A MULTI-LIMIT FORMULATION FOR THE EQUILIBRIUM DEPTH OF A STABLY STRATIFIED BOUNDARY LAYER

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Abstract. Currently no expression for the equilibrium depth of the turbulent stably-stratified boundary layer is available that accounts for the combined effects of rotation, surface buoyancy flux and static stability in the free flow. Various expressions proposed to date are reviewed in the light of what is meant by the stable boundary layer. Two major definitions are thoroughly discussed. The first emphasises turbulence and specifies the boundary layer as a continuously and vigorously turbulent layer adjacent to the surface. The second specifies the boundary layer in terms of the mean velocity profile, e.g. by the proximity of the actual velocity to the geostrophic velocity. It is shown that the expressions based on the second definition are relevant to the Ekman layer and portray the depth of the turbulence in the intermediate regimes, when the effects of static stability and rotation essentially interfere. Limiting asymptotic regimes dominated by either stratification or rotation are examined using the energy considerations. As a result, a simple equation for the depth of the equilibrium stable boundary layer is developed. It is valid throughout the range of stability conditions and remains in force in the limits of a perfectly neutral layer subjected to rotation and a rotation-free boundary layer dominated by surface buoyancy flux or stable density stratification at its outer edge. Dimensionless coefficients are estimated using data from observations and large-eddy simulations. Well-known and widely used formulae proposed earlier by Zilitinkevich and by Pollard, Rhines and Thompson are shown to be characteristic of the above interference regimes, when the effects of rotation and static stability (due to either surface buoyancy flux, or stratification at the outer edge of the boundary layer) are roughly equally important.

Key words: Stable Boundary Layer, Boundary-Layer Depth

1. Introduction

Stably stratified boundary layers (SBLs) are often encountered over a land surface during the night and in the upper ocean during periods of heating. One more geophysical example of the SBL is the bottom boundary layer in the ocean. It is only slightly, if at all, affected by the heat flux through the bottom but strongly influenced by stable density stratification aloft (this type in the atmosphere is known as an inversion-capped neutral layer). Although a lot of studies have been devoted to the SBL, it is not well understood. There is still confusion in the definition of its external boundary and thus in the definition of its depth. In this paper we consider various definitions of the SBL depth and show that different definitions should be used to examine different regimes. Then we attempt to develop a simple formula for the equilibrium depth of the stably (and neutrally) stratified boundary layer. The formula should be valid in specific cases of a truly neutral layer subject to rotation

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and rotation-free boundary layers influenced by stable density stratification near its bottom or/and top.

Strictly speaking, the SBL can hardly be in a perfectly steady state due to changes in large-scale flow. However, an inversion at the top of the atmospheric boundary layer, increasing in strength due to the cooling, would prevent strong energy and momentum exchange between the boundary layer and the rest of the troposphere. The SBL thus cannot grow unboundedly. Given enough time, it would evolve towards a quasi-equilibrium state. This is also true for the oceanic upper layer heated from above and affected by the underlying thermocline. Therefore, a diagnostic formula for the SBL depth is valid if synoptic variations are not too rapid (Derbyshire, 1990).

When external forcings, such as the pressure gradient and the surface heat flux, change rapidly, no diagnostic expression is valid. Then the boundary-layer depth should be determined from a rate equation that describes the SBL growth and decay. In this case, however, consideration of the equilibrium steady state is also useful. Although it does not define the instantaneous SBL depth, the equilibrium depth enters the rate equation for the boundary layer and sets the limit of a relaxation process. Without knowledge of the equilibrium solution, the rate equation is difficult to formulate. Note that, should this equilibrium solution depend on the instantaneous values of changing external parameters, it would itself be an implicit function of time and would, therefore, account for the complete time history of a relaxation process (see, e.g., Nieuwstadt and Tennekes, 1981).

2. Background

Clearly, modelling the equilibrium SBL depth requires a definition of what is meant by the boundary layer. Therefore, considering one or the other expression for the SBL depth, one must always keep in mind the definition this expression is based upon, or, if not clearly stated, implies. A number of definitions have been proposed. Among them, two should be highlighted, for they have very clear physical meaning and are fundamental for modelling the SBL depth.

(1) The first emphasises turbulence. The SBL is determined as a layer of continuous turbulence adjacent to the surface. Its external boundary is a level at which turbulence disappears or is a small portion of the surface value. Wyngaard (1983, 1988) adhered to this definition in his discussions of the atmospheric planetary boundary layer. It was used to estimate the SBL depth from measurements (e.g., Lenschow *et al.*, 1988a) and from results of numerical modelling (e.g., Richards, 1982). The energy considerations form the basis for several theoretical expressions for the equilibrium SBL depth. They have been proposed by Kitaigorodskii (1960), Kraus and Turner (1967), Deardorff (1972), Resnyansky (1975), Felzenbaum (1980), and Kitaigorodskii and Joffre (1988), to mention a few. Some of them, vitally important for the present analysis, are thoroughly discussed below.

(2) The second definition of the stable boundary layer employs the vertical profile of mean velocity. The SBL upper boundary is determined as a level at which the mean velocity approaches, or nearly approaches, the geostrophic velocity (in case of the wind-driven oceanic upper layer the formulation is given in terms of surface drift). This actually means that the boundary layer is identified with the Ekman layer. Since the Ekman layer owes its existence to the earth's rotation, any expression for the boundary-layer depth is likely to include the Coriolis parameter no matter how important the effect of rotation relative to that of static stability. This is indeed the case for the expressions proposed by Rossby and Montgomery (1935), Zilitinkevich (1972), Pollard *et al.* (1973), Weatherly and Martin (1978), and Nieuwstadt (1981), among others. It is shown below that the expressions based on this definition portray the depth of the turbulence in the intermediate regimes, when the effects of static stability and rotation essentially interfere.

When determining the SBL depth from measurements, a few alternative definitions have been invoked (Yu, 1978). Four of them are briefly discussed here. The boundary-layer top (bottom) was determined as a level at/to which

(3) the momentum flux (e.g., Businger and Arya, 1974), or,

(4) the heat flux (e.g., Caughey *et al.*, 1979) reduces to a small portion of its surface value,

(5) the lowest wind maximum occurs (e.g., Melgarejo and Deardorff, 1974, 1975),

(6) the surface temperature inversion extends (e.g., Yamada, 1976).

If the momentum flux is purely turbulent, (1) and (3) may give close results. It is not always so, however. Wavelike motions are often present in the stably stratified boundary layer. In the presence of shear these motions can carry momentum. Thus the depths determined from turbulence energy and momentum flux profiles may be different. Both the heat flux and temperature profile can vary due to the effects of radiation and may not correctly portray the depth of the turbulence (Mahrt, 1981). Definition (5), based on the mean velocity profile, is conceptually similar to (2).

We will not further discuss the alternative definitions of the SBL depth since they are not as important for the point in question as (1) and (2). It is these two definitions that stand behind the majority of theoretical formulae for the equilibrium SBL depth. And it is the difference between the two that leads to different formulations for seemingly similar cases. Let us now consider these formulations in some detail.

The following expression for the equilibrium boundary-layer depth was often used:

$$h = C_n \frac{u_*}{|f|}, \tag{1a}$$

where u_* is the surface friction velocity, f is the Coriolis parameter, and C_n is a proportionality factor whose estimates reported in literature range from 0.1 (Gill, 1967; Kitaigorodskii, 1970) to 0.5 (Mason and Thomson, 1987; Andrén and Moeng, 1993).

Equation (1a) was suggested by Rossby and Montgomery (1935) who examined the effect of friction on the structure of atmospheric and oceanic boundary layers subjected to the earth's rotation. In the light of the present discussion, it should have been based on definition (2) and must hold for the perfectly neutral Ekman layer. It is worth noting, however, that the mid-latitude neutral-static-stability boundary layer represents the simplest case where the Ekman layer depth is an appropriate scale for the depth of the boundary layer determined by the turbulence energy. The physical basis for this is quite clear. There are no parameters governing the steady turbulence regime in a perfectly neutral boundary layer, other than u_* and f, so that $u_*/|f|$ is the only length scale that can be composed out of the two. Therefore, the Ekman layer depth and the depth of the turbulence should be quite close.

Analysing data from measurements in the oceanic upper layer heated from above, Kitaigorodskii (1960) held the viewpoint that the SBL depth should be defined as the depth of the turbulence, i.e. using definition (1). He came to the conclusion that the Obukhov length, $L = -u_*^3/B_s$, where B_s is the surface buoyancy flux, is an appropriate scale for the depth of the surface-flux-dominated SBL:

$$h = C_s L. \tag{1b}$$

The dimensionless factor C_s was reported to range from 1.2 (Stigebrandt, 1985) to 100 (Kitaigorodskii and Joffre, 1988). Note that we do not include the von Karman constant in the definition of L. The reason Kitaigorodskii gives to justify the validity of Equation (1b) is that in the case of very strong static stability induced by the surface buoyancy flux the SBL is not deep enough for the Coriolis force to significantly affect the shear production of turbulence energy.

More recent attempts to model the depth of the surface-flux-dominated SBL using the balance equation for turbulence energy led to either Equation (1b) (e.g., Kraus and Turner, 1967; Niiler, 1975; Niiler and Kraus, 1977), or to expressions that combine Equations (1a) and (1b) (e.g., Resnyansky, 1975; Garwood, 1977). Deardorff (1972) proposed a reasonable interpolation between the two (the sum of inverse values) that includes an additional scale, the height of the tropopause, as an ultimate limit for the SBL depth. Stigebrandt (1985) simply chose the shortest of the lengths scales given by Equation (1a), Equation (1b) and the depth to the pycnocline as the SBL depth in his seasonal pycnocline model for the Baltic Sea proper.

Zilitinkevich (1972) adopted definition (2) to determine the equilibrium SBL depth in the case of strong static stability due to surface buoyancy flux. Using the Ekman length scale, $(K/|f|)^{1/2}$, where K is an effective eddy viscosity, and the concept of a limiting Richardson number at the boundary layer top to estimate K, he derived the expression

$$h = C_{sr} \frac{u_*^2}{|fB_s|^{1/2}}, \tag{1c}$$

where C_{sr} is a dimensionless parameter of order one (see the summary table in Zilitinkevich, 1989). Equation (1c) has been used for more than two decades, especially by meteorologists who often utilised it in observational and modelling studies (e.g., Nieuwstadt, 1981, 1984, 1985; Caughey, 1982; Byun, 1991; Grant, 1994). Interpolation formulae that reduce to either (1a) or (1c) in the limits of either no or very large surface buoyancy flux were suggested by Nieuwstadt (1981), Zilitinkevich (1989), Derbyshire (1990). An essential feature of Equation (1c) and the interpolations based on Equations (1a) and (1c) is that the Coriolis parameter is always present in the denominator. It is a direct consequence of the use of the definition (2) that the effect of rotation remains crucial for the SBL depth even in the limiting case of very strong stability.

Two more expressions could be mentioned here in relation to the Zilitinkevich formula. Brost and Wyngaard (1978) proposed $h \propto u_*^2/(fG\sin\alpha)$, where G is the magnitude of the geostrophic wind and α is the angle of wind turning in the boundary layer. Nieuwstadt and Tennekes (1981) proposed $h \propto fG^2 \sin\alpha\cos\alpha/(\beta \mathrm{d}\theta/\mathrm{d}t)$, where $\beta = g/T$ is the buoyancy parameter, g is the acceleration of gravity, T and θ are absolute temperature and potential temperature, and $\mathrm{d}\theta/\mathrm{d}t$ is the cooling rate. Using the geostrophic drag law and assuming a constant cooling rate within the bulk of the SBL (a frequently used assumption for the nocturnal boundary layer), the latter authors showed both expressions to be equivalent to Equation (1c) for large values of the surface buoyancy flux.

In an attempt to explore the response of the upper ocean to an imposed wind stress Pollard *et al.* (1973) adhered to definition (2). They used the Ekman equations, the heat conservation equation, and the overall Richardson number, Ri_0, stability criterion to close the problem. Assuming quite arbitrarily that the flow is marginally hydrodynamically unstable at Ri = 1, they simulated the mixed layer growth in a linearly stratified fluid with the buoyancy frequency N in response to the instantaneous onset of wind. Assuming further that the surface heating is small, they found that after one-half inertial period the deepening is arrested by rotation at the depth h determined as

$$h = C_{ir}\frac{u_*}{|fN|^{1/2}}, \tag{1d}$$

where C_{ir} is dimensionless parameter (equal to 1.7, according to Pollard *et al.*, 1973). It is worth noting that Equation (1d) can also be derived using the same arguments as those that led to Equation (1c). The only difference is that the static stability is due to the imposed density stratification rather than due to the surface buoyancy flux.

Using the assumption that the entrainment rate is the reciprocal of the overall Richardson number, Phillips (1977) obtained $h = 1.1u_*/(f^{2/3}N^{1/3})$, which is, in fact, a particular case of the more general theory considered by Pollard *et al.* (1973) in the Appendix to their paper. An interpolation between Equations (1a) and (1d) was suggested, e.g., by Weatherly and Martin (1978).

One more expression should be particularly mentioned. It reads

$$h = C_i \frac{u_*}{N}, \tag{1e}$$

where C_i is a dimensionless constant. To the best of our knowledge, Deardorff (1972) was the first to point out that the boundary-layer depth can be limited by the background density stratification but he did not suggest Equation (1e) in its explicit form. We also make reference to Kitaigorodskii and Joffre (1988; see also Kitaigorodskii, 1988) who adduced arguments in favour of Equation (1e) as the SBL depth scale. Considering the turbulence energy balance, i.e. using definition (1), they found that f drops out from the set of parameters governing the equilibrium SBL depth if the background density stratification is strong. More recent studies of the atmospheric boundary layer over sea ice (Overland and Davidson, 1992) and the benthic boundary layer (Rahm and Svensson, 1989; Kitaigorodskii, 1990, 1992) have added considerable support for Equation (1e). The estimates for C_i given by Kitaigorodskii and Joffre (1988) and Kitaigorodskii (1992) vary from 4 to 13. Data from Overland and Davidson (1992) suggest C_i of order 20, though with considerable scatter.

An interpolation equation that includes Equations (1a), (1b) and (1e), along with a number of asymptotic regimes of the mixed-layer deepening was derived by Felzenbaum (1980) from consideration of the turbulence energy budget in the oceanic upper layer.

Mahrt (1981) tried to avoid use of surface fluxes and formulated his model of the boundary-layer growth and decay in terms of bulk quantities. The equilibrium SBL depth is expressed through the wind at the boundary-layer top and the temperature difference across the layer. The validity of the model is still to be verified. We believe that models of that kind can be reformulated in terms of surface quantities through the use of appropriate resistance and heat transfer laws.

3. Motivation for the Present Study

As is evident from the above discussion, no consensus has been achieved so far as to the definition of the stable boundary layer. There is no expression for its equilibrium depth that is valid throughout the entire range of stability conditions and close to the equator. However, the SBL depth is required for many applications including air-sea/air-land interaction, pollution dispersion, and mesoscale and climate modelling. Here we attempt to derive an expression for the equilibrium depth of the SBL that holds in both the general case and in the limits of the rotation-free stable layer and perfectly neutral layer subjected to rotation.

Toward this goal we start with definition (1) considered above, i.e. we define the SBL as a continuously turbulent boundary layer adjacent to the surface. In support of our choice we refer to the fact that the turbulent boundary layer is universal in occurrence while the Ekman layer is peculiar to mid-latitudes. Using definition (1), we derive an expression for the equilibrium boundary-layer depth that holds in the limiting asymptotic cases. Then we classify the SBL regimes, show the place of the formulations based on definition (2) in the hierarchy of the SBL depth scales, and derive a general equation for the equilibrium SBL depth that includes both limiting and intermediate asymptotes.

We realise that the structure of the turbulent boundary layer may be complicated by longwave radiation (see e.g., Garratt and Brost, 1981; André and Mahrt, 1982) and internal gravity waves (e.g., Finnigan *et al.*, 1984). Consideration of these processes is basically beyond the scope of this paper, although the gross effect of waves on the integral turbulence energy budget in the SBL will be considered. We thus concentrate on the stable boundary layer dominated by turbulence. For the sake of clarity the derivations in the next section are carried out for a temperature stratified barotropic SBL.

4. Basic Model

4.1. Balance Equation for Turbulence Energy

Let us consider the turbulence energy budget for a horizontally homogeneous stable boundary layer of depth h. We utilise a right-hand Cartesian co-ordinate system with the origin at the surface, the z-axis vertically upward and the x-axis aligned with the surface stress. This corresponds to either the atmospheric or benthic boundary layer. The final result, however, is also valid for the oceanic upper layer. In the steady-state limit, the balance equation for turbulence energy integrated over the boundary layer depth reads

$$\int_0^h \left(\tau_x \frac{\partial u}{\partial z} + \tau_y \frac{\partial v}{\partial z}\right) \mathrm{d}z + \int_0^h B \,\mathrm{d}z + F_s - F_h - \int_0^h \varepsilon \,\mathrm{d}z = 0. \tag{2}$$

Here, u and v, τ_x and τ_y are horizontal components of the mean velocity and the shear stress, respectively; $B = \beta Q$ is the vertical buoyancy flux, Q is the potential temperature flux; F_s and F_h are the energy fluxes at $z = 0$ and $z = h$, respectively; and ε is the dissipation rate of turbulence energy. The first term on the l.h.s. of Equation (2) represents the work of the stress on the mean flow and is a source of turbulence energy. The buoyancy and dissipation terms are sinks in the SBL, while the flux terms describe the energy input/escape to/from the boundary layer. These terms are considered separately in the following.

4.2. Shear Production

Using the equations of motion for a rotating fluid, it can be shown that the total shear production of turbulence energy in the stationary horizontally homogeneous boundary layer of depth h is given by

$$\int_0^h \left(\tau_x \frac{\partial u}{\partial z} + \tau_y \frac{\partial v}{\partial z}\right) \mathrm{d}z = \tau_{xs}\bar{U} + \tau_{ys}\bar{V} = u_*^2\bar{U}, \tag{3}$$

where τ_{xs} and τ_{ys} are the surface values of the shear stress components along the x and y horizontal axes, respectively; and $\bar{U}$ and $\bar{V}$ are the u and v horizontal velocity components averaged over the boundary-layer depth. The stress components at the boundary-layer top are neglected as being small compared to the surface stresses (a usual simplification that may not, however, be possible in the case of strong entrainment). With the x-axis aligned with the surface stress, only the term with τ_{xs} remains, leading to the second equality on the r.h.s. of Equation (3). Exact manipulations that lead to Equation (3) are straightforward but cumbersome. We will not present them here. For the sake of simplicity, we illustrate the issue by considering a non-rotating boundary layer where both the velocity and the stress along the y-axis are zero. Integrating the l.h.s. of Equation (3) by parts with due regard to the lower boundary condition for u, then neglecting the stress at the boundary-layer top and estimating, to a leading order, the vertical gradient of τ_x as $\partial\tau_x/\partial z = -u_*^2/h$, the result given by Equation (3) is recovered.

The shear production of the turbulence energy, $u_*^2\bar{U}$, is conveniently determined from the resistance law. This law is derived by matching the velocity defect profile in the upper part of the boundary layer with the surface-layer velocity profile (see, e.g., Zilitinkevich, 1975; Tennekes, 1982). The resistance law is formulated either in terms of the velocity at the boundary-layer top U_h, or in terms of the layer-average velocity $\bar{U}$. These formulations differ in the values of dimensionless constants they include. The functional dependencies on the governing parameters are, however, the same. We consider here the formulation in terms of the layer- average velocity,

$$\bar{U} = \frac{u_*}{k}\left(\ln\frac{h}{z_0} - B_*\right), \tag{4}$$

where k is the von Karman constant, z_0 is the roughness length with respect to momentum, and B_* is a dimensionless function of the boundary-layer stability parameters. We use unconventional notation with the asterisk in lieu of simply B to avoid confusion with the vertical buoyancy flux. The resistance law contains the other equation that relates the mean v-component of velocity, or alternatively the veering angle, to the other dimensionless function, the so-called A_* function. We do not discuss it here since the v-component does not enter the r.h.s. of the second equality of Equation (3).

The B_* function for the near-neutral and surface-flux-dominated boundary layers has been evaluated by numerous researchers. A summary is given by Zilitinkevich (1989). Almost no attention has been paid to the inversion-capped neutral layer. Here we attempt to derive a reasonable approximation of this function that incorporates three basic limiting asymptotic regimes characteristic of the SBL.

In the case of static stability caused by the surface buoyancy flux the well-known log-linear velocity profile holds true in the surface layer:

$$u(z) = \frac{u_*}{k}\left(\ln\frac{z}{z_0} + \beta_{us}\frac{z}{L_*}\right), \tag{5}$$

where β_{us} is dimensionless constant of order 5 (e.g., Högström, 1988), and $L_* = L/k$ is the conventional Obukhov length. It is obtained by means of interpolation between the near-surface logarithmic profile and the Monin–Obukhov linear profile valid at some height above the surface, i.e. at $z/L_* \gg 1$ in terms of dimensionless height. An alternative interpolation that provides for the logarithmic and linear behaviour close to and away from the surface, respectively, reads

$$u(z) = \begin{cases} \dfrac{u_*}{k}\ln\dfrac{z}{z_0} & \text{at} \quad z_0 \le z \le \dfrac{L_*}{\beta_{us}} \\[2ex] \dfrac{u_*}{k}\left(\ln\dfrac{L_*}{\beta_{us}z_0} + \beta_{us}\dfrac{z}{L_*} - 1\right) & \text{at} \quad \dfrac{L_*}{\beta_{us}} \le z \ll h. \end{cases} \tag{6}$$

The second equation of (6) corresponds to the so-called limiting-Richardson-number stratification. It can be matched with the limiting-Richardson-number velocity defect law for the SBL interior. The latter is formulated in terms of either U_h or $\bar{U}$. We use the formulation in terms of the layer-average velocity[1]

$$\bar{U} - u(z) = \frac{u_* h}{L}\Phi_s\left(\frac{z}{h}\right), \tag{7}$$

where Φ_s is a dimensionless function. The reasons why Φ_s does not explicitly depend upon the Coriolis parameter and local stability, and is a function of z/h only, are as follows (see also Zilitinkevich, 1975). Since the equilibrium boundary layer depth is uniquely determined by the SBL governing parameters, h itself can be used instead of any one of those parameters. We eliminate f, keeping in mind that the effect of rotation is implicitly accounted for through h. As for local stability conditions, they are characterised by the Richardson number that increases with height in the surface layer but remains nearly constant (close to its limiting value Ri = 0.2) in the upper part of the SBL. Observational evidence for this is given by

[1] The two formulations are equivalent. Indeed, subtracting from Equation (7) this equation taken at $z = h$, we obtain the velocity defect law in terms of the velocity at the boundary-layer top, $U_h - u(z) = u_*(h/L)\Phi'_s(z/h)$, where the functions Φ'_s and Φ_s differ by dimensionless constant, $\Phi'_s(z/h) = \Phi_s(z/h) - \Phi_s(1)$.

Mahrt *et al.* (1979), Garratt (1982), and Nieuwstadt (1985). Due to this feature of the SBL, known as z-less (or limiting Richardson number) stratification, Φ_s ceases to depend on local stability in the explicit form.

Overlapping Equations (6b) and (7) in the region $z_0 \ll z \ll h$ results in the following expression for the B_* function:

$$B_* = \ln\left(k\beta_{us}\frac{h}{L}\right) - C_{us}\frac{h}{L} + 1 \sim -C_{us}\frac{h}{L} + \cdots, \tag{8}$$

where C_{us} is a dimensionless coefficient. The expression on the r.h.s. of Equation (8), retaining only the main term, is a familiar approximation that corresponds to the limit of strong stability induced by the surface buoyancy flux.

The resistance law for the inversion-capped neutral layer has not been thoroughly discussed in the literature. We could mention the work of Kitaigorodskii (1988) who pointed out that the B_* and A_* functions must depend on the background stratification but he did not suggest any specific form for that dependence.

Considering the inversion-capped SBL, it is advantageous to draw an analogy with the surface-flux-dominated SBL. We suppose that the nature of steady turbulence regimes in these cases is the same, with the only difference that the Obukhov length pertinent to the surface-flux-induced static stability is replaced by the length scale u_*/N characterising the effect of imposed stable stratification. Using the same arguments as those that led to Equation (7), the velocity defect for the inversion-capped layer is

$$\bar{U} - u(z) = Nh\Phi_i\left(\frac{z}{h}\right), \tag{9}$$

where Φ_i is a dimensionless function.

For the surface layer, again by analogy with the surface-flux-dominated SBL, the following velocity profile can be proposed:

$$u(z) = \frac{u_*}{k}\left(\ln\frac{z}{z_0} + \beta_{ui}\frac{zN}{u_*}\right), \tag{10}$$

where β_{ui} is dimensionless parameter. Equation (10) is a reasonable extension of the traditional Monin–Obukhov similarity to the case of the surface layer affected by the static stability in the free flow. Just as Equation (5), it is derived by means of interpolation between the logarithmic profile valid in the near vicinity of the surface, and the linear profile well above the surface at $zN/u_* \gg 1$, where the height is made dimensionless with the length scale u_*/N appropriate for the inversion-capped layer. Notice that away from the surface, Equation (10) implies that $\partial u/\partial z \propto N$, i.e. the limiting-Richardson-number stratification, in agreement with some observations (Mahrt, 1995, private communication) and results of large-eddy simulations (Derbyshire, 1995a,b).

The interpolation alternative to Equation (10) reads

$$u(z) = \begin{cases} \dfrac{u_*}{k} \ln \dfrac{z}{z_0} & \text{at} \quad z_0 \leq z \leq \dfrac{u_*}{\beta_{ui} N} \\[2ex] \dfrac{u_*}{k} \left(\ln \dfrac{u_*}{\beta_{ui} N z_0} + \beta_{ui} \dfrac{zN}{u_*} - 1 \right) & \text{at} \quad \dfrac{u_*}{\beta_{ui} N} \leq z \ll h. \end{cases} \tag{11}$$

Overlapping Equations (9) and (11b) at $z_0 \ll z \ll h$ yields the following expression for the B_* function:

$$B_* = \ln \left(\beta_{ui} \frac{hN}{u_*} \right) - C_{ui} \frac{hN}{u_*} + 1 \sim -C_{ui} \frac{hN}{u_*} + \cdots , \tag{12}$$

where C_{ui} is a dimensionless coefficient, and the approximation on the r.h.s. corresponds to the strong static stability at the SBL top.

Equations (9)–(12), obtained here from heuristic arguments, certainly require verification against observational data. Apart from references given above in support of Equation (10), we do not have data at the moment to quantitatively verify the components of the model, i.e. the velocity defect law Equation (9) and the surface layer profile Equation (10). However the final integrated result, to which the above relations contribute, will be validated. Nevertheless, Equations (9)–(12), which are the assumptions at present rather than established facts, seem to be a plausible generalisation of traditional similarity laws to the inversion-capped SBL.

The resistance law, Equation (4), should remain in force for the truly neutral boundary layer, one with zero buoyancy flux throughout. In this case, $u_*/|f|$ is the only appropriate length scale, and the Rossby–Montgomery formula (1a) holds true. Separating out the logarithmic term, the B_* function in the resistance law (the "correction function" as termed by Hinze, 1959) can be expressed as a power series in hf/u_*:

$$B_* = \ln \frac{h}{z_0} - k \frac{\bar{U}}{u_*} = C_{un} \left(\frac{hf}{u_*} \right)^2 + \cdots , \tag{13}$$

where C_{un} is a (positive) dimensionless constant, and dots on the r.h.s. indicate that only the leading order term is retained. The physical reason behind Equation (13) is as follows. The velocity component parallel to the surface stress, although dependent upon the rate of rotation, should be independent of the sign of the Coriolis parameter, and is, therefore, an even function of f (Long, 1974).

In order to account for the three basic asymptotic cases discussed above, we interpolate between Equations (8), (12), (13) and propose the B_* function in the form

$$B_* = C_* + C_{un} \left(\frac{hf}{u_*} \right)^2 - C_{us} \frac{h}{L} - C_{ui} \frac{hN}{u_*} , \tag{14}$$

where C_* is a dimensionless constant that accounts for the case of a non-rotating truly neutral boundary layer (e.g., the boundary layer over a flat surface in a non-rotating laboratory tank). Since its behaviour in the limiting cases is the major concern, only the main terms are retained in the expression for B_*, and the simplest linear interpolation between the limiting cases is used. The reason is also that we have used a method of asymptotic matching to derive the resistance law, i.e. an approximate method that cannot give an exact solution in intermediate regimes. The latter regimes are considered separately in Section 5.

In principle the same basic resistance law can be obtained using the momentum equations with any realistic turbulence closure. The simplest is the two-layer eddy-viscosity model with the eddy viscosity $K = ku_*z$ near the surface to account for the logarithmic velocity profile, and height-constant K dependent upon relevant stability parameters, such as L, u_*/N and the SBL depth h itself, far from the surface. The resulting expression for the B_* function would differ from Equation (14) in the intermediate regime, when the effects of rotation, surface cooling and stratification aloft are roughly equally important (and the terms containing both f and L, and/or f and N, would appear in the resistance law). Nevertheless, it would reveal the same asymptotic behaviour independent of any specific features of closure in all three limiting cases discussed above. We therefore adopt Equations (3), (4) and (14) to parameterize the shear production term in Equation (2).

4.3. Buoyancy Destruction

The vertical profile of the buoyancy flux $B = \beta Q$, where β is the buoyancy parameter and Q is the flux of virtual potential temperature, is obtained using the equation of heat transfer:

$$\frac{\partial \theta}{\partial t} = -\frac{\partial Q}{\partial z}. \tag{15}$$

We assume that the quasi-stationary stable boundary layer is characterised by a height-constant cooling rate. This assumption is often used to study the nocturnal boundary layer in the atmosphere (e.g., Nieuwstadt, 1985). Then differentiation and subsequent double integration of Equation (15) with respect to z results in the linear heat flux profile:

$$Q = Q_s\left(1 - \frac{z}{h}\right) + Q_h\frac{z}{h}, \tag{16}$$

where Q_s and Q_h are the fluxes at the boundary-layer bottom and top, respectively.

The latter flux is determined through the effective temperature conductivity and the temperature gradient at the SBL top, $Q_h = -K_h N^2/\beta$. The temperature conductivity can be estimated as $K_h \propto e_h/N$, where e_h is the turbulence energy at $z = h$ (e.g., Zilitinkevich and Mironov, 1992). Scaling e_h with the surface friction velocity, we get $Q_h \propto -u_*^2 N/\beta$. With this estimate, the buoyancy term in Equation (2) becomes

$$\int_0^h B\,dz = \frac{1}{2}h(\beta Q_s - C_{bh}u_*^2 N) = \frac{1}{2}h(B_s - C_{bh}u_*^2 N), \tag{17}$$

where C_{bh} is a dimensionless constant.

4.4. Energy Fluxes

The kinetic energy flux through the surface F_s is evidently zero in the atmospheric and benthic boundary layers. In the oceanic upper layer it may occur as an additional source of energy, e.g. as a result of breaking of the surface gravity waves. The simplest approximation in that case would be to relate F_s to the cube of surface friction velocity through an empirical coefficient. We do not pursue this issue further, however, as the velocity shear is typically a dominating source of energy in stably stratified boundary layers. Thus we take

$$F_s = 0. \tag{18}$$

The energy flux at the boundary-layer top F_h occurs due to internal gravity waves which may transfer energy from the turbulence to the stably stratified fluid aloft (Carruthers and Hunt, 1986). Following Thorpe (1973), this flux can be roughly estimated as

$$F_h \propto A^2 \lambda N^3, \tag{19}$$

where A and λ are a typical amplitude and a typical horizontal wave length, respectively of internal waves generated at the SBL top.

Kantha (1977) proposed $\lambda \propto h$. The assumption is plausible since large semi-organised eddies with the integral length scale of order h are often present in boundary layers. The wave amplitude should scale with the amplitude of disturbances caused by eddies impinging on a stably stratified fluid, and can be taken to be proportional to W_*/N, where W_* is a characteristic vertical velocity of turbulent eddies (Townsend, 1966; Kantha, 1977). Since the only source of turbulence energy in the stably stratified boundary layer is the instability of mean flow, the only velocity scale relevant to shear generated turbulence is u_*. We therefore take $W_* \propto u_*$ and hence $A \propto u_*/N$. With these estimates, the energy flux due to the radiation of internal gravity waves from the SBL upper boundary is

$$F_h = C_w u_*^2 h N, \tag{20}$$

where C_w is dimensionless coefficient.

In case when internal waves are already excited in the free flow (e.g., over complex terrain) and SBL is stratified strongly enough ($N_{\mathrm{SBL}} > N$), the wave energy could be pumped into the SBL, which could lead to its sporadic breakdowns (Nappo, 1991; Nappo and Eckman, 1995). This phenomenon, however, is not considered in the present paper.

4.5. Dissipation

The integral energy dissipation in the SBL is estimated as follows. We first consider the near-surface layer. The Monin–Obukhov similarity theory states that in the vicinity of the surface the balance is between the shear production of turbulence energy, the buoyancy destruction and the dissipation, the energy transport being of minor importance. Hence ε can be determined using the log-linear velocity profile and the surface values of momentum and buoyancy fluxes throughout the surface layer, a usual surface-layer approximation:

$$\varepsilon = \frac{u_*^3}{kz} + (1 - \beta_{us})B_s. \tag{21}$$

For the dissipation rate at the boundary layer top, the following expression can be deduced from dimensionality arguments [cf. the Brost and Wyngaard (1978) parameterization, and also the discussion of the potential temperature flux at the SBL top given with the derivation of Equation (17)]:

$$\varepsilon \propto u_*^2 N. \tag{22}$$

Equation (21) is valid at $z/h \ll 1$, while Equation (22) at $(h-z)/h \ll 1$. They can be generalised for the entire boundary layer as follows:

$$\varepsilon = \frac{u_*^3}{z}\Phi_{\varepsilon 1}(\zeta) - B_s\Phi_{\varepsilon 2}(\zeta) + u_*^2 N\Phi_{\varepsilon 3}(\zeta), \tag{23}$$

where $\Phi_{\varepsilon 1}$, $\Phi_{\varepsilon 2}$ and $\Phi_{\varepsilon 3}$ are dimensionless functions of $\zeta \equiv z/h$ obeying boundary conditions $\Phi_{\varepsilon 1}(0) = 1/k$, $\Phi_{\varepsilon 2}(0) = \beta_{us} - 1$ and $\Phi_{\varepsilon 3}(0) = 0$. Again, as in the case of the resistance law, only the main terms are included in the above expression, which is, in fact, an interpolation between the limiting asymptotic cases. In the intermediate regimes, the terms containing both f and L, and/or f and N, could also appear.

Integrating Equation (23) over the boundary layer depth and separating out the logarithmic term [through the use of z_0 as the lower limit when integrating the first term on the r.h.s. of Equation (23)], we get

$$\int_0^h \varepsilon\,\mathrm{d}z = \frac{u_*^3}{k}\ln\frac{h}{z_0} - C_{\varepsilon 1}u_*^3 - C_{\varepsilon 2}hB_s + C_{\varepsilon 3}u_*^2 hN, \tag{24}$$

where

$$
\begin{aligned}
C_{\varepsilon 1} &= \int_0^1 [\Phi_{\varepsilon 1}(0) - \Phi_{\varepsilon 1}(\zeta)] \frac{\mathrm{d}\zeta}{\zeta}, \quad C_{\varepsilon 2} = \int_0^1 \Phi_{\varepsilon 2}(\zeta)\,\mathrm{d}\zeta, \\
C_{\varepsilon 3} &= \int_0^1 \Phi_{\varepsilon 3}(\zeta)\,\mathrm{d}\zeta
\end{aligned} \tag{25}
$$

are dimensionless constants.

4.6. The Limiting Asymptote Equation for the SBL Depth

Substituting Equations (3), (4), (14), (17), (18), (20) and (24) into Equation (2) and rearranging the terms, we obtain the following equation for the equilibrium depth of the stably stratified boundary layer:

$$
\left(\frac{fh}{C_n u_*}\right)^2 + \frac{h}{C_s L} + \frac{Nh}{C_i u_*} = 1, \tag{26}
$$

with the following relations between dimensionless constants:

$$
\begin{aligned}
\frac{kC_{\varepsilon 1} - C_*}{C_{un}} = C_n^2, \quad \frac{kC_{\varepsilon 1} - C_*}{k\left(\frac{1}{2} + C_{\varepsilon 2}\right) - C_{us}} = C_s, \\
\frac{kC_{\varepsilon 1} - C_*}{k\left(\frac{1}{2}C_{bh} + C_w + C_{\varepsilon 3}\right) - C_{ui}} = C_i.
\end{aligned} \tag{27}
$$

One can see that Equations (1a), (1b) and (1e) hold true in the asymptotic cases of a truly neutral, surface-flux-dominated and inversion-capped boundary layers, respectively. The constants C_{un}, C_{us}, C_{ui}, C_*, C_{bh}, C_w, $C_{\varepsilon 1}$, $C_{\varepsilon 2}$, and $C_{\varepsilon 3}$ as such are not required. Only their combinations, as they appear in Equations (27), i.e. C_n, C_s and C_i, should be evaluated.

The truly neutral boundary layer is rarely met in nature (Wyngaard, 1988, 1992). Even in the case of negligibly small surface buoyancy flux, stability conditions aloft are very seldom neutral. In the upper ocean the boundary layer normally occurs with a background of stable density stratification in the underlying thermocline. Its atmospheric and benthic counterparts are affected by the capping inversion. This fact suggests a plausible explanation for substantial difference between estimates of C_n based on atmospheric and oceanic data and those derived from laboratory and numerical simulations, see Table I. Geophysical values average between 0.1 and 0.3 (Zilitinkevich, 1989) and are likely to implicitly account for the effect of background stable stratification. With neutral conditions, as it is in the laboratory experiments of Caldwell *et al.* (1972) and large-eddy simulations (LES) of Mason

Table I

Dimensionless constant C_n in the Rossby and Montgomery (1935) formula, Equation (1), for the depth of neutrally stratified boundary layers

Reference	C_n
Measurements in the atmosphere and the ocean, Zilitinkevich (1989)	0.1–0.3
Laboratory experiment, Caldwell *et al.* (1972)	0.4
Large-eddy simulations, Mason and Thomson (1987), Andrén and Moeng (1993)	0.5

and Thomson (1987) and Andrén and Moeng (1993), the C_n values prove to be higher, about 0.4 and 0.5, respectively.

We adopt the LES value $C_n = 0.5$ for the remainder of this study. The following points favour this estimate.

The Reynolds number and the radius of curvature of the flow in the laboratory experiments of Caldwell *et al.* (1972) are fairly small. The Rossby-number similarity assumes large Reynolds number and geostrophic balance outside the boundary layer. Due to a small radius of curvature of the laboratory apparatus, however, the gradient flow is significant and departures from geostrophy are large. Furthermore, Caldwell *et al.* defined the boundary-layer thickness as the height at which the magnitude of the velocity is 99% of its geostrophic value. Although the Ekman layer depth and the depth of the turbulence are of the same order for a truly neutral boundary layer, they do not necessarily coincide. No data are given in op. cit. to evaluate h by turbulence energy.

Mason and Thomson (1987) used a mesh of $40 \times 40 \times 48$ points to produce a series of LES runs, changing domain size and resolution simultaneously. Their case B10 is the best compromise that "considers the most relevant range of scales". Mason and Thomson do not present an estimate of the boundary-layer depth from the turbulence energy profile. They report that "the boundary-layer stresses extend up to the heights of about $0.5u_*/|f|$". This value is consistent with the estimate that can be obtained using vertical profiles of the velocity variances (Figures 13–16 of Mason and Thomson, 1987), e.g. at a level of 5% of the surface value judged by eye. Andrén and Moeng (1993) used a smaller domain (2 km $\times$ 2 km $\times$ 3.6 km as compared to 6 km $\times$ 3 km $\times$ 10 km in Mason and Thomson's case B10) but higher resolution ($80 \times 80 \times 120$ grid points). As their Figure 2 suggests, $C_n = 0.5$ gives a good estimate of the height at which turbulence energy is 5% of its surface value. Thus $C_n = 0.5$ is borne out by both LES data sets.

Estimates of the constant C_s reported in the literature vary over two orders of magnitude. By applying his one-dimensional model of the seasonal pycnocline to the Baltic Sea proper, Stigebrandt (1985) found the best overall agreement with observations at $C_s = 1.2$. Values as high as 100 were obtained by Kitaigorodskii and Joffre (1988) using the data of Wangara (Clarke *et al.*, 1971) and ICE-77

(Joffre, 1981). The value $C_s = 75$, close to that of Kitaigorodskii and Joffre, was proposed by Deardorff (1972). Resnyansky (1975) found $C_s = 12$. The LESs of Mason and Derbyshire (1990; see also Derbyshire, 1990) and Brown *et al.* (1994) suggest $C_s = 10$. The boundary layer in case B of Mason and Derbyshire and in the cases BA10, BA10HR, BC10(12k6) and BC10(28k8) of Brown *et al.* was, perhaps, the closest to the quasi-stationary SBL dominated by the surface flux [from the LES study of Brown *et al.* (1994) we use only the results consistent with this assumption]. By contrast, atmospheric measurements may not be free of the effects of non-stationarity and stable stratification aloft. This might be the reason why the points on the Kitaigorodskii and Joffre (1988) graphs reveal very large scatter. Stigebrandt (1985) estimated C_s by fitting the simulated yearly cycles of sea-surface temperature, heat content and potential energy to observations. His estimate is likely to include a number of side effects (horizontal inhomogeneity, uncertainties in the input data, and most probably, strong static stability in the thermocline) and is not quite appropriate for use in Equation (26). Thus, as a first approximation, we adopt the LES estimate $C_s = 10$.

Finally, we examine the inversion-capped neutral layer. Estimates of the constant C_i are scarce. We can refer to Kitaigorodskii and Joffre (1988) whose analysis of the atmospheric data suggests C_i of order 10, and to Overland and Davidson (1992) whose data are very scattered but support the value of order 20 on the average. A LES study of Andrén (1995) deals with the SBL affected by both surface buoyancy flux and static stability aloft. Using the above estimates $C_n = 0.5$ and $C_s = 10$, and the results of the simulation SGSM2 of Andrén (1995), we estimate C_i at 20. For lack of better data we adopt this estimate. More rigorous evaluation of C_i, and also of C_s and C_n, is needed.

5. Intermediate Asymptotes

We have developed a simple expression, Equation (26), for the equilibrium depth of the stably (and neutrally) stratified boundary layer. In doing so, we have adopted the definition of a boundary layer that emphasises turbulence, and considered the integral turbulence energy budget in the SBL. Equation (26) is, therefore, likely to correctly account for the asymptotic limiting cases. In the intermediate regimes, however, when the effects of rotation and static stability essentially interfere, a simple linear combination adopted in the resistance law, Equations (4) and (14), may not be the best functional form. As a result, Equation (26) does not incorporate the well-known scales proposed by Zilitinkevich (1972), Equation (1c), and Pollard *et al.* (1973), Equation (1d).

As distinct from the limiting cases, the Zilitinkevich and Pollard *et al.* formulae account for the interplay of rotation and stratification. In such regimes, small scale turbulence is suppressed predominantly by stratification, so that the eddy viscosity, K, does not immediately depend on the Coriolis parameter, f. Then, using the concept of a limiting Richardson number, K is scaled with $\tau^2/\beta Q$ (Zilitinkevich, 1972). However, the boundary layer in the interference regimes is still deep enough to be influenced by rotation. Hence, the SBL growth is arrested predominantly by rotation, and the Ekman scale, $(K/f)^{1/2}$, is an appropriate limit for h. As a result, for the surface-flux-dominated SBL, $\tau \propto u_*^2$ and $Q \propto Q_s$, which yields the Zilitinkevich formula, and for the inversion-capped SBL, $\tau \propto u_*^2$ and $\beta Q \propto u_*^2 N$, which yields the Pollard *et al.* formula. The question about their place in the hierarchy of the SBL depth scales can be considered using the following simple heuristic arguments.

For the rotating boundary layer affected by the surface buoyancy flux, Equation (26) predicts two asymptotic limits that follow from simple dimensionality arguments. The Rossby–Montgomery formula, Equation (1a), is valid when rotation strongly dominates over the stabilising effect of the surface buoyancy flux, and the Kitaigorodskii formula (which would, perhaps, be more correctly referred to as the Kitaigorodskii/Monin–Obukhov formula), Equation (1b), is valid in the opposite case. To put this another way, the SBL depth is simply given by the shortest of the above two scales. Let us examine whether there is a field of application for the Zilitinkevich (1972) formula, when the SBL depth given by Equation (1c) is much smaller than the depths given by Equations (1a) and (1b). Comparing these three expressions, we find that Equation (1c) holds as an appropriate SBL depth scale if

$$4 \ll \mu \ll 100, \tag{28}$$

where $\mu = u_*/|fL|$ is the conventional Kazanski and Monin (1960) stability parameter that accounts for the combined effects of rotation and surface buoyancy flux. Equation (28) gives typical "meteorological" values of μ very often encountered over the cooled land surface in mid-latitudes. It is, therefore, not surprising that the Zilitinkevich (1972) formula, Equation (1c), though not valid in the asymptotic limits of a truly neutral and surface-flux-dominated boundary layer, was fairly successful in numerous meteorological applications. Thus, Equation (1c) is an intermediate asymptote, whereas Equations (1a) and (1b) are the limiting asymptotes.

Using similar arguments to examine the applicability of the Pollard *et al.* (1973) expression to the rotating boundary layer affected by the imposed static stability, we find that Equation (1d) holds as an appropriate scale for the inversion-capped SBL [i.e. predicts smaller SBL depth than both Equations (1a) and (1e)] if

$$10 \ll N/|f| \ll 100, \tag{29}$$

where only one significant digit is kept in both lower and upper limits.

The ratio $N/|f|$ is typically of order 100 or greater in both the earth's atmosphere and the ocean, suggesting that Equation (1d), which is the intermediate asymptote for the inversion-capped SBL, is of comparatively minor practical interest. Note also that the limits of applicability set by Equations (28) and (29) for the Zilitinkevich

Table II
Dimensionless constants in Equations (26) and (30)

Constant	Estimate	Reference
C_n	0.5	LES data
C_s	10	LES data
C_i	20	LES data
C_{sr}	1.0	Atmospheric data (Zilitinkevich, 1989)
C_{ir}	1.7	Theoretical estimate (Pollard *et al.*, 1973)

(1972) and the Pollard *et al.* (1973) formulae, respectively, are not very wide. This suggests that the simple linear combination given by Equation (26) could serve as a sufficient approximation for many practical purposes even in the intermediate regimes (this is further discussed below and is illustrated in Figures 3).

In case one were to take better account of the intermediate regimes, Equation (26) is modified as follows in order to incorporate the intermediate asymptotes given by Equations (1c) and (1d):

$$\left(\frac{fh}{C_n u_*}\right)^2 + \frac{h}{C_s L} + \frac{Nh}{C_i u_*} + \frac{h|f|^{1/2}}{C_{sr}(u_* L)^{1/2}} + \frac{h|Nf|^{1/2}}{C_{ir} u_*} = 1. \tag{30}$$

This equation is a reasonable approximation over the entire range of stability conditions, from truly neutral to very stable due to surface flux or/and capping inversion. It gives a finite depth of the stably stratified boundary layer close to the equator where the Coriolis parameter tends to zero (a truly neutral equatorial boundary layer should be considered in the framework of a more sophisticated theory including horizontal inhomogeneity and non-stationarity). The equation remains in force in the asymptotic cases of a truly neutral rotating layer where it reduces to Equation (1a), surface-flux-dominated SBL, Equations (1b) and (1c), and inversion-capped neutral layer, Equations (1d) and (1e).

Estimates of dimensionless constants are summarised in Table II. Note, however, that for use in Equation (30) the C_{sr} and C_{ir} constants should be somewhat adjusted to our estimates of C_s and C_i so that the intermediate regimes are best described by a combination of scales in Equation (30), not by the Zilitinkevich (1972) or the Pollard *et al.* (1973) formula alone.

6. Concluding Remarks

As discussed above, the limits of applicability of the Zilitinkevich (1972) and especially the Pollard *et al.* (1973) formulae for the SBL depth are not too wide. For a number of practical purposes Equation (26) that does not incorporate these scales is a sufficient approximation. Its performance is illustrated by Figures 1–3.

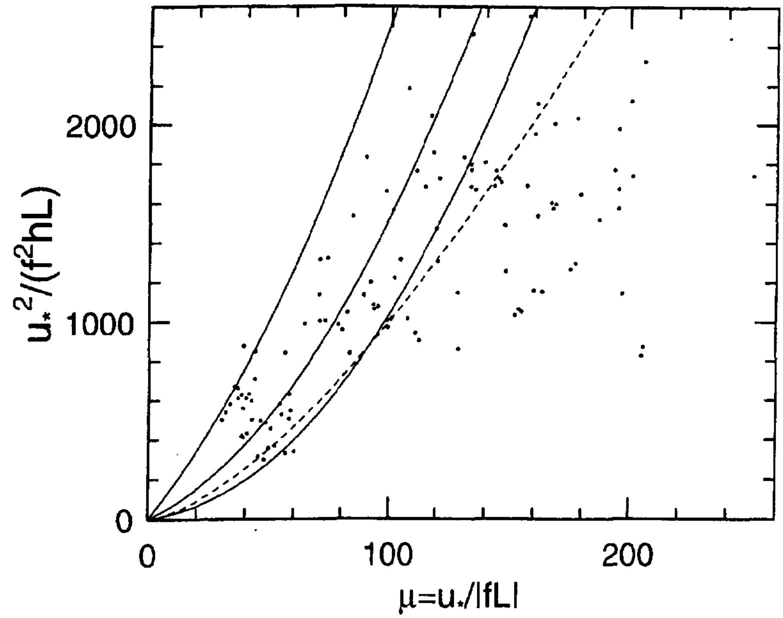

Figure 1. Reciprocal of the height of the surface-flux-dominated SBL made dimensionless with $u_*^2/(f^2L)$ versus conventional stability parameter $\mu = u_*/|fL|$. The data points are from Nieuwstadt (1981). Solid curves are obtained from Equation (26) with $C_n = 0.5$, $C_s = 10$ and $C_i = 20$ at $N/|f| = 0$ (right curve), $N/|f| = 100$ (middle) and $N/|f| = 300$ (left). Dashed curve corresponds to the Zilitinkevich (1972) formula, Equation (1c), with $C_{sr} = 1$.

In Figure 1 the dimensionless depth of the surface-flux-dominated SBL is plotted versus the conventional stability parameter $\mu = u_*/|fL|$. The unusual y-coordinate was used by Nieuwstadt (1981) to eliminate artificial correlation between dimensionless quantities containing common scaling variables. The scatter of data points is quite large. It may, however, be partly explained by static stability at the boundary-layer top. Indeed, the curves drawn using Equation (26) with different $N/|f|$ account for a considerable part of the spread in empirical points. The Zilitinkevich (1972) formula, Equation (1c), is also shown for comparison. Its overall agreement with observational data is about the same as that of Equation (26) at $N = 0$ (right solid curve). Note that both formulae prove to be somewhat inconsistent with the cloud of points to the right of the curves.

Figure 2 illustrates the effect of static stability aloft. Most data points lie to the right of the line corresponding to the inversion-capped SBL ($B_s = 0$ and $f = 0$). This suggests that the SBL was affected by the surface flux and rotation (the latter effect is relatively small when the static stability is strong). Equation (26) with non-zero terms that account for the effects of the surface buoyancy flux and rotation is able to cover the bulk of the area filled with empirical points.

Figure 3a shows the SBL depth h made dimensionless with the Obukhov length L as a function of the composite stability parameter $Lhf^2/(C_nu_*)^2 + LN/C_iu_*$.

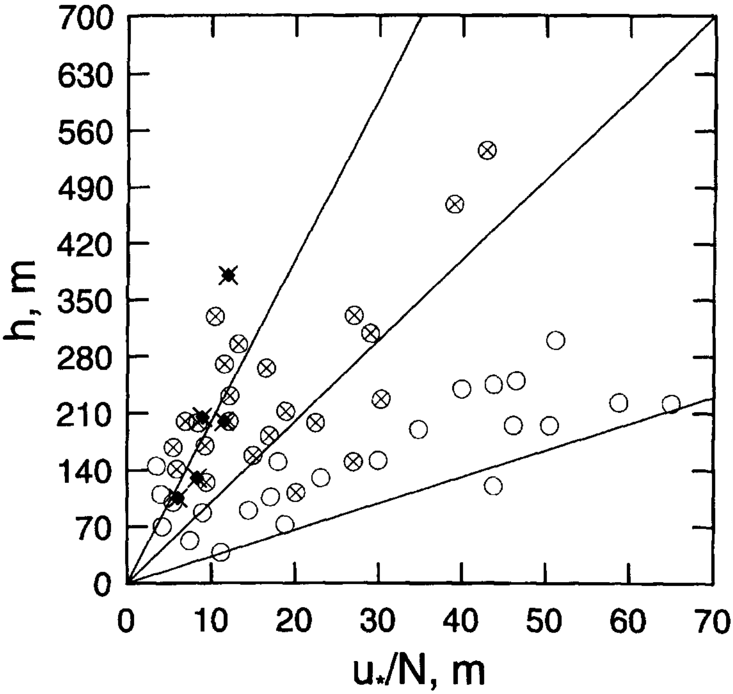

Figure 2. Depth of the inversion-capped SBL versus scaling depth u_*/N. Diamond crosses depict the data from Overland and Davidson (1992). Circled crosses and open circles are the Wangara data (Clarke *et al.*, 1971) and the ICE-77 data (Joffre, 1981), respectively, taken from Kitaigorodskii and Joffre (1988). Lines correspond to Equation (26), with $C_n = 0.5$, $C_s = 10$ and $c_i = 20$ at three different values of the stability parameter $C_i u_*/(C_s LN) + C_i hf^2/(C_n^2 N u_*)$: 0 (left line), 1 (middle) and 5 (right).

The LES data of Brown *et al.* (1994) represented by the group of circled crosses close to the ordinate correspond to the rotating SBL affected by the surface cooling. The numerically simulated SBL of Andrén (1995), as well as atmospheric boundary layers measured by Lenschow *et al.* (1988a, 1988b), correspond to the general case of a rotating SBL affected by both surface buoyancy flux and static stability aloft. The point from Weatherly and Martin (1978) shows the inversion-capped benthic boundary layer. The buoyancy flux through the bottom was not given by Weatherly and Martin (1978) but was presumably very small. Therefore, in order to show this data point together with the others, we use a very large value of the Monin–Obukhov length, thus eliminating the dependence of h on L [when L is large it drops out from Equation (26), and the position of this data point relative to the theoretical curve is independent of L]. Although the SBLs depicted in Figure 3a are quite different in nature, comparison between data and prediction from Equation (26) is obviously favourable. The Zilitinkevich (1972) formula, Equation (1c), is shown for comparison with the dashed line. In the intermediate regime, when the SBL is affected by both rotation and surface cooling, Equations (1c) and (26) give

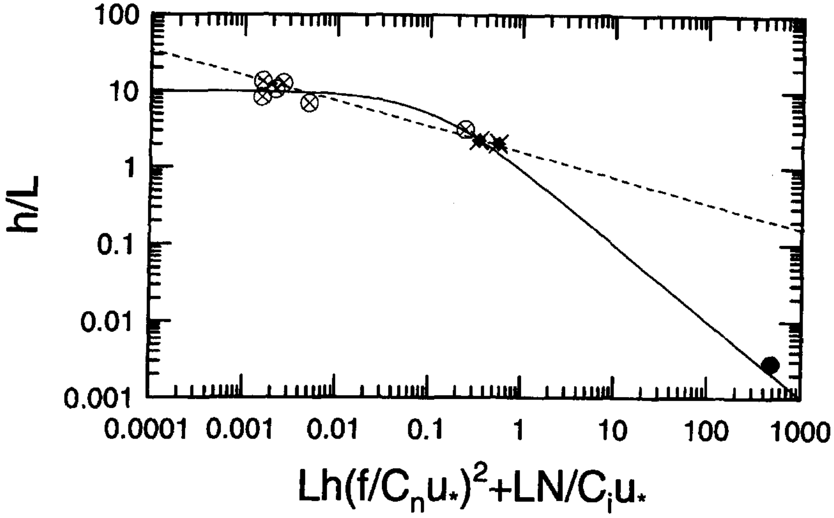

Figure 3a. Dimensionless SBL depth, h/L, as function of the composite stability parameter $Lh(f/C_n u_*)^2 + LN/C_i u_*$. Solid curve is obtained from Equation (26) with $C_n = 0.5$, $C_s = 10$ and $C_i = 20$. Dashed line corresponds to the Zilitinkevich (1972) formula, Equation (1c), with $C_{sr} = 1$. Circled crosses are the LES data of Brown *et al.* (1994) and Andrén (1995), diamond crosses are data from measurements in the atmospheric SBL taken from Lenschow *et al.* (1988a, 1988b), filled circle is the height of benthic boundary layer from Weatherly and Martin (1978).

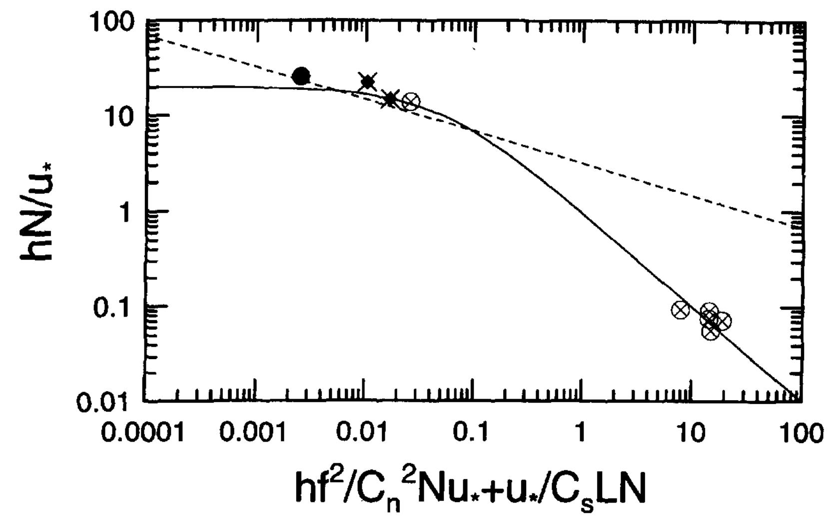

Figure 3b. Dimensionless SBL depth, hN/u_*, as a function of the composite stability parameter $hf^2/C_n^2 Nu_* + u_*/C_s LN$. Solid curve is obtained from Equation (26) with $C_n = 0.5$, $C_s = 10$ and $C_i = 20$. Dashed line corresponds to the Pollard *et al.* (1973) formula, Equation (1d), with $C_{ir} = 1.7$. Symbols are the same as in Figure 3a.

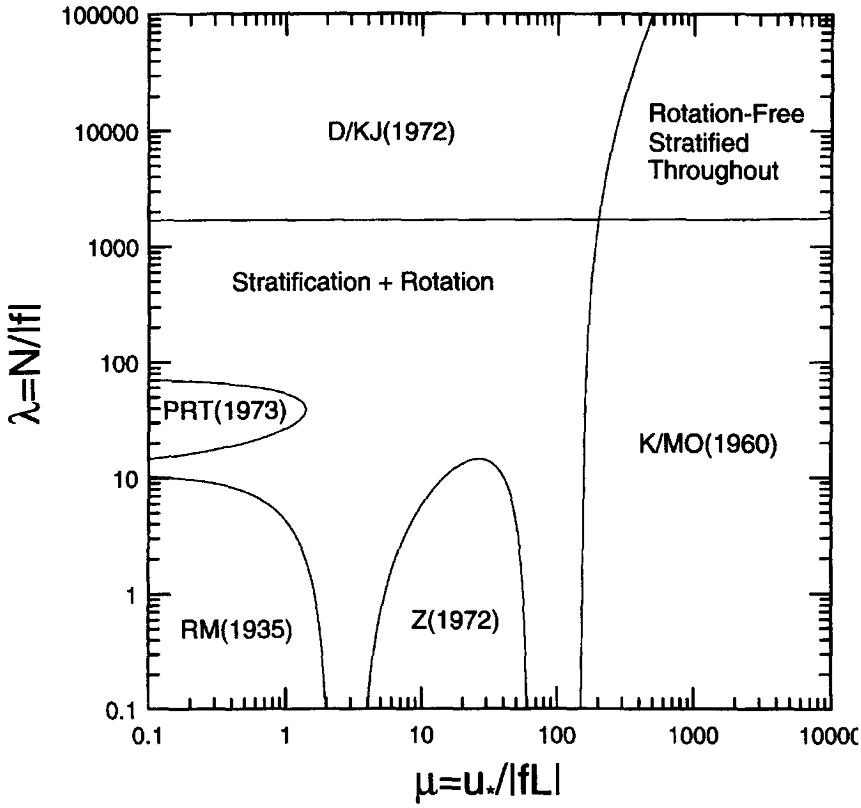

Figure 4. The SBL regime diagram. The regions of applicability of the asymptotic formulae, Equations (1a)–(1e), are marked with the abbreviations that refer to the authors of these formulae and are composed out of the first letters of the authors' names: **RM** is Rossby and Montgomery (1935), **Z** is Zilitinkevich (1972), **PRT** is Pollard, Rhines and Thompson (1973), **K/MO** is due to Kitaigorodskii (1960) and MO emphasises the fact that the depth of the surface-flux-dominated SBL scales on the Obukhov length, and **D/KJ** is due to Deardorff (1972) and KJ emphasises the contribution of Kitaigorodskii and Joffre (1988).

close results. In the limiting cases, the SBL depth computed from Equation (1c) is strongly overestimated, and it is infinite close to the equator where the Coriolis parameter tends to zero. For the stably stratified equatorial boundary layer, Equation (26) predicts finite depth. In case the equatorial SBL is strongly dominated by the surface buoyancy flux Equation (26) reduces to Equation (1b). Some experimental evidence for Equation (1b) as applied to low latitudes is provided by Garwood *et al.* (1985a, 1985b).

In Figure 3b the same data as in Figure 3a are presented in terms of the SBL depth h made dimensionless with the length scale u_*/N appropriate for the inversion-capped SBL, and the composite stability parameter $hf^2/C_n^2 N u_* + u_*/C_s LN$. The group of circled crosses close to the abscissa represent the LES data for the SBL with no imposed stratification. These data are shown by using a very large value of u_*/N, thus eliminating the dependence of h on N [when N is small it drops out from Equation (26), and the position of data points relative to the theoretical curve is independent of N]. The Pollard *et al.* (1973) formula, Equation (1d), is shown by the dashed line. Again, the limiting asymptotic regimes are not described by this formula, while its performance in the intermediate regime is very close to that of Equation (26). Thus, for practical purposes, Equation (26) seems to be a sufficient approximation of the SBL depth. It would be advantageous to further test Equation (26) against data from SBLs with different rotation rates (e.g. equatorial and laboratory boundary layers) and with different stability aloft.

Finally, Figure 4 presents the SBL regime in terms of the two stability parameters, where $\mu = u_*/|fL|$ accounts for the effect of the surface-flux-induced static stability relative to the effect of rotation and $\lambda = N/|f|$ accounts for the effects of imposed static stability relative to the effect of rotation. The diagram shows the regions of applicability of the asymptotic formulae, Equations (1a)–(1e). Each bounding curve corresponds to 55% contribution to h, Equation (30), from the individual depth scale given by one of the asymptotic formulae, Equations (1a)–(1e).

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References

André, J. C. and Mahrt, L.: 1982, 'The Nocturnal Surface Inversion and Influence of Clear-Air Radiative Cooling', *J. Atmos. Sci.* **39**, 864–878.

Andrén, A.: 1995, 'The Structure of Stably Stratified Atmospheric Boundary Layers. A Large-Eddy Simulation Study', *Quart. J. Roy. Meteorol. Soc.* **121**, 961–985.

Andrén, A. and Moeng, C.-H.: 1993, 'Single-Point Closures in a Neutrally Stratified Boundary Layer', *J. Atmos. Sci.* **50**, 3366–3379.

Brost, R. A. and Wyngaard, J. C.: 1978, 'A Model Study of the Stably Stratified Planetary Boundary Layer', *J. Atmos. Sci.* **35**, 1427–1440.

Brown, A. R., Derbyshire, S. H., and Mason, P. J.: 1994, 'Large-Eddy Simulation of Stable Atmospheric Boundary Layers with a Revised Stochastic Subgrid Model', *Quart. J. Roy. Meteorol. Soc.* **120**, 1485–1512.

Businger, J. A. and Arya, S. P. S.: 1974, 'Heights of The Mixed Layer in the Stably Stratified Planetary Boundary Layer', *Advances in Geophysics*, Vol. 18A, Academic Press, pp. 73–92.

Byun, D. W.: 1991, 'Determination of Similarity Functions of the Resistance Laws for the Planetary Boundary Layer Using Surface-Layer Similarity Functions', *Boundary-Layer Meteorol.* **57**, 17–48.

Caldwell, D. R., Van Atta, C. W., and Helland, K. N.: 1972, 'A Laboratory Study of the Turbulent Ekman Layers', *Geophys. Fluid Dyn.* **3**, 125–160.

Carruthers, D. J. and Hunt, J. C. R.: 1986, 'Velocity Fluctuations Near an Interface between a Turbulent Region and a Stably Stratified Layer', *J. Fluid Mech.* **165**, 475–501.

Caughey, S. J.: 1982, 'Observed Characteristics of the Atmospheric Boundary Layer', in F. T. M. Nieuwstadt and H. van Dop (eds.), *Atmospheric Turbulence and Air Pollution Modelling*, D. Reidel, Dordrecht, pp. 107–158.

Caughey, S. J., Wyngaard, J. C., and Kaimal, J. C.: 1979, 'Turbulence in the Evolving Stable Layer', *J. Atmos. Sci.* **36**, 1041–1052.

Clarke, R. H., Dyer, A. J., Brook, R. R., Reid, D. G., and Troup, A. J.: 1971, *The Wangara Experiment: Boundary Layer Data*. Techn. Pap. No. 19, CSIRO, Division of Meteorological Physics, Aspendale, Australia. 362 pp.

Deardorff, J. W.: 1972, 'Parameterization of the Planetary Boundary Layer for Use in General Circulation Models, *Mon. Wea. Rev.* **100**, 93–106.

Derbyshire, S. H.: 1990, 'Nieuwstadt's Stable Boundary Layer Revisited', *Quart. J. Roy. Meteorol. Soc.* **116**, 127–158.

Derbyshire, S. H.: 1995a, 'Stable Boundary Layers: Observations, Models and Variability. Part I: Modelling and Measurements', *Boundary-Layer Meteorol.* **74**, 19–54.

Derbyshire, S. H.: 1995b, 'Stable Boundary Layers: Observations, Models and Variability. Part II: Data Analysis and Averaging Effects', *Boundary-Layer Meteorol.* **75**, 1–24.

Felzenbaum, A. I.: 1980, 'Similarity Theory for the Oceanic Upper Layer', *Dokl. AN SSSR* **255**, 552–556.

Finnigan, J. J., Einaudi, F., and Fua, D.: 1984, 'The Interaction between an Internal Gravity Wave and Turbulence in the Stably-Stratified Nocturnal Boundary Layer', *J. Atmos. Sci.* **41**, 2409–2436.

Garratt, J. R.: 1982, 'Observations in the Nocturnal Boundary Layer', *Boundary-Layer Meteorol.* **22**, 21–48.

Garratt, J. R. and Brost, R. A.: 1981, 'Radiative Cooling Effects within and above the Nocturnal Boundary Layer', *J. Atmos. Sci.* **38**, 2730–2746.

Garwood, R. W.: 1977, 'An Oceanic Mixed Layer Model Capable of Simulating Cyclic States', *J. Phys. Oceanogr.* **7**, 455–468.

Garwood, R. W., Gallacher, P. C., and Muller, P.: 1985, 'Wind Direction and Equilibrium Mixed Layer Depth: General Theory', *J. Phys. Oceanogr.* **15**, 1325–1331.

Garwood, R. W., Muller, P., and Gallacher, P. C.: 1985, 'Wind Direction and Equilibrium Mixed Layer Depth in the Tropical Pacific Ocean', *J. Phys. Oceanogr.* **15**, 1332–1338.

Gill, A. E.: 1967, *The Turbulent Ekman Layer*, Preprint, Dept. Appl. Math. Theoret. Phys., Cambridge University.

Grant, A. L. M.: 1994, 'Wind Profiles in the Stable Boundary Layer, and the Effect of Low Relief', *Quart. J. Roy. Meteorol. Soc.* **120**, 27–46.

Hinze, J. O.: 1959, *Turbulence*, McGraw Hill Book Co., 517 pp.

Högström, U.: 1988, 'Non-Dimensional Wind and Temperature Profiles in the Atmospheric Surface Layer: A Re-Evaluation', *Boundary-Layer Meteorol.* **42**, 55–78.

Joffre, S. M.: 1981, *The Physics of the Mechanically Driven Atmospheric Boundary Layer as an Example of Air-Sea Ice Interactions*, Report No. 20, Dept. Meteorology, University of Helsinki, 75 pp.

Kantha, L. H.: 1977, 'Note on the Role of Internal Waves in Thermocline Erosion', in E. B. Kraus (ed.), *Modelling and Prediction of the Upper Layers of the Ocean*, Pergamon Press, Oxford, pp. 173–177.

Kazanski, A. B., and Monin, A. S.: 1960, 'On Turbulent Regime above the Atmospheric Surface Layer', *Izv. AN SSSR. Ser. geofiz.*, No. 1, 165–168.

Kitaigorodskii, S. A.: 1960, 'On the Computation of the Thickness of the Wind-Mixing Layer in the Ocean', *Izv. AN SSSR. Ser. geofiz.*, No. 3, 425–431.

Kitaigorodskii, S. A.: 1970, *The Physics of Air-Sea Interaction*, Gidrometeoizdat, Leningrad, 284 pp. (In Russian. English translation: Israel Progr. Scient. Translation, Jerusalem, 1973, 236 pp.)
Kitaigorodskii, S. A.: 1988, 'A Note on Similarity Theory for Atmospheric Boundary Layers in the Presence of Background Stable Stratification', *Tellus* **40A**, 434–438.
Kitaigorodskii, S. A.: 1990, 'A Note on the Variability of the Heights of Tidal Benthic Boundary Layers', *Geophysica* **26**, 37–44.
Kitaigorodskii, S. A.: 1992, 'The Location of Thermal Shelf Fronts and the Variability of the Heights of Tidal Benthic Boundary Layers', *Tellus* **44A**, 425–433.
Kitaigorodskii, S. A. and Joffre, S. M.: 1988, 'In Search of Simple Scaling for the Heights of the Stratified Atmospheric Boundary Layer', *Tellus* **40A**, 419–433.
Kraus, E. B. and Turner, J. C.: 1967, 'A One-Dimensional Model of the Seasonal Thermocline: Part II. The General Theory and its Consequences', *Tellus* **19**, 98–106.
Lenschow, D. H., Li, X. S., Zhu, C. J., and Stankov, B. B.: 1988a, 'The Stably Stratified Boundary Layer over the Great Plains. I. Mean and Turbulence Structure', *Boundary-Layer Meteorol.* **42**, 95–121.
Lenschow, D. H., Zhang, S. F., and Stankov, B. B.: 1988b, 'The Stably Stratified Boundary Layer over the Great Plains. II. Horizontal Variations and Spectra', *Boundary-Layer Meteorol.* **42**, 123–135.
Long, R. R.: 1974, 'Mean Stresses and Velocities in the Neutral, Barotropic Planetary Boundary Layer', *Boundary-Layer Meteorol.* **7**, 475–487.
Mahrt, L.: 1981, 'Modelling the Depth of the Stable Boundary-Layer', *Boundary-Layer Meteorol.* **21**, 3–19.
Mahrt, L., Heald, R. C., Lenschow, D. H., Stankov, B. B., and Troen, I.: 1979, 'An Observational Study of the Structure of the Nocturnal Boundary Layer', *Boundary-Layer Meteorol.* **17**, 247–264.
Mason, P. J. and Derbyshire, S. H.: 1990, 'Large-Eddy Simulation of the Stably Stratified Atmospheric Boundary Layer', *Boundary-Layer Meteorol.* **53**, 117–162.
Mason, P. J. and Thomson, D. J.: 1987, 'Large-Eddy Simulations of the Neutral-Static-Stability Planetary Boundary Layer', *Quart. J. Roy. Meteorol. Soc.* **113**, 413–443.
Melgarejo, J. W. and Deardorff, J. W.: 1974, 'Stability Functions for the Boundary Layer Resistance Laws Based upon Observed Boundary Layer Heights', *J. Atmos. Sci.* **31**, 1324–1333.
Melgarejo, J. W. and Deardorff, J. W.: 1975, 'Revision to "Stability Functions for the Boundary Layer Resistance Laws Based upon Observed Boundary Layer Heights" ', *J. Atmos. Sci.* **32**, 837–839.
Monin, A. S. and Obukhov, A. M.: 1954, 'Basic Laws of Turbulent Mixing in the Atmospheric Surface Layer', *Trudy Geofiz. Inst. Akad. Nauk SSSR*, **24**(151), 163–187.
Nappo, C. J.: 1991, 'Sporadic Breakdowns of Stability in the PBL over Simple and Complex Terrain', *Boundary-Layer Meteorol.* **54**, 69–75.
Nappo, C. J. and Eckman, R. M.: 1995, 'Breakdowns of the Nighttime Planetary Boundary Layer over Complex Terrain', *21st NATO/CCMS International Technical meeting on Air Pollution Modelling and its Application*, November 6–10, 1995, Baltimore, MD, USA, AMS, pp. 291–298.
Nieuwstadt, F. T. M.: 1981, 'The Steady-State Height and Resistance Laws of the Nocturnal Boundary Layer: Theory Compared with Cabauw Observations', *Boundary-Layer Meteorol.* **20**, 3–17.
Nieuwstadt, F. T. M.: 1984, 'The Turbulent Structure of the Stable, Nocturnal Boundary Layer', *J. Atmos. Sci.* **41**, 2202–2216.
Nieuwstadt, F. T. M.: 1985, 'A Model for the Stationary, Stable Boundary Layer', in J. C. R. Hunt (ed.), *Turbulence and Diffusion in Stable Environment*, Claredon Press, Oxford, pp. 149–179.
Nieuwstadt, F. T. M. and Tennekes, H.: 1981, 'A Rate Equation for the Nocturnal Boundary-Layer Height', *J. Atmos. Sci.* **38**, 1418–1428.
Niiler, P. P.: 1975, 'Deepening of the Wind-Mixed Layer', *J. Marine Res.* **33**, 405–422.
Niiler, P. P. and Kraus, E. B.: 1977, 'One-Dimensional Models of the Upper Ocean', in E. B. Kraus (ed.), *Modelling and Prediction of the Upper Layers of the Ocean*, Pergamon Press, Oxford, pp. 143–172.
Overland, J. E. and Davidson, K. L.: 1992, 'Geostrophic Drag Coefficient over Sea Ice', *Tellus* **44A**, 54–66.
Phillips, O. M.: 1977, 'Entrainment', in E. B. Kraus (ed.), *Modelling and Prediction of the Upper Layers of the Ocean*, Pergamon Press, Oxford, pp. 92–101.

Pollard, R. T., Rhines, P. B. and Thompson, R. O. R. Y.: 1973, 'The Deepening of the Wind-Mixed Layer', *Geophys. Fluid Dyn.* **3**, 381–404.

Rahm, L. and Svensson, U.: 1989, 'Dispersion in a Stratified Benthic Boundary Layer', *Tellus* **41A**, 148–161.

Resnyansky, Yu. D.: 1975, 'On the Parameterization of the Integral Turbulent Energy Dissipation in the Upper Ocean Quasi-Homogeneous Layer', *Izv. AN SSSR. Fizika Atmosfery i Okeana* **11**, 726–733.

Richards, K. J.: 1982, 'Modeling the Benthic Boundary Layer', *J. Phys. Oceanogr.* **12**, 428–439.

Rossby, C. G. and Montgomery, R. B.: 1935, 'The Layer of Frictional Influence in Wind and Ocean Currents', *Pap. Phys. Oceanogr. Meteorol.* **3**(3), 1–101. (M.I.T. and Woods Hole Oceanogr. Inst.)

Stigebrandt, A.: 1985, 'A Model of the Seasonal Pycnocline in Rotating Systems with Application to the Baltic Proper', *J. Phys. Oceanogr.* **15**, 1392–1404.

Tennekes, H.: 1982, 'Similarity Relations, Scaling Laws and Spectral Dynamics', in F. T. M. Nieuwstadt and H. van Dop (eds.), *Atmospheric Turbulence and Air Pollution Modelling*, D. Reidel, Dordrecht, pp. 37–68.

Thorpe, S. A.: 1973, 'Turbulence in Stably Stratified Fluids: A Review of Laboratory Experiments', *Boundary-Layer Meteorol.* **5**, 95–119.

Townsend, A. A.: 1966, 'Internal Waves Produced by a Convective Layer', *J. Fluid Mech.* **24**, 307–319.

Weatherly, G. L. and Martin, P. J.: 1978, 'On the Structure and Dynamics of the Oceanic Bottom Boundary Layer', *J. Phys. Oceanogr.* **8**, 557–570.

Wyngaard, J. C.: 1983, 'Lectures on the Planetary Boundary Layer', in D. K. Lilly and T. Gal-Chen (eds.), *Mesoscale Meteorology – Theories, Observations and Models*, NATO ASI Series, D. Reidel, Dordrecht, pp. 603–650.

Wyngaard, J. C.: 1988, 'Structure of the PBL', in A. Venkatram and J. C. Wyngaard (eds.), *Lectures on Air Pollution Modeling*, Amer. Meteor. Soc., Boston, pp. 9–61.

Wyngaard, J. C.: 1992, 'Atmospheric Turbulence', *Annu. Rev. Fluid Mech.* **24**, 205–233.

Yamada, T.: 1976, 'On the Similarity Functions *A*, *B*, and *C* of the Planetary Boundary Layer', *J. Atmos. Sci.* **33**, 781–793.

Yu, T.-W.: 1978, 'Determining the Height of the Nocturnal Boundary Layer', *J. Appl. Meteorol.* **17**, 28–33.

Zilitinkevich, S. S.: 1972, 'On the Determination of the Height of the Ekman Boundary Layer', *Boundary-Layer Meteorol.* **3**, 141–145.

Zilitinkevich, S. S.: 1975, 'Resistance Laws and Prediction Equation for the Depth of the Planetary Boundary Layer', *J. Atmos. Sci.* **32**, 741–752.

Zilitinkevich, S. S.: 1989, 'Velocity Profiles, Resistance Law and Dissipation Rate of Mean Flow Kinetic Energy in a Neutrally and Stably Stratified Planetary Boundary Layer', *Boundary-Layer Meteorol.* **46**, 367–387.

Zilitinkevich, S. S. and Mironov, D. V.: 1992, 'Theoretical Model of the Thermocline in a Freshwater Basin', *J. Phys. Oceanogr.* **22**, 988–996.