Spectral Moment Estimates from Correlated Pulse Pairs

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Abstract

Estimates statistics of the first two power spectrum moments from the pulse pair covariance are analyzed. The input signal is assumed to be colored Gaussian and the noise, white Gaussian. Perturbation formulas for the standard deviation of both mean frequency and spectrum width are applied to a Gaussian shaped power spectrum, and so is a perturbation formula for the bias in the width estimate.

Mean frequency estimation from interlaced pulse pairs is presented. Throughout this study, estimators from independent, spaced, and contiguous pulse pairs are compared to provide a continuum of statistics from equispaced tightly correlated to statistically independent pulse pairs.

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I. Introduction

Spectral moment estimation by covariance argument techniques is increasing as an important signal processing tool in the analysis of pulse Doppler echoes from distributed scatterers [1].

Hyde and Perry [2] suggested a version of the covariance technique for first moment estimation, and Woodman and Hagfors [3] used it to measure mean ionospheric motions. Independently, Rummler had proposed estimators of both mean and variance from pulse pair bursts [4], [5]. Hofstetter [6] and Miller and Rochwarger [7] established maximum likelihood properties of these estimators. Furthermore, Miller and Rochwarger analyzed the first and second order statistics of the estimators. The probability density of the mean frequency for a large number of independent pairs can be found in [3], while Berger [8] calculated the density for any arbitrary number of independent pairs.

To process independent echo sample pairs requires much longer observation time; hence it has been suggested [4] that independence in a uniform pulse train can be created by changing the carrier frequency on alternate pairs. This, of course, considerably complicates the system's hardware. Moment estimation from contiguous pairs with constant carrier frequency is the most direct technique to implement on conventional radars. Arithmetic units that calculate spectral moments using the fast Fourier transform algorithm also can be used. Although these units rapidly calculate the spectrum and its moments, all echo samples comprising the time series must be stored prior to calculations. When data fields consist of a thousand sample volumes along a radial and each volume generates a few hundred samples, the storage required normally exceeds a million bits.

Estimates from contiguous pairs (with a common pulse among each two pairs) were analyzed by Benham et al. [9], and Berger and Groginsky [10], who found the standard deviations of the mean and variance. In this paper the perturbation analysis of Benham et al. [Appendix A] is extended to include arbitrarily spaced pulse pairs. Under appropriate limiting conditions, it is shown how this more general situation reduces to previous results of Berger and Groginsky [10] (pair spacing equal to intrapair period), Miller and Rochwarger [7] (large spacing between pairs or uncorrelated pairs) and Lank et al. [11] (zero width spectrum or fully correlated signal samples). Moreover certain results not available in [10] are presented. Estimation of mean frequency from interlaced unequally spaced pulses is analyzed in detail. Further, the usefulness of such a technique for staggered PRF radars is discussed briefly.

II. Mean and Width from Correlated Pulse Pairs

Assume that the detected radar echo is a sample of the complex envelope $Z(t)$ with Gaussian in-phase and quadrature components that contain both colored signal and white noise contributions such that the autocorrelation function is

$$r(\tau) = s(\tau) + N\delta_{\tau,0} \tag{1}$$

where $s(\tau)$ is signal autocorrelation, N is the noise power per sample, and $\delta_{\tau,0}$ is the Kronecker delta symbol. Assume that staggered pulses are transmitted and spectral moments estimated from their sampled returns spaced as depicted by Fig. 1. These moments are related via the Doppler equation to the mean velocity and velocity spread.

The complex video signals at times $i(T_1 + T_2)$ and $i(T_1 + T_2) + T_1$ are

$$Z_{2i} = Z[i(T_1 + T_2)] = Z(iT)$$

$$Z_{2i+1} = Z[i(T_1 + T_2) + T_1] = Z(iT + T_1) \tag{2}$$

from which the autocorrelation function for a T_1 lag is estimated as

$$\hat{U} = M^{-1} \sum_{i=0}^{M-1} Z_{2i} Z^*_{2i+1} \tag{3}$$

where M is the number of pairs that are spaced T seconds apart. An autocorrelation estimate $\hat{W}$ for a T_2 lag is obtained from pairs also spaced T seconds apart:

$$\hat{W} = M^{-1} \sum_{i=0}^{M-1} Z_{2i+1} Z^*_{2i+2}. \tag{4}$$

Motivation behind this particular staggered pulse train and the consideration of two covariance estimates is twofold. First, by allowing the ratio T_1/T to vary from one to zero, all cases from contiguous pairs (with a common pulse) to independent pairs are treated in a single unifying theory. Second, Doviak et al. [12] have shown that from two covariance estimates at different lags some aliased mean velocities can be retrieved. Each individual unambigruous velocity is inversely proportional to the corresponding pulse separation (T_1 or T_2); the composite maximum unambiguous velocity obtained by comparing the two estimates is increased. An accurate mean velocity estimate can be obtained only if the difference between the velocities is largely due to different amounts of aliasing associated with the two time separations. The correlation between to mean velocity estimates relates to accuracy of ambiguity resolution; this correlation is evaluated in Section IIB. The following two mean frequency estimates and a spectrum width estimate are considered:

frequency:

$$\hat{f}_1 = (2\pi T_1)^{-1} \arg \hat{U} \tag{5.a}$$

$$\hat{f}_2 = (2\pi T_2)^{-1} \arg \hat{W} \tag{5.b}$$

width:

$$\hat{w}_2 = (2\pi^2 T_1^2)^{-1} [1 - |\hat{U}|/(\hat{Y} - N)] \tag{6}$$

where $\hat{Y}$ is the total power estimate of the returned pulse train

$$\hat{Y} = L^{-1} \sum_{k=0}^{L-1} |Z_k|^2 \tag{7}$$

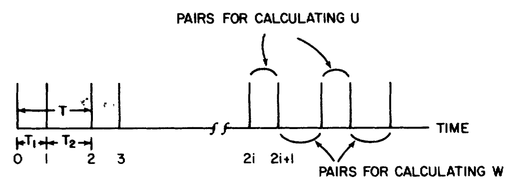

Fig. 1. Staggered pulse train. The number of pairs with T_1 and with T_2 separation between pulses is M.

and L is the number of pulses ($L = 2M$ for independent or spaced pairs; $L = M + 1$ for contiguous pairs).

The rationale behind these estimators can be found in Woodman and Hagfors [3], Rummler [4], [5], or Miller and Rochwarger [7]. Signal spectrum widths and receiver noise progressively degrade the estimates.

A. Mean Frequency Estimate Variance

To proceed further with the analysis, we need specific autocorrelation functions. A Gaussian shaped power spectrum represents various turbulent media very well and hence will be used throughout this study. Its autocorrelation is

$$r(\tau) = S\beta(\tau)e^{j\omega_0\tau} + N\delta_{\tau,0} \tag{8}$$

where

$$\beta(\tau) = e^{-2\pi^2 w^2 \tau^2}.$$

Here S and N are the signal and noise power per sample, $\omega_0 = 2\pi f_0$ the mean radian frequency, and w the spectral width. With this autocorrelation the variance [see Appendix (B7)] becomes

$$\begin{aligned}\operatorname{var} \hat{f}_1 = {} & [8\pi^2 T_1^2 \beta^2(T_1)]^{-1} \{M^{-2} [1 - \beta^2(T_1)] \\ & \cdot \sum_{-(M-1)}^{M-1} \beta^2(mT)(M - |m|) + N^2/MS^2 + (2N/MS) \\ & \cdot [1 - \beta(2T)\delta_{T-T_1,0} + [\beta(2T)/M]\,\delta_{T-T_1,0}]\}.\end{aligned} \tag{9}$$

Note that the Kronecker delta symbol in (9) is always zero for the spacing of pairs as in Fig. 1. The reason we include it is to show that (9) reduces exactly to Berger and Groginsky [10, eq. (45)] when pairs are contiguous ($T_1 = T$). For independent pairs ($T \Rightarrow \infty$), (9) simplifies to

$$\operatorname{var} \hat{f}_1 = (8\pi^2 T_1^2 M)^{-1} [(1 + N/S)^2 - \beta^2(T_1)]/\beta^2(T_1) \tag{10}$$

which is identical to Miller and Rochwarger [7, eq. (17)]. Expression (9) can be evaluated exactly for various number of pairs M; normalized widths wT_1 and various signal-to-noise ratios S/N. For a large number of pulse pairs, the only significant term in the summation (9) is the one

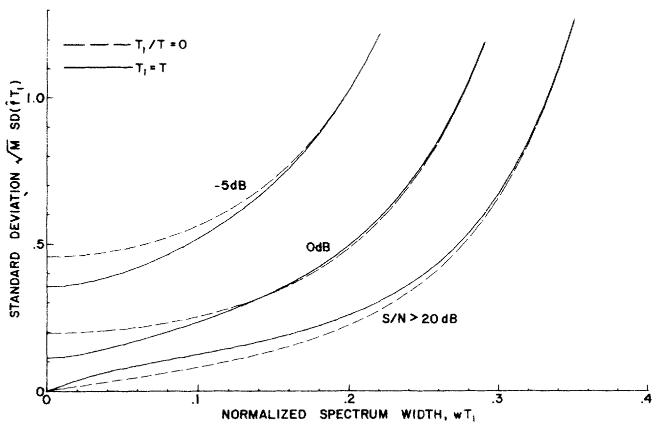

Fig. 2. Standard deviations of the mean frequency estimates for both independent (dashed line) and contiguous (solid line) pulse pairs. Note the almost constant deviation values between 0 and 0.1 and the doubling in value at about 0.2.

multiplied by M, and the variance is

$$\operatorname{var} \hat{f}_1 = [8M\pi^2 T_1^2 \beta^2(T_1)]^{-1} \{[1 - \beta^2(T_1)] \sum_{-(M-1)}^{M-1} \cdot \delta_{T-T_1,0} \beta^2(mT) + N^2/S^2 + 2(N/S) \cdot [1 - \beta(2T)\delta_{T-T_1,0}]\} + O(M^{-2}). \tag{11}$$

Furthermore, when $wT < (2\pi)^{-1}$ but $MwT \gg (2\pi)^{-1}$, the summation can be approximated with the integral

$$\Sigma \beta^2(mT) \approx \int_{-\infty}^{\infty} \exp(-2 \cdot 2\pi^2 w^2 m^2 T^2)\, dm = (2\sqrt{\pi} wT)^{-1} \tag{12}$$

to yield the following approximate variance equation:

$$\operatorname{var} \hat{f}_1 \approx [8\pi^2 M \beta^2(T_1) T_1^2]^{-1} \{2\pi^{3/2} w T_1^2/T + N^2/S^2 + 2(N/S)[1 - \beta(2T)\delta_{T-T_1,0}]\}. \tag{13}$$

At large signal-to-noise ratios (13) becomes

$$\operatorname{var} \hat{f}_1 \approx w/4\sqrt{\pi} MT\beta^2 (T_1). \tag{14}$$

Under noiseless conditions, it is instructive to compare the number of samples needed to achieve equal variance reduction (of the mean estimate) when contiguous pairs are used with the number when spaced but correlated pairs are employed (sample pair separation equals T_1). The ratio of the two numbers is

$$M_c/M_s = T/T_1 \tag{15}$$

where M_c is the number of contiguous pairs and M_s is the number of spaced pairs. Since the total dwell times are equal, $M_c T_1 = M_s T$, spaced pairs (which are fewer in number) achieve the same variance reduction during the dwell time as do the contiguous pairs. This is true only when spaced pairs are well correlated, the signal-to-noise ratio is large, and $wT < (2\pi)^{-1}$.

The normalized standard deviation, $\sqrt{M}\ \mathrm{SD}\ (\hat{f}_1 T_1)$, given in (9) is plotted in Fig. 2 which reveals that normalized spectrum widths wT_1 should be less than about 0.2 if variance is not to be strongly dependent on width. The curves for contiguous pairs give a standard deviation that is larger (about 3 times at $wT_1 = 0.3$) than those plotted by Berger and Groginsky [10], because the approximate formula (13) used in [10] is only valid for $wT_1 < (2\pi)^{-1}$. The estimate standard deviation is smaller for independent pairs when the signal-to-noise ratio (S/N) is large, but with increased spectrum widths both deviations are nearly equal. However, at low signal-to-noise ratios, mean frequency estimates from contiguous pairs are better. This behavior perhaps can be best understood if one considers a perfect sinusoid imbedded in noise at the input. Substitute the autocorrelation

$$r(mT_1) = S e^{j\omega_0 m T_1} + N\delta_{m,0} \tag{16}$$

into the variance expression (B7) to obtain*

$$\operatorname{var} (\hat{f}_1 T_1) = (8\pi^2 M)^{-1} \{2(N/S) - 2(N/S)[(M-1)/M]\, \delta_{T-T_1,0} + N^2/S^2\}. \tag{17}$$

For contiguous pairs ($T_1 = T$) this reduces to

$$\operatorname{var} (\hat{f}_1 T_1) = (8\pi^2 M)^{-1} [(2/M)(N/S) + N^2/S^2] \tag{18}$$

which is identical to Lank et al. [11, eq. (75)]. When pulse pairs are spaced, the variance is larger:

$$\operatorname{var} (\hat{f}_i T_i) = (8\pi^2 M)^{-1} [2(N/S) + N^2/S^2] \tag{19}$$

because the noise decorrelates more the sample pairs, i.e., there is not a common pulse which partially cancels the noise effects.

Various T_1/T values (Fig. 3) show the standard deviation behavior between the two extremes, independent and contiguous pairs. As expected, when the signal-to-noise ratios are large, the standard deviation improves with an increase in separation between pairs T (i.e., longer dwell or observation time). At low S/N ($S/N < 10$ dB) two regions on the wT_1 axis are evident: in one where $wT_1 < 0.1$, contiguous pairs yield better mean estimates, while with larger wT_1, independent pairs are superior.

Perturbation analysis shows a variance that monotonically increases whereas it is known that the upper bound for the variance (white noise input $wT_1 \gg 1$) is $(12)^{-1}$. The discrepancy is due to the fact that the error $|\epsilon(T_1)| > |r(T_1)|$ so that the perturbation expansion [see Appendix (A3a)] is not valid.

*Note that (B5) and (B8) are not valid for sinusoidal signals because expansion into moments (B3) is not directly applicable; it doubles the power of deterministic signals. However in all the derived statistics, differences of (B5) and (B8), (and similar expressions) are taken so that the excess sinusoidal power cancels out.

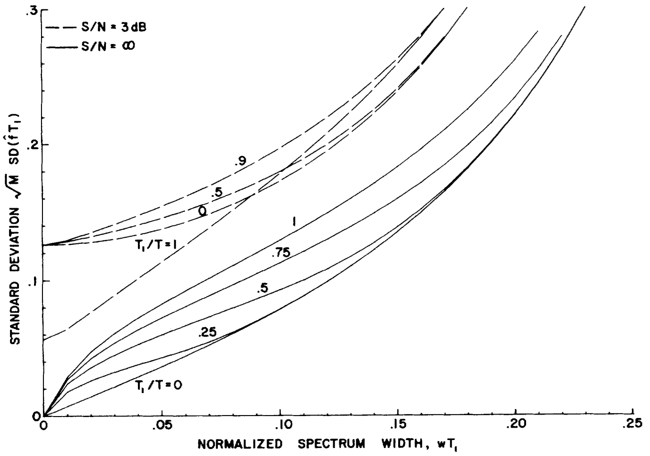

Fig. 3. Mean frequency estimate standard deviations for all three pulse pairs: independent, spaced, and contiguous pairs. At infinite signal-to-noise ratios, independent pairs are best at all wT_1 values. At lower ratios ($S/N < 3$ dB) the contiguous pairs are superior in the region $wT_1 < 0.1$. The improvement in standard deviation can be over 50 percent and is largest for sinusoidal signals.

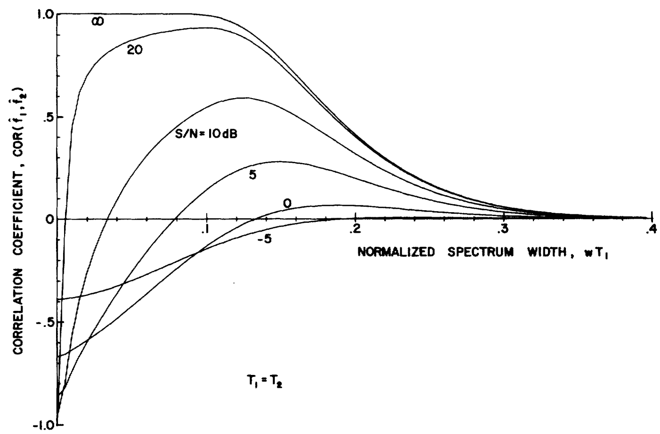

Fig. 4. Correlation coefficient between two mean frequency estimates obtained from interlaced pulse pairs ($T_1 = T_2$). The correlation for a sinusoid at an infinite signal-to-noise ratio is -1. Note that for $wT_1 > 0.3$ correlation is virtually zero.

B. Correlation Between Mean Frequency Estimates

The correlation between two frequency estimates $\hat{f}_1$ and $\hat{f}_2$ (5a), (5b) and the standard deviation of their difference is computed in this section.

With a Gaussian spectrum shape (8), the correlation equation (B10) is

$$E(\delta\hat{\theta}_1\,\delta\hat{\theta}_2) = [2M^2\beta(T_1)\beta(T_2)]^{-1}\Big\{\sum_{m=-(M-1)}^{M-1} \cdot [\beta(mT+T_1)\,\beta(mT+T_2) - \beta[(m+1)T]\,\beta(mT)]\,(M-|m|) - (2M-1)(N/S)\,\beta(T)\Big\}. \tag{20}$$

For large M, we retain terms next to M in the summation (20) to obtain

$$E(\delta\hat{\theta}_1\,\delta\hat{\theta}_2) = [2M\beta(T_1)\beta(T_2)]^{-1}\{\exp[-\pi^2w^2(T_1-T_2)^2] \cdot [1-\exp(-4\pi^2w^2T_1T_2)] \cdot \sum_{-(M-1)}^{M-1} \beta^2[(m+0.5)T] - 2\beta(T)\,(N/S)\} + O(M^{-2}) \tag{21}$$

which yields the correlation coefficient

$$\mathrm{cor}\,(f_1,f_2) \approx \{\exp[-\pi^2w^2(T_1-T_2)^2] \cdot [1-\exp(-4\pi^2w^2T_1T_2)] \sum_{-(M-1)}^{M-1} \beta^2[(m+0.5)T] - 2\beta(T)\,(N/S)\}/\{[[1-\beta^2(T_1)] \sum_{-(M-1)}^{M-1} \beta^2(mT) + N^2/S^2 + 2N/S]^{1/2}\, [[1-\beta^2(T_2)] \sum_{-(M-1)}^{M-1} \beta^2(mT) + N^2/S^2 + 2N/S]^{1/2}\}. \tag{22}$$

Further simplification of (22) is possible when wT_1 and wT_2 are small:

$$\mathrm{cor}\,(\hat{f}_1,\hat{f}_2) \approx \{\exp[-\pi^2w^2(T_1-T_2)^2]2\pi^{3/2}\,wT_1T_2/T - 2\exp(-2\pi^2w^2T^2)(N/S)\}/(2\pi^{3/2}wT_1T_2/T + N^2/S^2 + 2N/S). \tag{23}$$

When the input signal is sinusoidal, the correlation [from (20) and (18)] simplifies to

$$\mathrm{cor}\,(\hat{f}_1,\hat{f}_2) = -(1-1/2M)/(1-N/2S). \tag{24}$$

It is anticipated that in most applications T_1 and T_2 will not differ by much; hence the correlation (21) for $T_1 = T_2$ was computed. At $wT_1 > 0.3$ the two estimates (Fig. 4) are uncorrelated. Almost identical results were obtained for ratios $T_1/T = 4/9$ and $T_1/T = 3/7$. Next, the standard deviation SD is found:

$$\sqrt{M}\,\mathrm{SD}\,(\hat{f}_1-\hat{f}_2)T_1 = T_1\sqrt{M}\,[\mathrm{SD}^2\hat{f}_1 - 2\,\mathrm{cor}\,(\hat{f}_1,\hat{f}_2)\,\mathrm{SD}\hat{f}_1\,\mathrm{SD}\hat{f}_2 + \mathrm{SD}^2\hat{f}_2]^{1/2}. \tag{25}$$

From symmetry considerations it is deduced that, for T a constant, the $\mathrm{SD}(\hat{f}_1-\hat{f}_2)$ is smallest when $T_1 = T_2$. Also

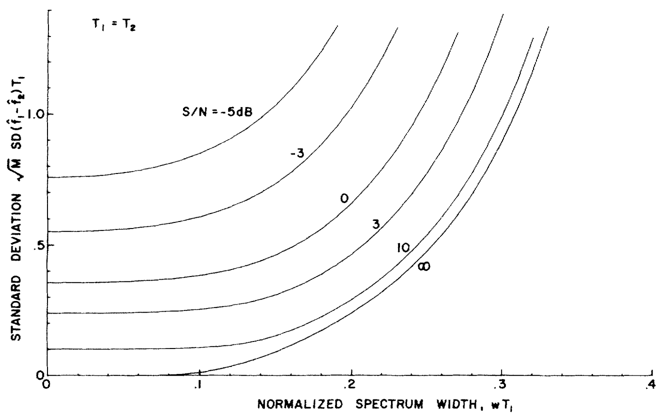

Fig. 5. Standard deviation of the difference between two mean frequency estimates obtained from interlaced pulse pairs ($T_1 = T_2$). The deviations are well behaved (constant) until $wT_1 \approx 0.12$ and are double in value at $wT_1 \approx 0.2$.

for this sample spacing the SD is almost constant at $wT_1 < 0.1$ (Fig. 5). Because the correlation between estimates is poor and each individual standard deviation increases with wT_1, the SD of the difference grows very rapidly for $wT_1 > 0.25$. Other ratios T_1/T yield somewhat larger standard deviations that still retain the above general behavior. When a sinusoid is corrupted with additive noise, the SD is found to be

$$\sqrt{M}\ \mathrm{SD}\,(\hat{f}_1 - \hat{f}_2)T_1 = (2\pi)^{-1}\{(N/S + N^2/2S^2)$$

$$[1 + T_1^2/T_2^2 + 2T_1(1 - 1/2M)/T_2(1 + N/2S)]\}^{1/2}. \quad (26)$$

As expected with no noise, the deviation is zero, and at low signal-to-noise ratios, it is inversely proportional to that ratio. Again the optimum time spacings are $T_1 = T_2$; at other spacings the variance of the difference is larger.

C. Variance of the Width Estimate

Spectrum width estimator (6) is independent of the mean frequency (A9); therefore, there is no motivation to consider samples spaced T_1 and T_2 seconds apart. Only the spacing T_1 with distance between pairs $T = T_1 + T_2$ is treated in this section.

A formula for the width variance is derived in (B15). Specifically for a Gaussian autocorrelation function (8), it is

$$M\,\mathrm{var}\,(\delta\hat{w}T_1) = \{16\pi^2[1 - \beta(T_1)]\}^{-1}$$

$$\cdot\{2[1 - (1 + \delta_{T-T_1,0})\beta^2(T^1)$$

$$+ \delta_{T-T_1,0}\beta^4(T_1)](N/S) + [1 + (1 + \delta_{T-T_1,0})$$

$$\cdot\,\beta^2(T_1)]\,N^2/S^2)$$

$$+ \beta^2(T_1)\sum_{m=-(M-1)}^{M-1}(\beta^2(mT)\,[2 + [\beta^{-2}(T_1)]$$

$$+ \beta^2(mT + T_1) - 4\beta(mT + T_1)\beta(mT)/\beta(T_1)$$

$$\cdot\,(1 - |m|/M)\}. \quad (27)$$

Note that (27) reduces exactly to Miller and Rochwarger [7, eq. (19)] when pairs are independent:

$$M\,\mathrm{var}\,(\delta\hat{w}T_1) = \{[1 - \beta^2(T_1)]^2 + 2[1 - \beta^2(T_1)](N/S)$$

$$+ [1 + \beta^2(T_1)](N^2/S^2)\}/\{16\pi^2[1 - \beta(T_1)]\}. \quad (28)$$

At narrow spectrum widths $[wT_1 < (2\pi)^{-1}]$, the following approximations for the variances are possible.

Independent Pairs:

$$M\,\mathrm{var}\,(\delta\hat{w}T_1) = \tfrac{1}{2}(wT_1)^2 + \tfrac{1}{4}[\pi^{-2} - (wT_1)^2](N/S)$$

$$+ \tfrac{1}{16}\{[\pi^4(wT_1)^2]^{-1} - \pi^{-2} + \tfrac{7}{3}(wT_1)^2\}(N^2/S^2)$$

$$+ O[(wT_1)^3]. \quad (29a)$$

Spaced Pairs:

$$M\,\mathrm{var}\,(\delta\hat{w}T_1) = 3wT_1^2/32\sqrt{\pi}\,T + \tfrac{1}{4}[\pi^{-2} - (wT_1)^2](N/S)$$

$$+ \tfrac{1}{16}\{[\pi^4(wT_1)^2]^{-1} - \pi^{-2} + \tfrac{7}{3}(wT_1)^2\}(N^2/S^2)$$

$$+ O[(wT_1)^3]. \quad (29b)$$

Contiguous Pairs:

$$M\,\mathrm{var}\,(\delta\hat{w}T_1) = (3/32\sqrt{\pi})\,wT_1 + (wT_1)^2(N/S)$$

$$+ \tfrac{1}{32}[3/\pi^4(wT_1)^2 - 5/\pi^2 + 9\,(wT_1)^2](N^2/S^2)$$

$$+ O[(wT_1)^3]. \quad (29c)$$

The last three expressions differ slightly in that the width estimates, for equal M, from independent pairs have the smallest variance while contiguous pairs produce the largest variance. Furthermore, the terms multiplying N/S and N^2/S^2 are equal in (29a) and (29b), whereas the terms next to wT_1 in (29b) and (29c) differ by a multiplying factor T_1/T.

Note, however, that the derived variance formulas can be used as long as their dominant terms are smaller than $M(wT_1)^2$; otherwise the perturbation equations take a different form (A12) through (A16). When the input signal is a sinusoid, one finds the variance from (A16b):

$$\sqrt{M}\ \mathrm{var}\,(\delta\hat{w}T_1) = (2\pi^{5/2})^{-1}(2 + \delta_{T-T_1,0})^{1/2}(N/S)$$

$$+ O(M^{-3/2}). \quad (29d)$$

We note with relief that the variance is bounded and for

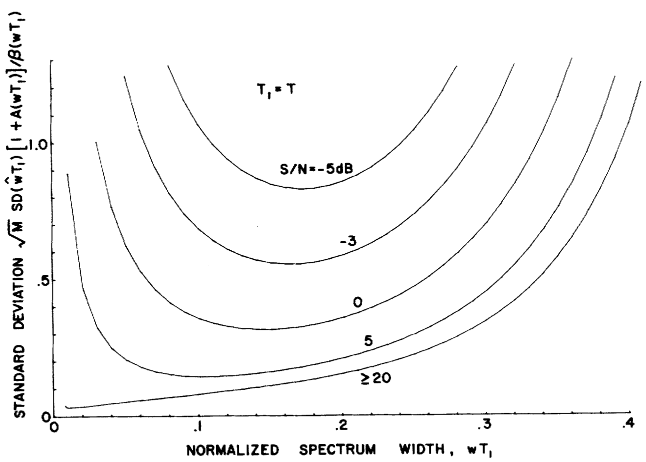

Fig. 6. Standard deviation of the asymptotically unbiased width estimate from contiguous pulse pairs. Note the poor quality of the estimators at widths $wT_1 \approx 2\pi^{-1}$ and low signal-to-noise ratios. The left portion of curves is valid when M is about 100 or larger and cannot be extended, for arbitrary large M, to the origin because the standard deviation there follows the $M^{-1/4}$ law.

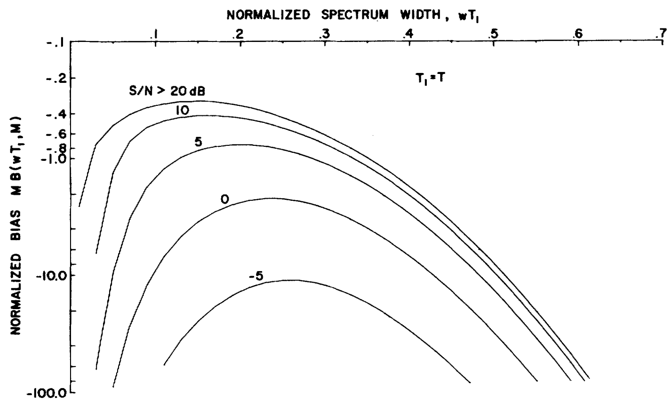

Fig. 7. Normalized M dependent bias in the width estimate from contiguous pulse pairs. This bias is very large at $wT_1 > 0.5$; hence the number of samples at these widths must be several hundred in order to obtain a useful estimate. Results towards the origin are valid for $M \geqslant 100$.

zero noise the estimate is perfect. Contrary to the M^{-1} dependence at larger widths the variance now has a $M^{-1/2}$ dependence. Equation (29d) for contiguous pairs $T = T_1$ was verified on a simulated time series.

Perturbation analysis determines a width estimate that has both an asymptotic (A9) and a number of samples M dependent bias (B16):

$$A(w) + B(w, M) = E(\hat{w})/w - 1 \tag{30a}$$

where we have defined the asymptotic bias as

$$A(w) = \lim_{M \Rightarrow \infty} E(\hat{w})/w - 1 = \bar{w}/w - 1$$

$$= [1 - \beta(T_1)]^{1/2}/\sqrt{2}\,\pi w T_1 - 1 \tag{30b}$$

and the M dependent bias as

$$B(w, M) = E(\delta\hat{w})/w. \tag{30c}$$

The asymptotically unbiased standard deviation

$$\sqrt{M}\ \mathrm{SD}\,(\delta\hat{w}T_1)[1 + A(wT_1)]\,/\beta(wT_1) \tag{31}$$

of the width estimate from contiguous pairs (27) shows (Fig. 6) the poor quality of the estimators at large spectrum widths $wT_1 > (2\pi)^{-1}$. At smaller widths, independent pairs [7] and spaced pairs are somewhat better, but with broad spectra all three have almost identical variances. Although percentage standard deviation $\mathrm{SD}(\delta\hat{w}T_1)/wT_1 - 1$ is large at small widths, it is not of great consequence. In practice it is possible for a moderate range of widths (10 to 1) to choose an interpulse spacing T_1 that yields acceptable measurement accuracy.

The M dependent bias of the width estimator (B16) is

$$\begin{aligned} B(wT_1, M) = &- \{\beta(T_1)/8M\pi w T_1\sqrt{2}[1 - \beta(T_1)]^{1/2}\} \\ &\cdot (2[\beta^{-2}(T_1) - \beta^2(T_1)\delta_{T-T_{1},0}](N/S) \\ &+ [\beta^{-2}(T_1) + 2(1 + \delta_{T-T_{1},0})]\ (N^2/S^2) \\ &+ \sum_{m=-(M-1)}^{M-1} \{\beta^2(mT)[1 + \beta^{-2}(T_1)] + 2\beta^2(mT + T_1) \\ &- 4\beta(mT + T_1)\ \beta(mT)\beta^{-1}(T_1)\}(1 - |m|/M)) \\ &- (2\bar{w}w)^{-1}\ \mathrm{var}\,(\delta\hat{w}). \end{aligned} \tag{32}$$

It is seen from (32) that $B(wT_1) \propto (M^{-1}) + O(M^{-2})$; thereby it is convenient to calculate $M\,B(wT_1, M)$. This was done for various pulse pair estimators. The results, when contiguous pairs are used (Fig. 7), suggest that this bias can ruin the estimate if the widths are $wT_1 > 0.3$ even for $M \gg 10$. Moreover, determination of the unbiased estimate is not accomplished easily because B is a complicated function of w.

For independent pairs, (32) simplifies to

$$\begin{aligned} B(wT_1, M) = &- [M\pi w T_1\ 16\sqrt{2}\,\beta(T_1)]^{-1} \\ &\{[1 - \beta(T_1)]^{3/2}[3\beta^2(T_1) + 5\beta(T_1) + 2] \\ &+ 2[1 - \beta(T_1)]^{-1/2}[\beta^2(T_1) + \beta(T_1) + 2](N/S) \\ &+ [1 - \beta(T_1)]^{-3/2}[-3\beta^3(T_1) + 4\beta^2(T_1) \\ &- \beta(T_1) + 2](N^2/S^2)\} + O(M^{-2}) \end{aligned} \tag{33}$$

which agrees with Miller and Rochwarger's results [7].

At small spectrum widths $[wT_1 < (2\pi)^{-1}]$, the following equations approximate the M dependent bias.

Independent Pairs:

$$M\,B(wT_1, M) = -\tfrac{5}{4}(\pi w T_1)^2 - \tfrac{1}{4}[(\pi w T_1)^{-2} + 1](N/S)$$
$$-\tfrac{1}{32}[(\pi w T_1)^{-4} + 11/2(\pi w T_1)^2 + \tfrac{3}{4}](N^2/S^2)$$
$$+ O[(wT_1)^2]. \tag{34a}$$

Spaced Pairs:

$$M\,B(wT_1, M) = -3/64\pi^{3/2} wT - \tfrac{1}{4}[(\pi w T_1)^{-2} + 1](N/S)$$
$$-\tfrac{1}{32}[(\pi w T_1)^{-4} + 11/2(\pi w T_1)^2 + \tfrac{3}{4}](N^2/S^2)$$
$$+ O[(wT_1)^2]. \tag{34b}$$

Contiguous Pairs:

$$M\,B(wT_1, M) = -3/64\pi^{3/2} wT_1 - \tfrac{3}{2}(N/S) - \tfrac{3}{64}[(\pi w T_1)^{-4}$$
$$+ 6/(\pi w T_1)^2 + \tfrac{1}{3}](N^2/S^2) + O[(wT_1)^2]. \tag{34c}$$

The similarity between (34a), (34b), and (34c) is evident. Terms next to N/S and N^2/S^2 in (34a) and (34b) are identical and slightly different from corresponding terms in (34c). The noise independent term in (34a) is proportional to w^2 while in (34b) and (34c) it is proportional to w^{-1}.

Again expressions for the bias are valid as long as $M(wT_1)^4\, S^2/N^2 \gg 3/(32\pi^4)$. A pure sinusoid has an infinite M dependent bias (30c) since the expected value $E(\delta\hat{w})$ differs from zero when even a slight noise is present. This expected value reads (A16a)

$$E(\delta \hat{w} T_1) = -\,[2(1 - \delta_{T-T_1,0})(N/S)^{1/2}$$
$$+ (3 + 2\delta_{T-T_1,0})(N/S)^{3/2}]\,/$$
$$[M^{3/4}(2\pi)^{1/4}\, 8\Gamma(3/4)\,(2 + \delta_{T-T_1,0})^{1/4}]$$
$$+ O(M^{-3/2})$$

and is not significant since it is proportional to $M^{-3/4}$, whereas standard deviation decays as $M^{-1/4}$.

III. Conclusions

Investigators developed formulas for the standard deviations of mean frequency and spectrum width estimates from both independent and contiguous samples pairs. More general expressions (for correlated pairs) are presented in this paper. In the limit when separation between pairs is very large, the formulas reduce to previous results [7] for independent pairs; at the other extreme when pairs are contiguous, the formulas agree exactly with the Berger and Groginsky [10] results. Throughout the study, pulse samples are assumed to form a narrowband complex Gaussian process. In order to obtain quantitative results, a signal with a Gaussian power spectrum of variable width immersed in white noise was analyzed. This signal often represents radar echoes from random media; hereafter these reported results can be used whenever mean velocity and variance from such media are estimated by means of "pulse pair technique."

It is shown that independent, spaced, and contiguous pulse pairs are very comparable mean frequency estimators. They are unbiased when the signal spectrum is symmetric. For noiseless signals, spectrum width is responsible for the mean frequency standard deviation. Therefore, estimates from independent pairs yield the smallest standard deviation (for fixed M) and those from contiguous pairs the largest. Furthermore, it is demonstrated that at small spectrum widths $[wT_1 < (2\pi)^{-1}]$ the total acquisition times for contiguous pairs and spaced pairs should be equal in order to achieve the same standard deviation. At low signal-to-noise ratios ($S/N < 10$ dB), the noise is the dominant factor in the measurement uncertainty; therefore, contiguous pairs with the noise effect partially cancelled are superior provided the spectrum width is small, $wT_1 < 0.1$. The improvement (over 50 percent) in the standard deviation depends on the spectrum width and is best for sinusoidal signals because errors are due exclusively to noise. All three estimators perform well (standard deviation gradually increases) for widths $wT_1 < 0.25$. At $wT_1 = 0.25$ the standard deviations are about twice the value at zero. With larger wT_1, the variance increases rapidly due to the loss of coherency, i.e., broader spectrum increases aliasing effects until in the limit the signal becomes white noise throughout the Nyquist interval.

The correlation coefficient between two mean frequency estimates from interlaced pulse pairs with variable separation between samples depends on the signal-to-noise ratio and the spectrum width. At widths $wT_1 > 0.3$, it is negligibly small; while at smaller widths and large signal-to-noise ratios, it is positive; otherwise the correlation is negative. When a frequency of a sinusoid is estimated, the correlation coefficient is negative for all signal-to-noise ratios. The standard deviations of the difference in the two mean frequency estimates are almost constant as long as $wT_1 < 0.1$; past that value they double at about $wT_1 \approx 0.2$. Equal pulse spacing among interlaced pairs is optimum in that it minimizes the difference standard deviation. It is anticipated that equally spaced interlaced pairs will not be used in practice since regular pulse pairs are simpler to implement. Unequally spaced pulse pairs do offer the advantage of larger maximum unambiguous range and velocity.

The spectrum width estimator from pulse pairs is not as well behaved as the mean frequency estimator. This is expected since the estimate of the autocorrelation second derivative is more prone to noise degradation than the first derivative. In addition, the width estimator is biased even when input signal spectrum is symmetric, whereas the mean frequency estimator is not. Standard deviations for all three (independent, spaced and contiguous pairs) width estimators are very close, yet independent pairs reduce most the standard deviation. Variance increased at small widths is due mainly to noise while at larger widths it depends about equally on noise and spectrum width. Perturbation equations for a degenerate case (sinusoidal signals) differ from those of regular narrowband signals. Sinusoids without noise produce zero variance; noise creates bounded variances that are inversely proportional to the square root of the number of pairs, in contrast to the M^{-1} dependence at larger widths.

The asymptotic bias, equal for all three estimators, is positive and increases with the spectrum width. On the other hand, the number of sample dependent bias is negative and also high at large spectrum widths. Although this bias is inversely proportional to the number of pairs, its high values (about -100) require a large number of pairs. With sinusoidal inputs and no noise, mean value for all three estimators is zero; additive white noise causes slight negative biases that are proportional to $M^{-3/4}$.

All calculations, for Figs. 2 through 7, were carried with at least 20 terms in the summations. Most curves in Figs. 2 through 6 were verified with simulated data [13] and some real data [1], [14].

In summary, independent, spaced, and contiguous sample pairs yield good mean frequency estimators. Due to ease of implementation and good properties at lower signal-to-noise ratios, contiguous pairs offer advantages. On the other hand, interlaced spaced pulse pairs can be used to increase the range velocity ambiguity region if signal spectra are not too broad. The spectrum width estimator from pulse pairs is weaker, especially at narrow widths ($wT_1 < 0.01$) where precision is poor, and large widths where the bias and standard deviation become excessive. Considerable caution must be exercised when data reduced by this estimator are interpreted. In order to get reliabile moments with the pulse pair technique, systems ought to be designed such that normalized input spectrum widths seldom exceed 0.25.

Appendix A

Perturbation Analysis

Perturbation analysis is applied to derive the variance of the spectrum mean and width estimators and the correlation between the two mean frequency estimates. Inherent assumptions for the validity of the perturbation statistics are that the probability densities of the estimates are smooth functions around the mean and that the perturbations are not excessive. The first condition is always true in our applications while the second improves with increased number of samples [see Appendix (B17)].

A. Mean Frequency Estimate Variance

Let the phase estimate be $\hat{\theta}_1 = 2\pi\hat{f}_1 T_1$ such that

$$\hat{\theta}_1 = \arg \hat{U}. \tag{A1}$$

Assume small perturbation around the mean phase $\bar{\theta}_1$ such that $\hat{\theta}_1 = \bar{\theta}_1 + \delta\hat{\theta}_1$ with the corresponding perturbation equation for the autocorrelation $\hat{r}(T_1) = r(T_1) + \epsilon(T_1)$. Recall that $\hat{U} = \hat{r}(T_1)$ and let $U = r(T_1)$. Both U and $\epsilon(T_1)$ are complex:

$$U = U_r + jU_i$$

$$\epsilon = \epsilon_r + j\epsilon_i. \tag{A2}$$

Benham et al. have shown [9] that second order expansion of arg $\hat{U}$, with respect to the real and imaginary error components ϵ_r and ϵ_i (with $\epsilon = \hat{U} - U$), yields the following mean frequency variance:

$$\operatorname{var}\hat{f}_1 = (2\pi T_1)^{-2}\, E[(\delta\hat{\theta}_1)^2]$$

$$= (2\pi T_1)^{-2}\, \tfrac{1}{2}\, \mathrm{Re}\{E[|\hat{U}/U|^2] - E[(\hat{U}/U)^2]\}. \tag{A3a}$$

Similarly, the variance of the second mean frequency estimate $[W = r(T_2)]$ is

$$\operatorname{var}\hat{f}_2 = (2\pi T_2)^{-2}\, \tfrac{1}{2}\, \mathrm{Re}\{E[|\hat{W}/W|^2] - E[(\hat{W}/W)^2]\}. \tag{A3b}$$

B. Correlation Between Two Mean Frequency Estimates

Perturbation analysis for the correlation between the two frequency estimates is very similar to (A1), (A2), and (A3) with the exception that now two estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ must be combined:

$$\hat{\theta}_1 = \tan^{-1}\, [(U_i + \epsilon_i)/(U_r + \epsilon_r)] \tag{A4a}$$

$$\hat{\theta}_2 = \tan^{-1}\, [(W_i + \xi_i)/(W_r + \xi_r)]. \tag{A4b}$$

The correlation coefficient, cor, can be found directly:

$$\mathrm{cor}\,(\hat{\theta}_1, \hat{\theta}_2) = E(\delta\hat{\theta}_1\, \delta\hat{\theta}_2)/\sqrt{[E(\delta\hat{\theta}_1^2)\, E(\delta\hat{\theta}_2^2)]}. \tag{A5}$$

Since the variances $E(\delta\hat{\theta}_1^2)$ and $E(\delta\hat{\theta}_2^2)$ are given in (A3), it is necessary to determine only $E(\delta\hat{\theta}_1\, \delta\hat{\theta}_2)$. From (A3) we have

$$\delta\hat{\theta}_1 = \mathrm{Im}\,(\hat{U}/U) \quad \text{and} \quad \delta\hat{\theta}_2 = \mathrm{Im}\,(\hat{W}/W) \tag{A6}$$

so that the expected value of the product becomes

$$E(\delta\hat{\theta}_1\, \delta\hat{\theta}_2) = \tfrac{1}{2}\, \mathrm{Re}[E(\hat{U}/U,\, \hat{W}^*/W^*) - E(\hat{U}/U,\, \hat{W}/W)]. \tag{A7}$$

C. Width Estimate Variance

The perturbation equation for width estimate is [9]

$$(\bar{w} + \delta\hat{w})^2 = (2\pi^2 T_1^2)^{-1}\, [1 - |U + \epsilon|/(S + \gamma)] \tag{A8}$$

where γ is the error in the mean signal power estimate and $\bar{w}$ is the asymptotic mean width

$$\overline{w} = \lim_{M \Rightarrow \infty} E(\hat{w}) = (\sqrt{2}\,\pi T_1)^{-1}\,[1 - |U|/S]^{1/2}. \tag{A9}$$

Note the width is independent of the arg (U) and thus is not affected by the spectrum position within the Nyquist interval.

From (A8) the variance of the width is obtained (neglecting M dependent bias terms):

$$(4\pi^2 T_1^2 \overline{w})^2 \ \mathrm{var}\,(\delta\hat{w}) = (|U|/S)^2\ E[\gamma/s - \mathrm{Re}(\hat{U}/U - 1)]^2$$

$$= |U/S|^2\ \{\mathrm{var}\,(\hat{Y}/S) - 2\ \mathrm{Re}[\mathrm{cov}\,(\hat{Y}/S,\ \hat{U}/S)]$$

$$+ \tfrac{1}{2}\ \mathrm{Re}\,E[|\hat{U}/U - 1|^2 + (\hat{U}/U - 1)^2]\}. \tag{A10}$$

Recall that $\hat{Y}$ is the total signal plus noise power estimate (7). Finally, the bias of the width to second order in ϵ and γ is also obtained from (A8):

$$E(\delta\hat{w}) = -(4\pi^2 T_1^2 \overline{w})^{-1}\ |U/S|\{\mathrm{var}\,(\hat{Y}/S) - \mathrm{Re}[\mathrm{cov}\,(\hat{Y}/S,\ \hat{U}/U)]$$

$$+ \tfrac{1}{4}\ \mathrm{Re}\,E[|\hat{U}/U - 1|^2 - (\hat{U}/U - 1)^2]\}$$

$$- (2\overline{w})^{-1}\ \mathrm{var}\,(\delta\hat{w}). \tag{A11}$$

For extremely small spectrum widths $[\overline{w} \ll E(\delta\overline{w})]$ the perturbation is singular and (A8) degenerates to

$$\delta\hat{w}^2 = (2\pi^2 T_1^2)^{-1}[1 - |U + \epsilon|/(S + \gamma)] \tag{A12}$$

from here:

$$E(\delta\hat{w})^2 = -(2\pi^2 T_1^2)^{-1}\ |U/S|\{\mathrm{var}\,(\hat{Y}/S) - \mathrm{Re}[\mathrm{cov}\,(\hat{Y}/S,\ \hat{U}/U)]$$

$$+ \tfrac{1}{4}\ \mathrm{Re}\,E[|\hat{U}/U - 1|^2 - (\hat{U}/U - 1)^2]\}. \tag{A13}$$

Note that the estimated width squared is negative, and thus the width estimate is defined as

$$\delta\hat{w} = |\delta\hat{w}|\ \mathrm{sign}\,(\delta\hat{w}). \tag{A14}$$

There is nothing unusual with negative width estimates; they are simply a predictable bias that can be accounted for.

With a large number of pairs, first order perturbations in δw^2 (A12) are Gaussian with mean $\mu = E(\delta\hat{w})^2$ (A13) and variance σ^2

$$\mathrm{var}\,(\delta w^2) = \sigma^2 = (4\pi^4 T_1^4)^{-1}(|U|/S)^2\ E[\gamma/S - \mathrm{Re}(\hat{U}/U - 1)]^2. \tag{A15}$$

The Gaussian assumption allows us to determine the bias and variance of the estimator at extremely narrow widths:

$$E(\delta\hat{w}) = [\mu 2^{1/4}\Gamma(3/2)/\sigma^{1/2}\Gamma(3/4)]\ \exp(-\mu^2/4\sigma^2) + O(M^{-3/2}) \tag{A16a}$$

$$\mathrm{var}\,(\delta\hat{w}) = (\sqrt{2}/\sqrt{\pi})\ \sigma \exp(-\mu^2/2\sigma^2)$$

$$+ \mu\ \mathrm{erf}(\mu/\sqrt{2}\,\sigma) - E^2(\delta\hat{w}) + O(M^{-3/2}) \tag{A16b}$$

where Γ is the gamma function and erf the error function defined by

$$\mathrm{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2}\ dt. \tag{A17}$$

Appendix B

Statistics of Pulse Pair Estimators for Correlated Data

In order to calculate various statistical parameters of the two mean frequency estimates and the width estimate, the following relationships for complex Gaussian processes are needed:

$$E(Z_{2i} Z_{2i+1}^*) = r(T_1) = U \tag{B1a}$$

$$E(Z_{2i+1} Z_{2i+2}^*) = r(T_2) = W \tag{B1b}$$

$$E(Z_k Z_n) = 0, \text{ for all } k \text{ and } n \tag{B1c}$$

$$E(Z_{2i} Z_{2j}^*) = r[(j - i)T] \tag{B1d}$$

$$E(Z_{2i+1} Z_{2j+1}^*) = r[(j - i)T] \tag{B1e}$$

$$E(Z_{2i} Z_{2j+1}^*) = r[(j - i)T + T_1] \tag{B1f}$$

$$E(Z_{2i+1} Z_{2j+2}^*) = r[(j - i)T + T_2]. \tag{B1g}$$

These expressions are used to evaluate terms that enter into equations (A3), (A7), (A10), and (A11). We start by finding $E|\hat{U}|^2$ and $E(\hat{U})^2$:

$$E|\hat{U}|^2 = M^{-2} \sum_{i,j} E(Z_{2i}^* Z_{2i+1} Z_{2j} Z_{2j+1}^*). \tag{B2}$$

The expected value of the product of four complex Gaussian variables can be expressed in terms of second order moments:

$$E(Z_{2i}^* Z_{2i+1} Z_{2j} Z_{2j+1}^*) = E(Z_{2i}^* Z_{2i+1}) E(Z_{2j} Z_{2j+1}^*)$$

$$+ E(Z_{2i}^* Z_{2j}) E(Z_{2i+1} Z_{2j+1}^*). \tag{B3}$$

Insert (B1a), (B1d), and (B1e) in (B3) to obtain

$$E(Z_{2i}^* Z_{2i+1} Z_{2j} Z_{2j+1}^*) = |r(T_1)|^2$$

$$+ r^*[(j - i)T]\ r[(j - i)T]. \tag{B4}$$

Finally the sum (B2) yields

$$E(|\hat{U}|^2) = |r(T_1)|^2 + M^{-2} \sum_{m=-(M-1)}^{M-1} |r(mT)|^2 (M - |m|). \tag{B5}$$

Calculation of $E(U^2)$ parallels the derivation leading to (B5):

$$E(\hat{U}^2) = r^2(T_1) + M^{-2} \sum_{m=-(M-1)}^{M-1} r(mT + T_1) \cdot r^*(mT - T_1)\ (M - |m|). \tag{B6}$$

Insert (B5) and (B6) into (A3a) to find the variance directly:

$$\mathrm{var}\,\hat{f}_1 = (2\pi T_1)^{-2}\,\tfrac{1}{2}\,\mathrm{Re}\{\frac{1}{M^2}\sum_{m=-(M-1)}^{M-1}[|r(mT)/r(T_1)|^2 - r(mT+T_1)\,r^*(mT-T_1)/r^2(T_1)](M-|m|)\}. \quad (B7)$$

The second mean frequency estimate variance is found by substituting T_2 for T_1 in (B7).

To evaluate the correlation (A7), we need the two expected values $E(\hat{U}\hat{W}^*)$ and $E(\hat{U}\hat{W})$ which are found following the previous procedure:

$$E(\hat{U}\hat{W}) = r(T_1)\,r(T_2) + M^{-2}\sum_{-(M-1)}^{M-1} \cdot r^*[(m+1)T]\,r(mT)\,(M-|m|) \quad (B8)$$

$$E(\hat{U}\hat{W}^*) = r(T_1)\,r^*(T_2) + M^{-2}\sum_{m=-(M-1)}^{M-1} \cdot r(mT+T_1)\,r^*(mT+T_2)\,(M-|m|). \quad (B9)$$

Insert (B8) and (B9) in (A7) to obtain

$$E(\delta\hat{\theta}_1\,\delta\hat{\theta}_2) = (2M^2)^{-1}\,\mathrm{Re}\{\sum_{m=-(M-1)}^{M-1}[r(mT+T_1) \cdot r^*(mT+T_2)/r(T_1)\,r^*(T_2) - r[(m+1)T]\,r^*(mT)/r(T_1)\,r(T_2)](M-|m|)\}. \quad (B10)$$

The width estimate variance (A10) and bias (A11) are considered next. For both, the following four quantities are needed:

$$\mathrm{var}\,(\hat{Y}/S) = (2M^2S^2)^{-1}\{\sum_{-(M-1)}^{M-1}[|r(mT)|^2 + |r(mT+T_1)|^2](M-|m|)\} \quad (B11)$$

$$\mathrm{cov}\,(\hat{Y}/S,\,\hat{U}/U) = [M^2 S r(T_1)]^{-1}\sum_{-(M-1)}^{M-1} \cdot r(mT+T_1)\,r^*(mT)\,(M-|m|) \quad (B12)$$

$$E|\hat{U}/U - 1|^2 = E|\hat{U}/U|^2 - 1 = M^{-2}\sum_{-(M-1)}^{M-1} \cdot |r(mT)/r(T_1)|^2\,(M-|m|) \quad (B13)$$

$$E(\hat{U}/U - 1)^2 = E(\hat{U}/U)^2 - 1 = M^{-2}\sum_{-(M-1)}^{M-1} \cdot [r(mT+T_1)\,r^*(mT-T_1)/r^2(T_1)]\,(M-|m|). \quad (B14)$$

Finally, the variance (A10) and the bias (A11) are as follows:

variance:

$$(4\pi^2 T_1^2 \overline{w})^2\ \mathrm{var}\,(\delta\hat{w}) = \tfrac{1}{2}\,(|r(T_1)|/MS)^2\ \mathrm{Re}\{\sum_{-(M-1)}^{M-1}[|r(mT)|^2/S^2 + |r(mT+T_1)|^2/S^2 + |r(mT)/r(T_1)|^2 + r(mT+T_1)\,r^*(mT-T_1)/r^2(T_1) - 4\,r(mT+T_1)\,r^*(mT)/Sr(T_1)]\,(M-|m|)\} \quad (B15)$$

bias:

$$E(\delta\hat{w}) = -[|r(T_1)|/4M^2\pi^2T_1^2\overline{w}S]\ \mathrm{Re}\{\sum_{-(M-1)}^{M-1}[|r(mT)|^2/S^2 + |r(mT+T_1)|^2/S^2 + \tfrac{1}{2}|r(mT)/r(T_1)|^2 - \tfrac{1}{2}\,r(mT+T_1)\,r^*(mT-T_1)/r^2(T_1) - 2\,r(mT+T_1)\,r^*(mT)/Sr(T_1)]\,(M-|m|)\} - (2\overline{w})^{-1}\ \mathrm{var}\,(\delta\hat{w}). \quad (B16)$$

The derived statistics are correct when the perturbation variables ϵ and γ (Appendix A) are concentrated near their origin. A large number of samples and a good signal-to-noise ratio help the concentration. To insure a sufficient number of independent samples the dwell time must be long compared to decorrelation time:

$$2\pi MwT \gg 1. \quad (B17a)$$

Relationships between S, N, and M which insure the derived statistics are valid will now be established. Note that the variances of the normalized errors are given by (B11), (B13), and (B14). It can be shown those variances will be much smaller than one if

$$[\beta^2(T_1)M]^{-1}\ (N/S+1)^2 \ll 1. \quad (B17b)$$

The lower limit on wT_1 for which the statistics are usable is given by

$$(wT_1)^2 \gg \mathrm{var}\,(\delta\hat{w}T_1) \quad (B18a)$$

which when used in conjunction with the dominant term of (29c) becomes

$$M(S^2/N^2)\,(wT_1)^4 \gg 3/32\pi^4. \quad (B18b)$$

Extremely small widths for which this inequality does not hold require use of (A16a) and (A16b) for the bias and standard deviation of the width estimate.

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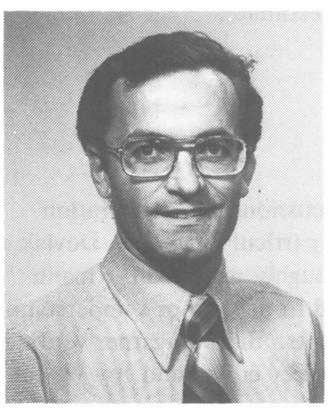

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