions about the stability of travelling waves of permanent form. Saffman (1985), using Zakharov's (1968) Hamiltonian formulation, proved analytically (thereby verifying Tanaka's (1985*a*) computations) that an exchange of stability occurs in finite-amplitude water waves of permanent form on deep water for a superharmonic disturbance to every wave whose total energy is stationary. Later Zufiria & Saffman (1985) extended this result, using Hamiltonian theory again, to periodic waves of finite depth and the solitary wave (again verifying Tanaka's (1985*b*) results for this case). Also MacKay & Saffman (1986) determined under which conditions the crossing of eigenvalues for the linearized problem about a wave of permanent form leads to loss of stability.

In the present paper a weakly nonlinear Hamiltonian model is developed for water waves on finite depth by a direct truncation of the complete Hamiltonian for water waves. With this model all the subharmonic bifurcations found by Chen & Saffman (1980) and Saffman (1980) on deep water and Vanden-Broeck (1983) on finite-depth water are reproduced. The model shows also the possibility of more general bifurcations.

Based on some similarities with area preserving maps that were suggested to us by Dr Robert MacKay, the bifurcation tree for the Stokes family is studied in more detail, and a symmetry-breaking bifurcation is found that leads to non-symmetric waves.

2. Weakly nonlinear Hamiltonian theory

Consider two-dimensional irrotational water waves in a laterally unbounded domain of constant depth. Miles (1977), extending Zakharov's (1968) and Broer's (1974) work, proved that this system has the following Hamiltonian structure

$$\left.\begin{aligned}\frac{\partial \eta}{\partial t} &= \frac{\delta \mathscr{H}}{\delta \psi},\\ \frac{\partial \psi}{\partial t} &= -\frac{\delta \mathscr{H}}{\delta \eta},\end{aligned}\right\} \tag{1}$$

where $\mathscr{H}$ is the total energy of the waves and δ represents a functional derivative (see

Goldstein 1980). The canonical variables $\eta(x,t)$ and $\psi(x,t)$ are the surface shape and the velocity potential evaluated at the surface.

$$\psi(x,t) = \phi(x,\eta(x,t),t), \tag{2}$$

where $\phi(x,y,t)$ is the velocity potential.

$\mathscr{H}$ is given by

$$\mathscr{H} = \int H(\eta,\psi)\,\mathrm{d}x, \tag{3}$$

where the integral extends over one period in the case of periodic waves. The energy density H is given by

$$H = \frac{1}{2} = \int_{-h}^{\eta} (\nabla\phi)^2\,\mathrm{d}y + \tfrac{1}{2}g\eta^2. \tag{4}$$

g is the gravitational acceleration and h the distance from the origin to the bottom.

To make this formulation useful we have to express the energy density in terms of the canonical variables. This transformation leads to very complicated expressions involving integral operators. Considering small-amplitude and long waves, a simple expansion for H can be obtained. In the present case of finite depth, this expansion contains only spatial derivatives of the canonical variables.

In order to obtain this expansion consider the following dimensionless variables,

$$\left.\begin{aligned} x' &= \frac{x}{\lambda}, \quad t' = t\left(\frac{g}{h}\right)^{\frac{1}{2}}, \quad H' = \frac{\beta^{\frac{1}{2}}}{\alpha^2}\frac{H}{gh^2},\\ y' &= \frac{y}{h}, \quad \phi' = \frac{h\phi}{\lambda a (gh)^{\frac{1}{2}}},\\ \eta' &= \frac{\eta}{a}, \quad \psi' = \frac{h\psi}{\lambda a (gh)^{\frac{1}{2}}}, \end{aligned}\right\} \tag{5}$$

where a represents a measure of the amplitude of the waves and λ the typical wavelength for the considered waves (see figure 1). The two parameters α and β are defined as follows

$$\alpha = \frac{a}{h}, \quad \beta = \left(\frac{h}{\lambda}\right)^2.$$

α measures the amplitude of the wave and β the dispersion (Whitham 1974).

If the motion is studied in the frame of reference in which the mean horizontal velocity is zero then the dimensionless variables are all of order unity. This change of variable can be also considered as a canonical transformation (Radder & Dingemans 1985). Dropping the primes the new energy density will be

$$H = \frac{1}{\beta^{\frac{1}{2}}}\left\{\frac{1}{2}\int_{-1}^{\alpha\eta}\left(\phi_x^2 + \frac{1}{\beta}\phi_y^2\right)\mathrm{d}y + \tfrac{1}{2}\eta^2\right\}. \tag{6}$$

If we assume that α and β are small, it is possible to obtain an expansion of ϕ_x and ϕ_y in terms of ψ, η and their spatial derivatives.

Taking the Fourier transform, it is clear that

$$\phi(x,y,t) = \int_{-\infty}^{\infty} \frac{\cosh\,[k\beta^{\frac{1}{2}}(1+y)]}{\cosh\,[k\beta^{\frac{1}{2}}]}\,\hat{\phi}(k,t)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \tag{7}$$

where $\hat{\phi}(k,t)$ is the Fourier transform of the value of $\phi(x,y,t)$ at $y = 0$.

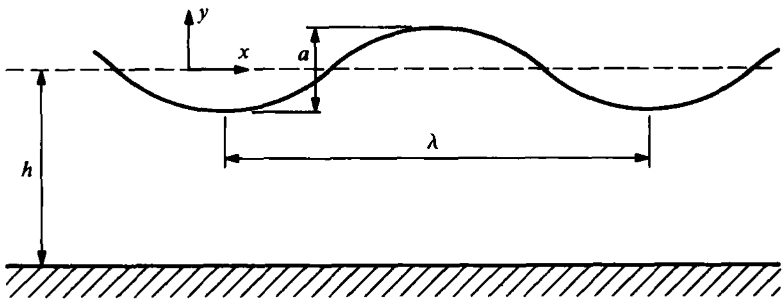

FIGURE 1. Some notation for periodic travelling waves of permanent form on water of finite depth.

Then $\psi(x,t)$ is given by

$$\psi(x,t) = \phi(x,\eta(x,t),t) = \int_{-\infty}^{\infty} \frac{\cosh[k\beta^{\frac{1}{2}}(1+\alpha\eta)]}{\cosh[k\beta^{\frac{1}{2}}]}\hat{\phi}(k,t)\,e^{ikx}\,dk. \tag{8}$$

Expanding the kernel of the integral in powers of α and inverting the operator it follows

$$\begin{aligned}\hat{\phi}(k,t) &= \hat{\psi}(k,t) - \alpha\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\eta}(k-k_1)\hat{M}(k_1)\hat{\psi}(k_1,t)\,dk_1\\ &\quad+\alpha^2\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\eta}(k-k_2)\hat{\eta}(k_2-k_1)\hat{M}(k_1)\hat{\psi}(k_1,t)\,dk_1\,dk_2\\ &\quad-\tfrac{1}{2}\alpha^2\beta\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\eta}(k-k_1-k_2)\hat{\eta}(k_2)k_1^2\hat{\psi}(k_1,t)\,dk_1\,dk_2+O(\alpha^3),\end{aligned} \tag{9}$$

where $\hat{\eta}$ and $\hat{\psi}$ are the Fourier transforms of η and ψ respectively, and

$$\hat{M}(k) = \beta^{\frac{1}{2}}k\tanh(k\beta^{\frac{1}{2}}). \tag{10}$$

Assuming $\beta \ll 1$, equation (7) can be expanded in Taylor series giving

$$\phi(x,y,t) = \int_{-\infty}^{\infty}[1+y\hat{M}(k)+\beta^2k^2\tfrac{1}{2}y^2+\beta k^2\hat{M}(k)\tfrac{1}{6}y^3+\beta^2k^4\tfrac{1}{24}y^4]\hat{\phi}(k,t)\,e^{ikx}\,dk+O(\beta^3). \tag{11}$$

From (9) the previous relation can be rewritten as

$$\begin{aligned}\phi(x,y,t) = \psi+(y-\alpha\eta)M\cdot\psi-\tfrac{1}{2}\beta(y^2-\alpha^2\eta^2)\psi_{xx}-\alpha(y-\alpha\eta)M\cdot[\eta M\cdot\psi]\\ -\tfrac{1}{6}\beta y^3M\cdot\psi_{xx}+\tfrac{1}{24}\beta^2y^4\psi_{xxxx}+O(\beta^3,\alpha^3),\end{aligned} \tag{12}$$

where $M\cdot\psi$ is defined as follows

$$\begin{aligned}M\cdot\psi &= \int_{-\infty}^{\infty}k\beta^{\frac{1}{2}}\tanh(k\beta^{\frac{1}{2}})\hat{\psi}(k,t)\,e^{ikx}\,dk\\ &= -\beta\psi_{xx}-\tfrac{1}{3}\beta^2\psi_{xxxx}+O(\beta^3).\end{aligned} \tag{13}$$

Finally introducing (12) and (13) into (6) the following Hamiltonian density is obtained

$$\begin{aligned}\beta^{\frac{1}{2}}H &= \tfrac{1}{2}\eta^2+\{\tfrac{1}{2}\psi_x^2+\tfrac{1}{6}\beta[\psi_{xx}^2+2\psi_x\psi_{xxx}]+\tfrac{1}{15}\beta^2[\psi_{xxx}^2+\tfrac{5}{2}(\psi_{xx}\psi_{xxxx}+\psi_x\psi_{xxxxx})]\\ &\quad+O(\beta^3)]\}+\alpha\{\tfrac{1}{2}\eta\psi_x^2+\tfrac{1}{2}\beta[\eta(\psi_{xx}^2+2\psi_x\psi_{xxx})+2\psi_x\psi_{xx}\eta_x]\\ &\quad+\tfrac{1}{3}\beta^2[\eta(\psi_{xxx}^2+\psi_{xx}\psi_{xxxx}+\psi_x\psi_{xxxxx})+\eta_x(\psi_x\psi_{xxxx}+\psi_{xx}\psi_{xxx})]\\ &\quad+\tfrac{1}{2}\beta^2[(\psi_{xx}\eta)_{xxx}\psi_x+(\psi_{xx}\eta)_{xx}\psi_{xx}]+O(\beta^3)\}+O(\alpha^2).\end{aligned} \tag{14}$$

This expansion is accurate up to terms of order $\alpha\beta^2$. The evolution equations that can be obtained from this energy density agree with those obtained by Miles (1977) following a similar expansion, and also with the equations obtained by Hunter & Vanden-Broeck (1983) using a direct expansion of the classical water-waves equations for α and β small.

Keeping only first-order terms in α and β in the expansion of H, the famous Boussinesq approximation is reproduced. Dropping all terms in α the linear water-wave equations are obtained.

In the present analysis the following truncation of energy density will be considered

$$\beta^{\frac{1}{2}}H = \tfrac{1}{2}\psi_x^2(1+\alpha\eta)+\tfrac{1}{2}\eta^2+\tfrac{1}{6}\beta[\psi_{xx}^2+2\psi_x\psi_{xxx}]\tfrac{1}{15}\beta^2[\psi_{xxx}^2+\tfrac{5}{2}(\psi_{xx}\psi_{xxxx}+\psi_x\psi_{xxxxx})]. \tag{15}$$

This energy density contains one more term $O(\beta^2)$ in the expansion of the dispersion than the Boussinesq approximation in the theory of weakly nonlinear water waves.

3. Stokes waves model

The dimensionless variables considered in the previous section are good choices for showing the relative magnitude of the different terms that appear in the equations. In order to write the energy density in a form that is independent of the lengthscale of the waves, it is convenient instead to use the dimensionless variables

$$\left.\begin{aligned} \eta' &= \frac{\eta}{h}, \quad \psi' = \frac{\psi}{h(gh)^{\frac{1}{2}}}, \\ x' &= \frac{x}{h}, \quad H' = \frac{H}{gh^2}. \end{aligned}\right\} \tag{16}$$

With these new variables (15) can be written as follows

$$H = \tfrac{1}{2}\psi_x^2(1+\eta)+\tfrac{1}{2}\eta^2+\tfrac{1}{6}[\psi_{xx}^2+2\psi_x\psi_{xxx}]\tfrac{1}{15}[\psi_{xxx}^2+\tfrac{5}{2}(\psi_{xx}\psi_{xxxx}+\psi_x\psi_{xxxxx})], \tag{17}$$

where the primes have been dropped.

The evolution equations corresponding to the energy density given by (17) are

$$\left.\begin{aligned} \eta_t &= \frac{\delta\mathscr{H}}{\delta\psi} = -\psi_{xx}-\tfrac{1}{3}\psi_{xxxx}-\tfrac{2}{15}\psi_{xxxxxx}-(\eta\psi_x)_x, \\ \psi_t &= -\frac{\delta\mathscr{H}}{\delta\eta} = -\eta-\tfrac{1}{2}\psi_x^2. \end{aligned}\right\} \tag{18}$$

Travelling waves of permanent form of this system are solutions of the form

$$\left.\begin{aligned} \eta(x,t) &= \eta(x'), \\ \psi(x,t) &= \psi(x'), \quad x' = x-ct, \end{aligned}\right\} \tag{19}$$

where c is the phase speed of the train waves.

Using relations (19) and dropping the primes, the evolution equations (18) can be rewritten as

$$c\eta_x = \psi_{xx}+\tfrac{1}{3}\psi_{xxxx}+\tfrac{2}{15}\psi_{xxxxxx}+(\eta\psi_x)_x, \tag{20a}$$

$$c\psi_x = \eta+\tfrac{1}{2}\psi_x^2. \tag{20b}$$

Equation (20a) can be integrated once with respect to x giving

$$(c-\psi_x)\,\eta = \psi_x+\tfrac{1}{3}\psi_{xxx}+\tfrac{2}{15}\psi_{xxxxx}+C. \tag{21}$$

The vertical position of the origin of the frame of reference for the waves can always be chosen such that $C = 0$. Thus in the following we will assume $C = 0$ without loss of generality.

With the change $u = \psi_x$ and using (20*b*), the problem can be reduced to the single equation

$$\tfrac{1}{3}u_{xx}+\tfrac{2}{15}u_{xxxx} = u[(c^2-1)-\tfrac{1}{2}u(3c-u)]. \tag{22}$$

This fourth-order equation can be rewritten as the following dynamical system of four first-order equations

$$\left.\begin{aligned} \frac{\mathrm{d}u_1}{\mathrm{d}x} &= u_2, \\ \frac{\mathrm{d}u_2}{\mathrm{d}x} &= u_3, \\ \frac{\mathrm{d}u_3}{\mathrm{d}x} &= u_4, \\ \frac{\mathrm{d}u_4}{\mathrm{d}x} &= \tfrac{15}{2}u_1[(c^2-1)-\tfrac{1}{2}u_1(3c-u_1)]-\tfrac{15}{6}u_3, \end{aligned}\right\} \tag{23}$$

where $\quad u_1 = u, \quad u_2 = u_x, \quad u_3 = u_{xx}, \quad u_4 = u_{xxx}.$

Equation (22) has an energy integral that is

$$E = \tfrac{2}{15}(u_x u_{xxx}-\tfrac{1}{2}u_{xx}^2)+\tfrac{1}{6}u_x^2+\tfrac{1}{2}u^2[(c^2-1)-u(c-\tfrac{1}{4}u)]. \tag{24}$$

Introducing the change of variables

$$\left.\begin{aligned} q_1 &= u, & p_1 &= -(\tfrac{2}{15}u_{xxx}+\tfrac{1}{3}u_x), \\ q_2 &= \tfrac{2}{15}u_{xx}, & p_2 &= u_x, \end{aligned}\right\} \tag{25}$$

the system (23) can be written in the following form

$$Q_t = \mathbf{J}D_Q E, \tag{26}$$

where $Q = (p_1, p_2, q_1, q_2)$ and $D_Q E$ is the gradient of E with respect to Q. $\mathbf{J}$ is the following symplectic linear operator (Arnol'd 1978)

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{27}$$

Therefore the system is Hamiltonian with a parameter c. The Hamiltonian is the energy

$$E(p_1, p_2, q_1, q_2) = -p_1 p_2-\tfrac{1}{2}(\tfrac{15}{2}q_2^2+\tfrac{1}{3}p_2^2)+\tfrac{1}{2}q_1^2[(c^2-1)-q_1(c-\tfrac{1}{4}q_1)]. \tag{28}$$

Notice that we started with a continuous Hamiltonian system evolving in time. Looking for travelling waves of that system we reduced the equations to a fourth-order discrete system evolving in x, and we have found that this new system is also Hamiltonian. Benjamin (1984) has shown that this is a general property for travelling waves of continuous Hamiltonian systems for which Hamiltonian densities can be expressed in terms of finite-order derivatives of the canonical variables, as happens in our model.

Our aim is to study periodic orbits of the system (23). These periodic orbits correspond to periodic travelling waves of permanent form. The system has three fixed points for all values of the parameter c. These fixed points are

$$u_1 = \begin{cases} 0 \\ \frac{1}{2}(3c+(c^2+8)^{\frac{1}{2}}) \\ \frac{1}{2}(3c-(c^2+8)^{\frac{1}{2}}) \end{cases} \quad u_2 = u_3 = u_4 = 0. \tag{29}$$

Studying the eigenvalues of the linearized systems around the fixed points, it is possible to see that there are several families of periodic orbits around the fixed points. We are interested in the Stokes waves, and these waves correspond to periodic orbits around the fixed point $u_1 = u_2 = u_3 = u_4 = 0$ for $c > 1$. This family exists in the present model.

It is important to notice that in the case of deep water Stokes waves form a one-parameter family of waves. In the finite-depth case there is an additional parameter that is the depth of the fluid. As we have normalized the depth to 1, the additional parameter in our formulation is the period L of the considered periodic wave. We are interested in studying the properties of the Stokes waves as the phase speed c is changed, therefore we will fix the period and will consider the solutions depending only on one parameter. A change of the phase speed c corresponds to a change of the wave amplitude.

4. Computations and results

To perform the continuation in c along the family of periodic orbits we have used the program AUTO developed by Doedel & Kernevez (1986). This program can locate Hopf bifurcation points in dynamical systems, and continue the periodic orbits that appear at those bifurcation points in two parameters using a collocation method.

The code AUTO continues periodic orbits of dynamical systems by writing the problem as a boundary-value problem for the set of ordinary differential equations that describe the system and imposing periodic boundary conditions on all the variables. The interval of computation is scaled to $(0,1)$ by introducing the period L of the orbit as a new parameter. To solve the boundary-value problem numerically the differential equations are approximated by the method of collocation at m Gauss points with piecewise Lagrange polynomials that belong to the class $C[0,1]$. This approach is equivalent to an implicit Runge–Kutta method. With this discretization a pseudo-arclength continuation is used for the computations of the solution branches. Even though the code allows non-uniform meshes, in the present computations the mesh for the collocation was chosen to be regular with $m = 4$. Up to a maximum of 80 mesh points were used to ensure the convergence in the region of interest.

Hamiltonian systems are degenerate dynamical systems, in the sense that they have families of periodic orbits for fixed values of the parameters (Abraham & Marsden 1978). For example our system has infinitely many periodic orbits for a given value of the parameter c. These periodic orbits are not isolated. They form a continuous family depending on the period L. In a generic dynamical system the periodic orbits are isolated, and for given values of the parameters the orbit and the period are determined. Because of this degeneracy in computing periodic solutions of Hamiltonian systems singular Jacobians arise. To avoid this condition it is better to perturb the system for numerical proposes. For the computations we cons˙der the following perturbed system

$$\left.\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}x} &= u_2-\epsilon u_1,\\ \frac{\mathrm{d}u_2}{\mathrm{d}x} &= u_3,\\ \frac{\mathrm{d}u_3}{\mathrm{d}x} &= u_4,\\ \frac{\mathrm{d}u_4}{\mathrm{d}x} &= \tfrac{15}{2}u_1[(c^2-1)-\tfrac{1}{2}u_1(3c-u_1)]-\tfrac{15}{6}u_3.\end{aligned}\right\} \tag{30}$$

This is a two parameter (c, ϵ) non-degenerate dynamical system. For $\epsilon = 0$ we reproduce our original system.

We want to continue a travelling wave of fixed period L in the parameter c. The additional constraint of fixing the period L imposes a relation between the two parameters c and ϵ (a codimension one situation). In our case for a given c and a period L, the periodic wave and also ϵ are determined. Actually we should find that $\epsilon \equiv 0$ for all solutions because these are in fact solutions of the Hamiltonian system. The value obtained for ϵ will serve as a check of the accuracy of the computations.

Looking at the eigenvalues of the linearization of the system (30) around the fixed points, it is possible to see that fixed points of our Hamiltonian system (23) are Hopf bifurcations in the perturbed system (30) when $\epsilon = 0$. The bifurcations are in fact vertical Hopf bifurcations (Guckenheimer & Holmes 1983).

To generate a starting orbit for the continuation process, consider the fixed point $u_1 = u_2 = u_3 = u_4 = 0$ of the perturbed system (30). For a fixed value of c there is a Hopf bifurcation at $\epsilon = 0$. We take the branch of periodic orbits that is created at this bifurcation point, and we continue it with ϵ as parameter without fixing the period. Notice that the continuation can be done with only one parameter, keeping the other fixed, because the period is not fixed. We find that the branch is vertical, in the sense that $\epsilon \equiv 0$ in all the branch as we expected. Any point of this branch can be a starting orbit to be continued in c and ϵ. The family of orbits corresponding to this branch are shown in figure 2.

In all our computations ϵ was less than 10^{-10}. In the following we will say that we do the continuation in only one parameter c, but actually we continue in c and ϵ, and find that $\epsilon \equiv 0$ for all the solutions.

The program also computes the Floquet multipliers of the orbits. The multipliers give information about the possible bifurcation points and the stability of the orbits. An orbit is said to be stable if all the Floquet multipliers lie inside the unit circle, and unstable if at least one multiplier lies outside the unit circle. If all Floquet multipliers lie on the unit circle the system is said to be marginally stable, and to determine the stability it is necessary to go to a nonlinear analysis.

It is important to notice that the concept of stability related to the periodic orbits of the dynamical system (23) is not the same as the stability of the travelling waves, which is determined by the time-dependent analysis of the evolution equations (18). The multipliers can be thought of, however, as the magnification of a perturbation in one period.

A periodic orbit always has a multiplier $+1$ corresponding to sliding a little along the orbit. For Hamiltonian systems E is conserved so there is always another multiplier $+1$. Also we know that the product of the four multipliers has to be $+1$

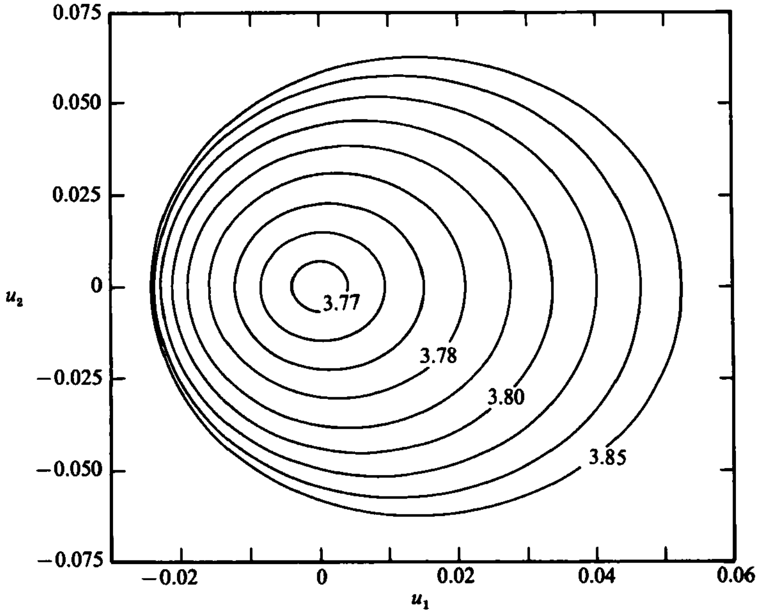

FIGURE 2. Periodic orbits for $c = 1.05$ demonstrating degeneracy of Hamiltonian systems. Numbers denote wavelength.

because the system is Hamiltonian. Together with the reality of E, the two other multipliers will be conjugate points $\sigma, \bar{\sigma}$ on the unit circle or reciprocal points $\sigma, 1/\sigma$ on the real axis (Green *et al.* 1981). So our system actually will have two Floquet multipliers not equal to $+1$. For proofs of these results refer to Abraham & Marsden (1978).

A discrete Hamiltonian system of two degrees of freedom (q_1, q_2), *via* surface section (Arnol'd & Avez 1968) can be reduced to a two-dimensional area-preserving map. Green *et al.* (1981), carrying out numerical experiments, found some universal behaviour in families of area-preserving maps with symmetries. In particular they were able to locate symmetry-breaking bifurcations in the bifurcation tree of a universal one-parameter map. Because of the analogy with the problem of travelling water waves of permanent form, we expect a similar behaviour to occur in the water-waves problem. So we study the bifurcation tree of our system with c as a parameter.

The way in which the starting point was generated fixed the period to be $L = 3.8$. With this value of the period we compute the Stokes waves for different values of c. This branch will be referred to as a period-1 branch (P-1). In this branch for small values of c the two Floquet multipliers are real and positive. As c is increased the multipliers go through $+1$ and lie on the unit circle for a range of values of c. They then go through -1 and lie again on the real axis, being now negative (see figure 3).

When the two multipliers lie on the unit circle, we have possible bifurcation points to new orbits. For example if $\sigma = e^{2\pi i m/n}$, at that point there is a possible bifurcation to a period-n wave†. The bifurcations correspond to the subharmonic bifurcations that Saffman (1980) found for finite amplitude Stokes waves on deep water. In particular if $m = 1$ and $n = 2$ then $\sigma = -1$ and we have a period doubling bifurcation.

† The possibilities of these bifurcations are limited by the conservation of the *Poincaré index* (Finn 1974).

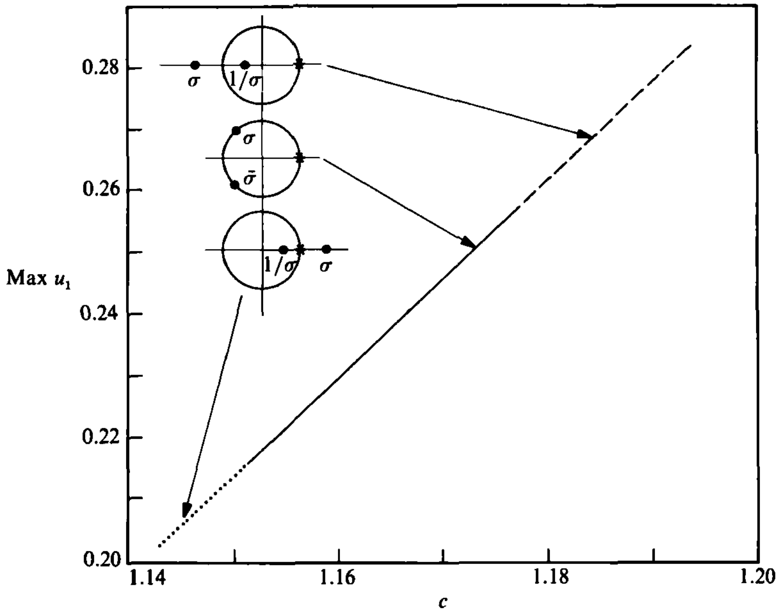

FIGURE 3. Maximum of $u_1(x)$ versus the phase speed for the period L branch (P-1). Dashed line denotes σ real and negative, dotted line σ real and positive, and solid line σ on the unit circle.

The new bifurcated branch corresponds to the Class 2 waves that Chen & Saffman (1980) and Vanden-Broeck (1983) computed.

As our model is qualitatively correct, in the sense that it has the same symmetries and Hamiltonian structure as the original water-wave problem, the qualitative agreement with the results given by Saffman (1980), Chen & Saffman (1980) and Vanden-Broeck (1983) is very good. Our results show the same structure in the sequence of different crests. However, there are quantitative differences, expected because the model is only an approximation. We find from the results given by Vanden-Broeck (1983) and Cokelet (1977), for the value of the period considered in our computations, the period-doubling bifurcation in the full equations occurs at an amplitude $a/\lambda \approx 0.122$, whereas the value obtained in our computations is $a/\lambda \approx 0.095$.

It is important to say at this point that the Boussinesq approximation (one term less in the expansion of the dispersion) can be reduced to a one-degree-of-freedom Hamiltonian system. In that case there are only two Floquet multipliers and as we stated above they have to stay at $+1$ for all values of the parameter c. So the Boussinesq model does not have enough structure to show all this behaviour.

Consider now the case $\sigma = e^{2\pi i/3}$ corresponding to a period-tripling bifurcation (point A figure 4). We computed the new branch of period $3L$. This new branch (P-3) corresponds to the Chen & Saffman Class 3 waves. The waves are still symmetric, but the crests and troughs have unequal levels. A characteristic of this branch is that it exists for smaller and larger values of the phase speed than the value of c at the period-tripling bifurcation point.

Computing the P-3 branch with the new period $3L$ we found that at the bifurcation point the Floquet multipliers are at $+1$, because for this branch the period-tripling bifurcation is a bifurcation point from a wave with the three equal crests to a wave with unequal crests. We found that following the P-3 branch in direction of increasing

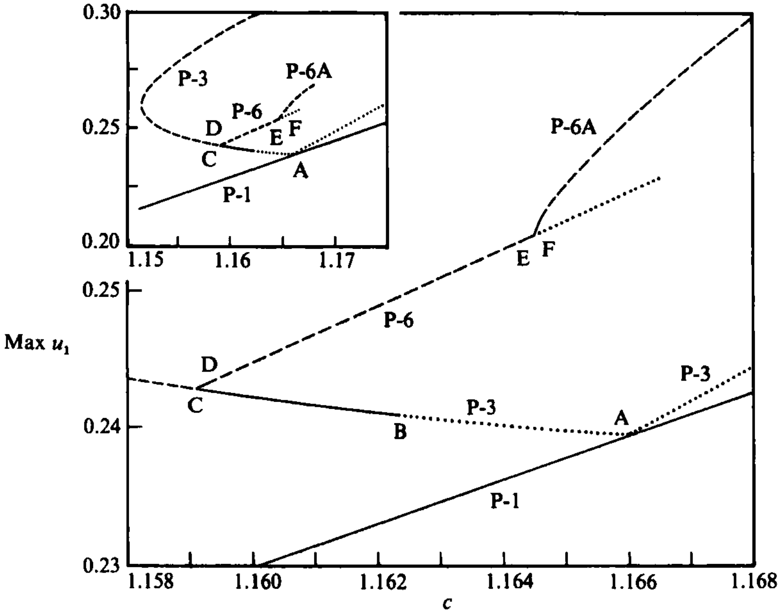

FIGURE 4. Bifurcation diagram for the family of travelling waves. Insert shows the variation of P-3 branch for larger variation of c.

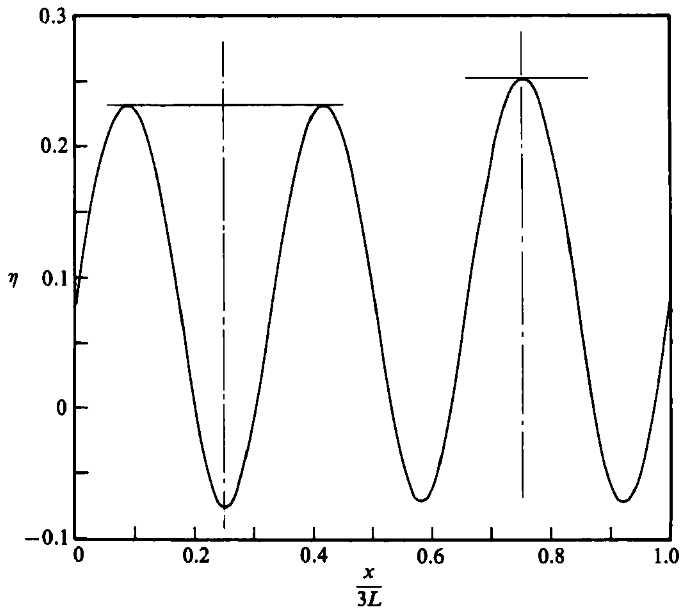

FIGURE 5. Symmetric travelling wave of wavelength $3L$ corresponding to point C in figure 4.

c the Floquet multipliers are real positive, and one of them increases (σ) as the other goes to zero ($1/\sigma$). If the branch is continued in the other direction (decreasing c) first the behaviour is the same, but σ reaches a maximum and decreases going again through $+1$ (point B figure 4). Then the multipliers move along the unit circle, and later leave the unit circle through -1 becoming real and negative. When $\sigma = -1$,

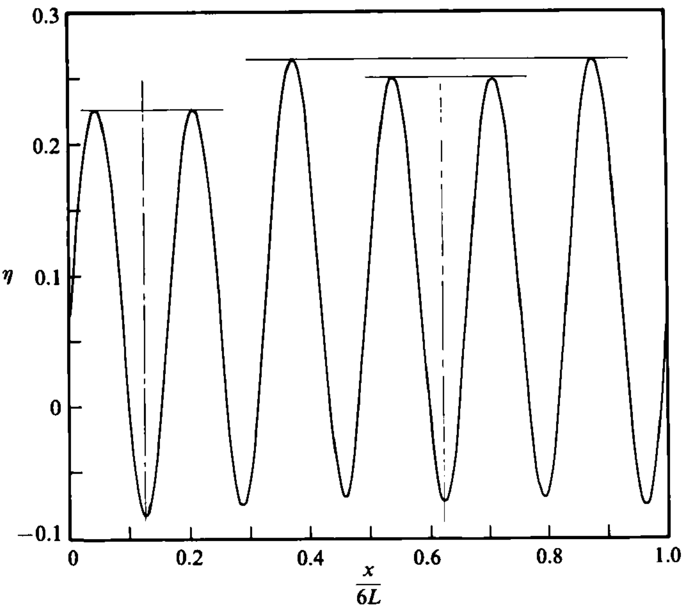

FIGURE 6. Symmetric travelling wave of wavelength $6L$ corresponding to point F in figure 4.

we have another period-doubling bifurcation (point C figure 4). This is a bifurcation from a P-3 wave to a P-6 wave. This new bifurcation corresponds to what Chen & Saffman called a secondary bifurcation. Figure 5 shows the P-3 wave at point C.

The two branches that are created at the period doubling correspond to the same wave just shifted. Computing with the new period $6L$ along P-6 we find again that $\sigma = +1$ at the bifurcation point corresponding to a bifurcation in the P-6 branch. As c is changed the Floquet multipliers move along the unit circle and go through -1 giving a new period-doubling bifurcation to a period $12L$ wave (point D figure 4). Beyond this period doubling on the P-6 branch, the multipliers are real negative. As c increases σ increases in absolute value up to a point where σ reaches a maximum and starts decreasing again. The multipliers go again to the unit circle through -1. So we have a restabilization through an inverse period doubling (point E figure 4).

If we keep going on the P-6 branch we find that σ goes again through $+1$ giving a new bifurcation in which all multipliers are at $+1$ (point F figure 4). This kind of bifurcation was studied by Rimmer (1978). The bifurcation can be of two types: a saddle-node or a symmetry-breaking bifurcation. To determine whether we are in one case or in the other, it is necessary to perform a nonlinear stability analysis of the periodic orbit. A saddle-node bifurcation is not actually a bifurcation point in the sense that two families of orbits cross at the bifurcation point, but it indicates an exchange of stability. A saddle node corresponds to a fold on the bifurcation diagram when this is plotted using an appropriated characteristic parameter that the problem has. In general this parameter is the energy of the orbit, and the fold corresponds to an extremum of the energy.

An example of a saddle-mode bifurcation was found by Tanaka (1985*a*) at the point of maximum energy for periodic travelling waves of permanent form on deep water (however Tanaka's concept of stability is different from the one used here). He found that a change of superharmonic stability happens for the wave of maximum energy, and at this point there is no non-trivial bifurcation since the eigenvector, corre-

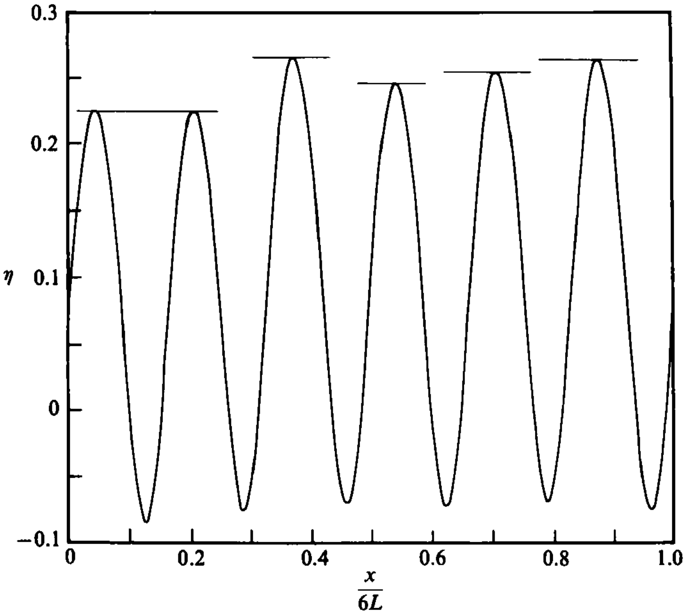

FIGURE 7. Non-symmetric travelling wave of wavelength $6L$ for $c = 1.1646$.

sponding to the zero eigenvalue, is the eigenvalue corresponding to the horizontal displacement allowed by Galilean invariance. This result was later proved analytically by Saffman (1985) using the Hamiltonian formulations for water waves.

The two previous cases in which we found that all the Floquet multipliers were at $+1$ (branches P-3 (point B figure 4) and P-1) are actually saddle-node bifurcation points. In the present case (point F figure 4) we have found a symmetry-breaking bifurcation. The new branch that bifurcates from this point (P-6A) corresponds to period $6L$ waves, but these are not symmetric any more. Figure 6 shows the travelling wave at point F.

In the P-3 branch the sequence of crests was XXY, in the P-6 branch it was $XXYZZY$. These two sequencies are symmetric. In the non-symmetric branch the sequence is $XXYZWY$ which is not symmetric.

Figure 7 shows a wave on the non-symmetric P-6A branch.

The present study is evidence that the behaviour of continuous systems can in some cases be described by a discrete dynamical system of only a few degrees of freedom.

The discovery of non-symmetric water waves in the truncated model leads us to postulate the existence of non-symmetric solutions in the full equations for water waves. In particular the existence of these waves on deep water is presently under study.

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