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Third-order transport due to internal waves and non-local turbulence in the stably stratified surface layer

By SERGEJ S. ZILITINKEVICH*
Uppsala University, Sweden

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SUMMARY

Until recently the concern of the traditional theory of the atmospheric stable boundary layer (SBL) was, almost without exception, the *nocturnal* SBL developing after sunset on the background of a neutral or slightly stable residual layer. In the nocturnal SBLs the nature of turbulence is basically local. Its lower portion is well described by the classical Monin–Obukhov surface-layer similarity theory. Things are different in *long-lived* SBLs situated immediately below the stably stratified free flow. Here, the surface-layer turbulence is affected by the free-flow Brunt–Väisälä frequency, N. The surface layer represents approximately one-tenth of the SBL, so that it is separated from the free atmosphere by the upper nine-tenths of the SBL comprising hundreds of metres. Traditional concepts fail to explain such distant links. Zilitinkevich and Calanca extended the traditional Monin–Obukhov similarity theory by including N in the surface-layer scaling, and provided experimental evidence in support of this extension. In the present paper, physical mechanisms responsible for non-local features of the long-lived SBL turbulence are identified as: radiation of internal waves from the SBL upper boundary to the free atmosphere, and the internal-wave transport of the squared fluctuations of velocity and potential temperature. The third-order wave-induced fluxes are included in an advanced turbulence-closure model to correct the wind and temperature profiles in the surface layer. The model explains why developed turbulence in the surface layer can exist at much larger Richardson numbers than the classical theory predicts. Results from the new model are in good agreement with the extended similarity theory and experimental data.

KEYWORDS: Free atmosphere Internal gravity waves Residual layer Stable stratification

1. INTRODUCTION

An attempt is made to develop a theoretical model which would explain non-local features of the surface-layer turbulence disclosed recently by Zilitinkevich and Calanca (2000; henceforth Z&C). In this approach two essentially different types of stable boundary layer (SBL) are distinguished, namely *nocturnal* and *long-lived* (see Fig. 1).

The nocturnal midlatitudinal SBLs are disconnected from the stably stratified free atmosphere by a thick neutral or weakly stable residual layer (e.g. Kim and Mahrt 1992). The latter results from the daytime convection and is usually kept well-mixed during the first hours of the night. Clearly, the residual layer prevents internal-wave interactions between the SBL and the free atmosphere. As a result, the nocturnal SBL is basically governed by locally generated small-scale turbulence. Then the nature of turbulence is basically local, the structure of the surface layer obeys the Monin–Obukhov similarity theory (Obukhov 1946; Monin and Obukhov 1954) and the SBL is successfully modelled employing local-scaling reasoning or turbulence-closure schemes (Brost and Wyngaard 1978; Nieuwstadt 1984; Derbyshire 1990). It is believed that non-local features, although occasionally observed (Cuxart *et al.* 2000), are rather exceptional for nocturnal SBLs.

On the contrary, in persisting stable stratification typical of cold weather and often observed in coastal zones, the SBLs live long enough to achieve immediate contact with the stably stratified free atmosphere. In this case the SBL and the free atmosphere could interact due to the vertical propagation of internal gravity waves (IGWs) and non-local IGW-induced transports. The key parameter characterizing this mechanism is the

* Corresponding address: Department of Earth Sciences–Meteorology (MIUU), Uppsala University, Villavägen 16, SE-752 36 Uppsala, Sweden. e-mail: sergej@met.uu.se

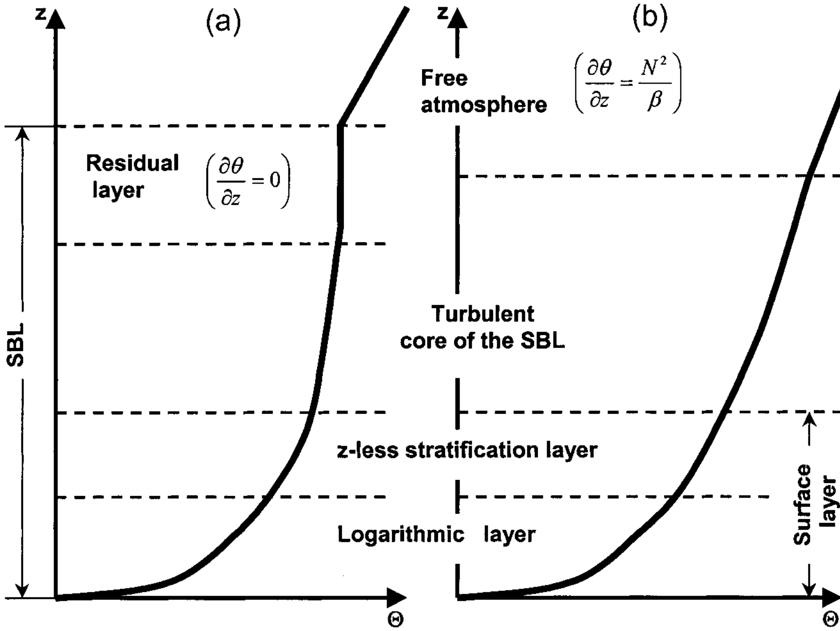

Figure 1. A sketch of the potential-temperature profiles and relevant sub-layers in (a) the nocturnal, and (b) the long-lived stable boundary layers.

Brunt–Väisälä frequency in the free atmosphere, N. Traditional local theories are insufficiently advanced to reproduce realistically such long-lived SBLs. They systematically overestimate the SBL height (Kitaigorodskii and Joffre 1988; Zilitinkevich and Mironov 1996; Zilitinkevich *et al.* 2002) and in some cases dramatically underestimate the level of turbulent mixing in the surface layer (Sempreviva *et al.* 1992; Z&C).

The fact that the free-flow stability generally affects the SBL is well documented. The effect of N on the structure of the Antarctic SBL was recognized by King (1990). Olesen *et al.* (1984) and Larsen *et al.* (1985, 1990) disclosed that the velocity spectra in the lower part of the SBL depend on N. One could consider these observations as indirectly indicative of the role of the vertical propagation of IGWs and the IGW–turbulence interaction in the physical nature of long-lived SBLs. Some mechanisms of possible influence of IGWs on SBLs have already been considered (Finnigan *et al.* 1984; Hooke and Jones 1986; Weinstock 1987; Finnigan 1998). Mahrt (1998, 1999) gave a comprehensive presentation of the modern state-of-the-art in this field.

In a new theoretical model presented below in sections 2 and 3 the IGW-induced third-order fluxes within the long-lived SBL are parametrized and incorporated in the context of a turbulence-closure model. This allows the earlier Z&C heuristic arguments and experimental findings to be justified.

Z&C extended the classical Monin–Obukhov similarity theory to the surface layers within long-lived SBLs. They provided a qualitative physical reasoning for including N in the list of the governing parameters and received convincing experimental support for this extension. The Z&C similarity theory includes, besides the familiar Monin–Obukhov length-scale, L, one more length-scale, L_N, and consequently a dimensionless number, Fi, which can be treated as an inverse Froude number:

$$L = \frac{-u_*^3}{B}, \quad L_N = \frac{u_*}{N}, \quad \mathrm{Fi} = \frac{L}{L_N} = \frac{LN}{u_*}. \tag{1}$$

Here, $u_* \equiv \sqrt{|\boldsymbol{\tau}_\mathrm{s}|}$ is the friction velocity; $\boldsymbol{\tau}_\mathrm{s}$, $B = \beta F_{\theta\mathrm{s}}$ and $F_{\theta\mathrm{s}}$ are the near-surface values of vertical turbulent fluxes of momentum, buoyancy and virtual potential temperature (here and below the subscript s stands for near-surface values); $\beta = g/T$ is the buoyancy parameter; g is the acceleration due to gravity; T is a reference value of the absolute temperature; N is given by:

$$N^2 = \beta \left. \frac{\partial\theta}{\partial z} \right|_{z>h}, \tag{2}$$

where $\partial\theta/\partial z$ is the vertical gradient of potential temperature, and h is the boundary-layer height. The inverse Froude number, Fi, quantifies the extent to which the surface-layer turbulence is affected by the free-flow stability.

In further analysis the surface layer is considered as an integral part of the SBL. Here, the vertical turbulent fluxes of momentum, $\boldsymbol{\tau} = (\tau_x, \tau_y)$, and potential temperature, F_θ, decrease towards the SBL upper boundary:

$$|\boldsymbol{\tau}| = u_*^2(1-\varsigma)^m, \quad F_\theta = F_{\theta\mathrm{s}}(1-\varsigma)^n \quad \text{where } \varsigma \equiv \frac{z}{h}. \tag{3}$$

The surface layer is the lower portion of the SBL where $z \ll h$ and consequently $|\boldsymbol{\tau}| \approx u_*^2$ and $F_\theta \approx F_{\theta\mathrm{s}}$.

For the exponents in Eq. (3), Nieuwstadt (1984) suggested $n = 1$ and deduced $m = 3/2$ from his nocturnal SBL model. Derbyshire (1990) observed that $n = 1$ is inconsistent with his large-eddy simulations and hardly realistic. Indeed, taking $n = 1$, the heat conductivity equation, $\partial\theta/\partial t = -\partial F_\theta/\partial z$, immediately suggests that the cooling rate is constant with depth, $\partial\theta/\partial t = F_{\theta\mathrm{s}}/h < 0$. If so, the potential temperature immediately below the SBL height, θ_{h-0}, should decrease linearly with time. Then, bearing in mind that the temperature immediately above the SBL, θ_{h-0}, is kept constant, a temperature jump at the top of the SBL should develop. In reality no jumps of this sort are observed. To avoid this inconsistency, n should be taken larger than unity.

2. Velocity Gradient

In stable stratification, the shear generation of turbulent kinetic energy (TKE), $E_K = \frac{1}{2}\overline{(u'^2 + v'^2 + w'^2)}$, is the major (positive) term in the TKE budget. This makes it reasonable to derive the velocity gradient from the TKE budget equation:

$$\boldsymbol{\tau} \cdot \frac{\partial \mathbf{u}}{\partial z} \approx u_*^2 \frac{\partial u}{\partial z} = -B + \varepsilon + \frac{\partial F_{\mathrm{KE}}}{\partial z}. \tag{4}$$

Here, $\mathbf{u} = (u, v)$ is the wind velocity vector, ε is the dissipation rate of the TKE, $F_{\mathrm{KE}} = \frac{1}{2}\overline{(u'^2 + v'^2 + w'^2)w'} + \rho_0^{-1}\overline{p'w'}$ is the sum of the third moments representing vertical fluxes of the TKE and the fluctuations of pressure, p'; ρ_0 is the reference density. The terms $\boldsymbol{\tau} \cdot \partial\mathbf{u}/\partial z$ and B represent, respectively, the generation rate of the TKE due to the shear and the TKE loss due to the negative buoyancy forces. The x-axis is aligned with the near-surface wind; hence $\mathbf{u} = (u, 0)$ and $\boldsymbol{\tau} = (\tau_\mathrm{s}, 0)$. Assuming that $z \ll h$, the surface-layer approximation is applied to the fluxes of momentum and potential temperature, $\boldsymbol{\tau}(u_*^2, 0)$ and $B = \beta F_{\theta s}$, such that height-independent values are used. Certainly, when vertically differentiating, the height-dependence of the fluxes should be taken into account.

For the dissipation rate of TKE, the conventional Kolmogorov–Heisenberg closure hypothesis reads:

$$\varepsilon = \frac{E_K^{3/2}}{l_E} = \frac{E_K}{t_E}, \tag{5}$$

where l_E and $t_E = l_E/E_K^{1/2}$ are the energy-dissipation length- and time-scales, respectively.

In the stably stratified surface layer, the ratio of the TKE to the momentum flux modulus does not exhibit any dependence on z/L:

$$E_K = C_K|\tau| \approx C_K u_*^2, \tag{6}$$

where $C_K \approx 5$ is an empirical constant (e.g. Fig. 1.24 in Zilitinkevich 1970; Fig. 75 in Monin and Yaglom 1971).

For the length- and time-scales, two asymptotic formulas are known: $l_E \sim z$, $t_E \sim z/u_*$ close to the surface, and $l_E \sim L$, $t_E \sim L/u_*$ far away from the surface—in the so-called z-less stratification layer (e.g. section 7 in Monin and Yaglom 1971). Interpolation between the above asymptotes for t_E^{-1} yields:

$$\frac{1}{t_E} = C_{t1}\frac{u_*}{z} + C_{t2}\frac{u_*}{L}, \tag{7}$$

where C_{t1} and C_{t2} are dimensionless coefficients. Then the dissipation rate of the TKE becomes:

$$\varepsilon = C_K C_{t1}\frac{u_*^3}{z}\left(1 + \frac{C_{t2}}{C_{t1}}\frac{z}{L}\right). \tag{8}$$

Substituting Eq. (8) for ε and neglecting $\partial F_{\mathrm{KE}}/\partial z$, Eq. (4) would immediately yield the familiar log–linear velocity profile:

$$\frac{\partial u}{\partial z} = \frac{u_*}{kz}\left(1 + C_u\frac{z}{L}\right). \tag{9}$$

Here, $k = 1/C_K C_{t1} \approx 0.4$ is the von Karman constant and $C_u = k(C_K C_{t2} + 1) \approx 2.1$ is one more empirical constant (taken after Högström 1995). Remember that L is specified here by Eq. (1) and does not include k. Accordingly the above estimate of $C_u \approx 2.1$ is consistent with the habitual value of the constant in the log–linear wind profile, which is equal to $C_u/k \approx 5$. Equation (9) is known to be a reasonable approximation for the surface layers within the nocturnal SBLs.

To extend the analysis to long-lived SBLs, consider the TKE flux divergence. Close to the surface (in the logarithmic boundary layer) the local energy generation $u_*^2\partial u/\partial z$ is, to a high accuracy, balanced by its local dissipation ε, so that the buoyancy and the flux divergence terms in Eq. (4) become negligible. Moreover, at the very surface the total energy flux should turn to zero. Therefore the lower boundary condition for this flux reads:

$$F_{\mathrm{KE}}|_{z=0} \equiv F_{\mathrm{KEs}} = 0. \tag{10}$$

This condition is consistent with the widely used turbulent-diffusion approximation for F_{KE}, namely $F_{\mathrm{KE}} = -K\partial E_K/\partial z$, assuming that the eddy diffusivity K is proportional to the eddy viscosity, $K \sim u_* z$, and taking the kinetic energy E_K after Eqs. (6) and (3).

At the upper boundary of the long-lived SBL, turbulent disturbances may interact with the IGW fields inherent in the free atmosphere. Consider the case when the IGWs are excited by the above disturbances*. Then the IGWs propagate upward into adjacent stably stratified layer of the free atmosphere and generate their vertical fluxes of the kinetic energy and pressure fluctuations. Consider first the wave-induced kinetic energy flux. Let the typical horizontal scale of disturbances be λ, therefore the typical horizontal wave number of the IGW is $2\pi/\lambda$. Generally, a variety of wave harmonics with a wide range of wave numbers is produced. Each of these components is responsible for a certain vertical energy transport. The maximum energy flux caused by a single wave harmonic is $(3\pi\sqrt{3})^{-1} l^2 \lambda N^3$, where l is the wave amplitude (e.g. Thorpe 1973). The wave-induced vertical flux of the pressure disturbances is given by precisely the same expression (Soomere and Zilitinkevich 2001). Hence the total flux afforded by the entire wave spectrum can be taken to be $\sim l^2 \lambda N^3$, with the proportionality coefficient generally larger than $(3\pi\sqrt{3})^{-1}$. The wave length λ and amplitude l typical of the dominant waves can be taken to avoid integration over the spectrum (cf. Carruthers and Hunt 1986).

On the requirement of continuity at $z = h$, the energy flux provided by joint actions of the IGW and turbulence immediately below the SBL height should coincide with the above wave-induced flux in the free atmosphere:

$$F_{\mathrm{KE}}|_{z=h} \equiv F_{\mathrm{KE}h} \sim l^2 \lambda N^3. \tag{11}$$

The role of this flux in the TKE budget is shown schematically in Fig. 2.

A natural scale for the horizontal wavelength, λ, is given by the horizontal sizes of the largest eddies in the SBL. The latter are controlled by the SBL height h. Then a simple estimate can be taken as a first approximation:

$$\lambda \sim h. \tag{12}$$

The omitted 'anisotropy factor' on the right-hand side (r.h.s.) of Eq. (12) is of order ten or even larger (e.g. Mason and Thomson 1987). Strong anisotropy of turbulent eddies in boundary layers due to asymmetry of the stretching mechanism is clearly understood. It is characterized in the vertical by the turbulent velocity, u_*, and in the horizontal by the increment in mean wind velocity, u, across the layer. Then the omitted factor on the r.h.s. of Eq. (12) should be of the order of the inverse drag coefficient, u/u_*, which is large but not very variable†.

Obviously l should be close to the vertical scale of disturbances at the SBL upper boundary. The latter is immediately estimated from simple energy budget reasoning. Indeed, the potential energy acquired by a portion of fluid displaced in the vertical by the distance l is $\sim l^2 \beta \partial\theta/\partial z = l^2 N^2$, whereas its initial kinetic energy is $\sim u_*^2$. Equating the two energies yields:

$$l \sim \frac{u_*}{N}. \tag{13}$$

* Clearly, an alternative process when the IGW field inherent in the free atmosphere generates turbulent disturbances in the SBL would affect the SBL structure in an essentially different way (see Hooke and Jones 1986; Weinstock 1987). This interesting regime is not considered in the present paper and left for future work.

† A more advanced estimate of λ accounting for the effect of the static stability on horizontal stretching of large eddies is left for future work (see section 6).

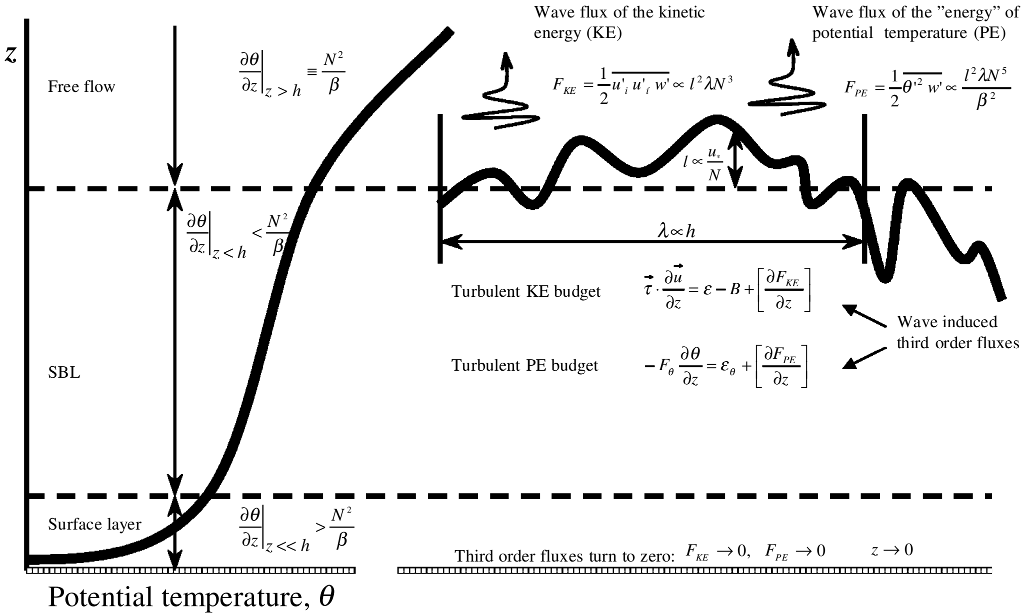

Figure 2. Vertical structure and budgets of the kinetic energy (KE) and the ‘energy' of potential-temperature fluctuations (PE) in the long-lived stable boundary layer. See text for discussion.

The r.h.s. of Eq. (13) is precisely the length-scale L_N, Eq. (1).

Equations (11)–(13) specify the energy flux at the upper boundary of the long-lived SBL. It is worth mentioning that nothing of this kind could be derived from traditional energy-flux parametrizations currently used within turbulence closure schemes, in particular from the turbulent-diffusion approximation already mentioned in the discussion of Eq. (10). For better understanding and parametrization of the turbulence–IGW interactions within the SBL, the divergence of the energy flux can be roughly approximated through the finite difference across the SBL as $\partial F_{\mathrm{KE}}/\partial z \sim (F_{\mathrm{KE}h} - F_{\mathrm{KEs}})/h$. Substituting here F_{KEs} after Eq. (10) and $F_{\mathrm{KE}h}$ after Eqs. (11)–(13) yields:

$$\frac{\partial F_{\mathrm{KE}}}{\partial z} = C_{\mathrm{WK}} u_*^2 N, \tag{14}$$

where C_{WK} is a dimensionless coefficient to be determined empirically.

Now, substituting Eq. (14) for $\partial F_E/\partial z$ and Eq. (8) for ε in Eq. (4), yields precisely the same velocity-gradient formulation as that derived heuristically by Z&C:

$$\frac{\partial u}{\partial z} = \frac{u_*}{kz}\left\{1 + C_u(1 + C_{uN}\mathrm{Fi})\frac{z}{L}\right\}. \tag{15}$$

Here, $C_{uN} = C_{\mathrm{WK}}/(C_K C_{t2} + 1) \approx 0.1 \div 0.4$ is a new coefficient introduced and roughly estimated by Z&C*. At $\mathrm{Fi} = 0$, Eq. (15) reduces to the traditional Monin–Obukhov theory formulation.

Figure 3(a) shows the Z&C plot of the wind slope factor, $s_u = (kL/u_*)\partial u/\partial z - L/z$, versus Fi. According to the classical theory, Eq. (9), this factor should be a

* In Z&C, the notations C_{u1} and C_{u2} were used instead of C_u and C_{uN}. In the present paper the notations have been changed to avoid confusion, insofar that the temperature-gradient formulation derived here differs from the Z&C formulation.

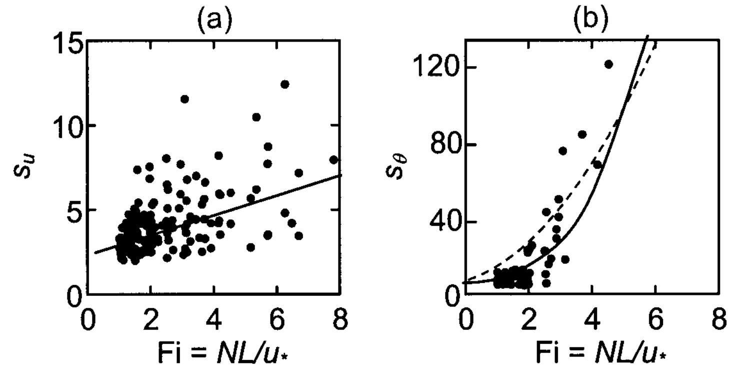

Figure 3. Slope factors: (a) $s_u = (kL/u_*)\partial u/\partial z - L/z$ for the wind profile; and (b) $s_\theta \equiv (k_T L/\theta_*)\partial\theta/\partial z$ for the temperature profile, versus the inverse Froude number Fi. See text for details.

universal constant, $s_u = C_u$, which evidently contradicts the data. Equation (15) predicts a generally realistic behaviour, $s_u = C_u + C_u C_{uN}\mathrm{Fi}$. The line is calculated for $C_u = 2$ and $C_{uN} = 0.3$. The wide spread of data is probably caused by the variability of the 'anisotropy factor' neglected in Eq. (12).

3. Temperature gradient

In the same way as the velocity gradient is derived in section 2 from the TKE budget equation, the potential-temperature gradient is derived below from the budget equation for the squared potential-temperature fluctuations, in other words for the 'energy' of these fluctuations, $E_P = \frac{1}{2}\overline{\theta'^2}$. In the surface layer, this equation reads:

$$-F_{\theta \mathrm{s}}\frac{\partial\theta}{\partial z} = \varepsilon_\theta + \frac{\partial F_{\mathrm{PE}}}{\partial z}. \tag{16}$$

Here, the left-hand side (l.h.s.) represents the production of the potential-temperature fluctuations. On the r.h.s. ε_θ is their decay rate (dissipation), and the second term is the divergence of their vertical flux represented by the third moment $F_{\mathrm{PE}} = \frac{1}{2}\overline{\theta'^2 w'}$. When multiplied by β^2, Eq. (16) becomes the budget equation for squared fluctuations of buoyancy.

For ε_θ, the conventional parametrization reads:

$$\varepsilon_\theta = \frac{E_P}{t_\theta}, \tag{17}$$

where t_θ is the dissipation time-scale for the temperature fluctuations (cf. Eq. (5) for ε). In the stably stratified surface layer, the ratio of E_P to the squared potential-temperature-scale θ_*^2 does not exhibit any dependence on z/L:

$$E_P = C_P\frac{F_\theta^2}{|\tau|} \approx C_P\theta_*^2, \quad \theta_* \equiv \frac{-F_{\theta \mathrm{s}}}{u_*}, \tag{18}$$

where $C_P \approx 0.3$ is an empirical constant (e.g. Fig. 1.25 in Zilitinkevich 1970; Fig. 76 in Monin and Yaglom 1971). Similarly to Eq. (7), the time-scale t_θ is given by the interpolation formula:

$$\frac{1}{t_\theta} = C_{t3}\frac{u_*}{z} + C_{t4}\frac{u_*}{L}, \tag{19}$$

where C_{t3} and C_{t4} are dimensionless coefficients. Then the temperature-fluctuation decay rate becomes:

$$\varepsilon_\theta = C_P C_{t3}\frac{\theta_*^2 u_*}{z}\left(1 + \frac{C_{t4}}{C_{t3}}\frac{z}{L}\right). \tag{20}$$

Neglecting $\partial F_{\mathrm{PE}}/\partial z$, Eqs. (16) and (20) yield the familiar log–linear temperature profile:

$$\frac{\partial\theta}{\partial z} = \frac{\theta_*}{k_T z}\left(1 + C_\theta\frac{z}{L}\right), \tag{21}$$

where $k_T = 1/C_P C_{t3} \approx 0.42$ is the von Karman constant for temperature, and $C_\theta = C_{t4}/C_{t3} \approx 3.2$ is the same type of constant as C_u in Eq. (9). Its value (taken here after Högström 1995) differs from the usual one by the factor k, as the Monin–Obukhov length specified by Eqs. (1) does not include k. Equation (21) is essentially the local formulation applicable to the surface layers within nocturnal SBLs.

To account for the effect of IGWs, consider the vertical flux of the squared potential-temperature fluctuations, F_{PE}. Close to the surface its role in Eq. (16) is negligible. At the very surface this flux tends to zero for the same reasons as the kinetic energy flux:

$$F_{\mathrm{PE}}|_{z=0} \equiv F_{\mathrm{PEs}} = 0. \tag{22}$$

In the free atmosphere, the 'energy' of the buoyancy fluctuations, $\frac{1}{2}\overline{(\beta\theta')^2}$, is transported by the same internal-wave mechanism as the kinetic energy. The linear wave theory offers that the maximum 'buoyancy-energy' flux due to a single wave component is $(3\pi\sqrt{3})^{-1}l^2\lambda N^5$ (Soomere and Zilitinkevich 2001). Then the total squared buoyancy fluctuation flux is $\sim l^2\lambda N^5$, and the total squared potential-temperature fluctuation flux is $\sim\beta^{-2}l^2\lambda N^5$, with a proportionality coefficient generally larger than $(3\pi\sqrt{3})^{-1}$. Employing Eq. (12) for λ and Eq. (13) for l, the total vertical flux of squared potential-temperature fluctuations due to internal waves radiated from the SBL upper boundary becomes:

$$F_{\mathrm{PE}h} \sim \frac{l^2\lambda N^5}{\beta^2} \sim \frac{u_*^2 h N^3}{\beta^2}. \tag{23}$$

Then the flux divergence approximated through the finite difference across the SBL, $\partial F_{\mathrm{PE}}/\partial z \sim (F_{\mathrm{PE}h} - F_{\mathrm{PEs}})/h$, is immediately deduced from Eqs. (22) and (23):

$$\frac{\partial F_{\mathrm{PE}}}{\partial z} = C_{\mathrm{WP}}\frac{u_*^2 N^3}{\beta^2}, \tag{24}$$

where C_{WP} is the same type of dimensionless coefficient as C_{WK} in Eq. (14) (generally these coefficients can differ due to the difference in the shape of the energy and the squared buoyancy spectra).

Clearly, Eq. (22) is consistent with the turbulent-diffusion approximation, $F_{\mathrm{PE}} = -K\partial E_P/\partial z$, whereas Eq. (23) and consequently Eq. (24) cannot be derived from traditional turbulence closures (see Fig. 2).

Substituting Eq. (20) for ε_θ and Eq. (24) for $\partial F_{\mathrm{PE}}/\partial z$ in Eq. (16) yields the potential-temperature-gradient equation:

$$\frac{\partial \theta}{\partial z} = \frac{\theta_*}{k_T z}\{1 + C_\theta(1 + C_{\theta N}\mathrm{Fi}^3)z/L\}, \tag{25}$$

where $C_{\theta N} = C_{\mathrm{WP}}/C_P C_{t4}$ is a new coefficient to be determined empirically.

Equation (25) specifies the temperature slope factor, $s_\theta \equiv (k_T L/\theta_*)\partial\theta/\partial z - L/z$, as a cubic function of Fi, $s_\theta = C_\theta + C_\theta C_{\theta N}\mathrm{Fi}^3$, rather than the quadratic function proposed by Z&C. In this context it should be remembered that dimensional analysis used by Z&C did not pursue determination of the exponent. Moreover, the Z&C empirical data on s_θ versus Fi presented in Fig. 3(b) definitely fit the Fi^3 dependence (solid line calculated for $C_{\theta N} = 0.3$) better than the Z&C Fi^2 dependence (dotted line). Thus, in terms of the Z&C similarity theory, Eq. (25) suggests that the coefficient $C_{\theta 2}$ considered by Z&C as an unknown function of Fi, is in fact a linear function, $C_{\theta 2} = C_{\theta N}\mathrm{Fi}$. It is worth mentioning that the classical theory predicts $s_\theta = C_\theta =$ constant, which becomes absolutely inconsistent with experimental data at Fi > 2.

4. The Richardson and the Prandtl numbers

As follows from Eqs. (15) and (25), the eddy viscosity, K_M, and the eddy conductivity, K_H, in the surface layer are:

$$K_M = \frac{\tau}{\partial u/\partial z} = \frac{k u_* z}{1 + C_u(1 + C_{uN}\mathrm{Fi})z/L} \sim \frac{k u_* L}{C_u(1 + C_{uN}\mathrm{Fi})}, \tag{26}$$

$$K_H = \frac{-F_{\theta \mathrm{s}}}{\partial\theta/\partial z} = \frac{k_T u_* z}{1 + C_\theta(1 + C_{\theta N}\mathrm{Fi}^3)z/L} \sim \frac{k_T u_* L}{C_\theta(1 + C_{\theta N}\mathrm{Fi}^3)}. \tag{27}$$

Here, approximations on the r.h.s. of both equations hold true at large values of z/L, in the z-less stratification layer.

In the nocturnal SBL regimes (Fi = 0), these approximations reduce to $K_M = k u_* L/C_u$ and $K_H = k_T u_* L/C_\theta$. Then the Prandtl number, Pr, and Richardson number, Ri, become constants:

$$\mathrm{Pr} \equiv \frac{K_M}{K_H} \sim \frac{k C_\theta}{k_T C_u} \approx 1.5, \quad \mathrm{Ri} \equiv \frac{\beta\,\partial\theta/\partial z}{(\partial u/\partial z)^2} \sim \frac{k^2 C_\theta}{k_T C_u^2} \approx 0.28. \tag{28}$$

By contrast, in the surface layers within long-lived SBLs, when both the dimensionless height z/L and Fi (Eq. (1)) are large, Pr and Ri become functions of Fi:

$$\mathrm{Pr} \equiv \frac{K_M}{K_H} \sim \frac{k C_\theta C_{\theta N}}{k_T C_u C_{uN}}\mathrm{Fi}^2, \quad \mathrm{Ri} \equiv \frac{\beta\,\partial\theta/\partial z}{(\partial u/\partial z)^2} \sim \frac{k^2 C_\theta C_{\theta N}}{k_T C_u^2 C_{uN}^2}\mathrm{Fi}, \tag{29}$$

which suggest a quadratic dependence $\mathrm{Pr} \sim \mathrm{Ri}^2$. In other words, in strongly stable long-lived boundary layers, the eddy viscosity becomes Ri^2 times larger than the eddy conductivity. This conclusion is consistent with the quite natural expectation that the IGWs in long-lived SBLs contribute more to the momentum transport than to the heat transport (cf. Schumann 1991).

Thus, in contrast with the classical local theory, the proposed model suggests that developed turbulence in the stably stratified atmospheric surface layer can exist at very large Richardson numbers. This conclusion is confirmed by recent experimental evidence (e.g. Sempreviva *et al.* 1992; Mahrt *et al.* 1998; Z&C). A rather common expectation that the surface-layer turbulence should decay at Richardson numbers larger than the critical one ($\mathrm{Ri}_c \sim 0.25 \div 0.3$) is neither supported by data nor theoretically founded. Indeed, the well-known upper limit, $\mathrm{Ri} < \mathrm{Ri}_c = 0.25$, for flow instability and the generation of turbulence holds true for essentially *local* consideration relevant to a *homogeneous* stratified sheared flow. There are no reasons to extend this limit to the surface layer, representing a part of the SBL, which is *inhomogeneous* in the vertical and whose nature can be essentially *non-local.* Notice that the proposed model does not exclude the existence of critical Richardson numbers for the surface layer; it only indicates that these numbers should not be confused with the conventional flow instability limit $\mathrm{Ri}_c = 0.25$.

5. Limits of applicability

Integrating Eqs. (15) and (25) over z, the wind and temperature profiles become

$$u = \frac{u_*}{k}\left(\ln\frac{z}{z_{0u}} + C_u\frac{z}{L}\right) + a_u N z, \tag{30}$$

$$\theta - \theta_{\mathrm{s}} = \frac{\theta_*}{k_T}\left(\ln\frac{z}{z_{0T}} + C_\theta\frac{z}{L}\right) + a_\theta\frac{N^2}{\beta}Sz. \tag{31}$$

Here, z_{0u} and z_{0T} are the roughness lengths for momentum and temperature, respectively, θ_{s} is the potential temperature at the very surface; $a_u = C_u C_{uN}/k$ and $a_\theta = C_\theta C_{\theta N}/k_T$ are combinations of empirical constants considered above.

As seen from Eqs. (30) and (31), the proposed model does not cover the whole range of the SBL regimes. It addresses a specific type of long-lived SBL overlooked in the classical local theory. An obvious requirement for consistency of Eqs. (30) and (31) is:

$$u(z) > a_u N z, \qquad \theta(z) - \theta_{\mathrm{s}} > a_\theta\frac{N^2}{\beta}Sz. \tag{32}$$

The above restriction on the wind speed argues that the wind shear is sufficient to generate turbulence only if $a_u N < u/z$. The latter inequality is nothing but a non-local Richardson number criterion:

$$\frac{\beta(\partial\theta/\partial z)_{\text{free-flow}}}{(u/z)^2_{\text{surface-layer}}} < a_u^2. \tag{33}$$

One more requirement should be imposed to guarantee the very fact of radiation of internal waves from the SBL upper boundary to the free atmosphere. It takes place only when the Brunt–Väisälä frequency is larger in the layer immediately above the SBL than in the upper portion of the SBL, namely, when $N^2 > \beta(\partial\theta/\partial z)_{z<h}$. Although the potential-temperature gradient usually decreases towards the SBL upper boundary, this requirement should be checked before the model is applied.

Thus Eqs. (30) and (31) represent a model which is complementary rather than alternative to the classical formulation. Besides the above restrictions, these equations do not reduce to the conventional neutral stability formulation when $L \to \infty$ and $N \neq 0$. For modelling applications, a reasonable combination of the classical and the proposed formulations is needed to cover the range of stability conditions including both nocturnal and long-lived SBLs.

6. Concluding remarks

The proposed theoretical model identifies key mechanisms responsible for the recently discovered effect of the free-flow stability on the wind and temperature profiles in the surface layer. These are the radiation of IGWs from the SBL upper boundary into the stably stratified free atmosphere, and the wave-induced third-order transports within the SBL.

An important result from the new model is a theoretical explanation of the well-known empirical fact that developed turbulence in the stably stratified surface layers can exist at much larger Richardson numbers than the classical theories predict. This 'drawing' of developed turbulence towards very strong temperature gradients is shown to be inherent in long-lived SBLs. Here, radiation of internal waves from the SBL upper boundary results in a very efficient withdrawal of the potential-temperature variance from the surface layer ($\propto N^5$), which in turn essentially reduces the negative buoyancy flux and eventually gives more room for generation of turbulence by the velocity shear. Accompanying internal-wave withdrawal of TKE is much less efficient ($\propto N^3$), so that the overall effect of internal waves results in the strengthening of turbulence in the lower portions of long-lived SBLs.

The model allows clarification of the physical meaning of the two length-scales, L and L_N (Eq. (1)) relevant to the surface-layer scaling. The Monin–Obukhov-scale, L, imposes an upper limit on vertical displacements of chaotically moving fluid particles affected by the negative buoyancy forces. It represents an essentially local scale based on the near-surface buoyancy flux, B, and the friction velocity, u_*. The scale L_N imposes a limit on the amplitudes of internal waves radiated from the SBL upper boundary to the stably stratified non-turbulent layer aloft. It is based on the Brunt–Väisälä frequency in the free atmosphere, N, and the friction velocity (which appears as a turbulence velocity-scale for the entire SBL). It is precisely the length-scale that reflects the links between the surface layer and the free atmosphere.

As specified by Eq. (32), the model is not applicable when the wind speed in the surface layer is very low whereas the static stability in the free atmosphere is very strong. It is also not applicable when the temperature gradient in the free atmosphere is essentially stronger than the bulk temperature gradient in the near-surface layer. The model does not reduce to the conventional neutral stability formulation when $L \to \infty$ and $N \neq 0$. These restrictions call for further investigation.

The proposed approach is an attempt to incorporate the internal-wave transport of kinetic energy and potential-temperature variance in the context of turbulence closures for stably stratified sheared flows. The present version of the model is deliberately made very simple—to illustrate the basic idea clearly, and to create links between the earlier analysis (Z&C) and more rigorous physical theory.

In future work a more advanced model should be developed. First of all, Eqs. (12) and (13) for the horizontal and vertical length-scales, λ and l, need refinement. In the present theory they are identified with the scales of large eddies inherent to the upper portion of the SBL.

Equation (13) for the vertical scale of disturbances at the SBL upper boundary, $\lambda \sim u_*/N$, employs u_* as a turbulent-velocity-scale. This is justified in barotropic flows. In baroclinic flows, an alternative velocity-scale accounting for the geostrophic wind shear $|\partial \mathbf{u}_g/\partial z|$ should be used in the upper portion of the SBL instead of u_*.

Equation (12) for the horizontal length-scale of the largest eddies in the SBL, $\lambda \sim h$, is probably an acceptable approximation in near-neutral stratification. Here, the ratio of the horizontal to vertical extension of eddies, λ/h, should be proportional to the inverse drag coefficient $C_D^{-1} = u/u_*$, which is not too variable ($C_D^{-1} \approx 30$). Then the volume of a typical large eddy is $\lambda^2 h \sim h^3 C_D^{-2}$. In essentially stable stratification, large eddies which initially have the same volume squeeze in the vertical and extend in the horizontal. Taking instead of h the barotropic SBL vertical-turbulence-scale $l \sim u_*/N$, the conservation of the eddy volume suggests $\lambda^2 u_*/N \sim h^3 C_D^{-2}$. Consequently, the horizontal length-scale becomes $\lambda \sim h^{3/2} C_D^{-1} (N/u_*)^{1/2}$. The horizontal length-scale for baroclinic SBLs could be derived similarly by employing a corrected turbulent-velocity-scale.

One more reasonable refinement could concern the divergences of the third-order fluxes, $\partial F_{\mathrm{KE/PE}}/\partial z$, in the energy budget equations, Eqs. (4) and (16). In the present simple model they are approximated through finite differences across the SBL, as $\partial F_{\mathrm{KE/PE}}/\partial z = F_{\mathrm{KE}h/\mathrm{PE}h}/h$. It follows that the divergences are constant with depth throughout the SBL. However, at the SBL upper boundary they should tend to zero together with all other terms in Eqs. (4) and (16), otherwise the equations would become inconsistent. Moreover, in the logarithmic layer close to the surface the third-order fluxes are maintained by the small-scale turbulence rather than by internal waves. Accordingly, not only the fluxes but the flux divergences should vanish as $z \to 0$. Hence a more realistic approximation of $\partial F_{\mathrm{KE/PE}}/\partial z$ in the SBL should be applied, with particular attention to the accuracy of approximation in the logarithmic and the z-less stratification layers. This would hopefully allow extension of the model to a wider range of the SBL regimes, including the neutral stability limit.

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