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A unified linear theory of wavelike perturbations under general ocean conditions

Yeli Yuan[a], Lei Han[a,b,*], Fangli Qiao[a], Yongzeng Yang[a], Ming Lu[a]

[a] First Institute of Oceanography, State Oceanic Administration, Qingdao, China
[b] Institute of Oceanography, China Academy of Science, Qingdao, China

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ABSTRACT

A linearized instability analysis model with five unknowns was proposed to describe disturbance motions under general oceanic background conditions, including large-scale current shear, density stratification, frontal zone, and arbitrary topography. A unified linear theory of wavelike perturbations for surface gravity waves, internal gravity waves and inertial gravity waves was derived for the adiabatic case, and the solution was then found using Fourier integrals. In this theory, we discarded the assumptions widely accepted in the literature concerning derivations of wave motions such as the irrotationality assumption for surface gravity waves, the rigid-lid approximation for internal gravity waves, and the long-wave approximation for inertial gravity waves. Analytical solutions based on this theory indicate that the complex dispersion relationships between frequency and wave-number describing the propagation and development of the three types of wavelike perturbation motions include three components: complex dispersion relationships at the sea surface; vertical invariance of the complex frequency; and expressions of the vertical wave-number (phase). Classical results of both surface waves and internal waves were reproduced from the unified theory under idealized conditions. The unified wave theory can be applied in the dynamical explanation of the generation and propagation properties of internal waves that are visible in the satellite SAR images in the southern part of the China Seas. It can also serve as the theoretical basis for both a numerical internal-wave model and analytical estimation of the ocean fluxes transported by wavelike perturbations.

* Corresponding author at: First Institute of Oceanography, State Oceanic Administration, Room 611, 6 XianXiaLing Road, Qingdao, China.
E-mail address: hanlei@fio.org.cn (L. Han).

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1. Introduction

Gravity waves in the ocean can be generally categorized into three types differentiated by temporal and spatial scales: small scale surface gravity waves, sub-mesoscale internal gravity waves, and large-scale inertial gravity waves. In the existing theories, these three kinds of oceanic gravity waves have been studied independently under different assumptions or approximations such as the irrotationality assumption for surface gravity waves, the rigid-lid approximation for internal gravity waves, and the long-wave approximation for the inertial gravity waves (Phillips, 1966; Pedlosky, 2003). However, for all three kinds of wave motions gravity is the common restoring force, and as a result, they are supposed to follow the same governing equations. This common characteristic impels us to develop a unified wave theory. Since their scales are significantly smaller than those of the ocean circulation, gravity waves behave like perturbations relative to large-scale motions or oceanic background. Therefore, in this article we treat gravity waves as perturbations, and study their influence in the presence of the factors imposed by the large-scale circulation. To this end, we propose a unified theory of wavelike perturbations.

One application example of the unified theory is the dynamical explanation of oceanic wave phenomenon from the observations. Fig. 1 reveals the internal-wave distribution and propagation in the

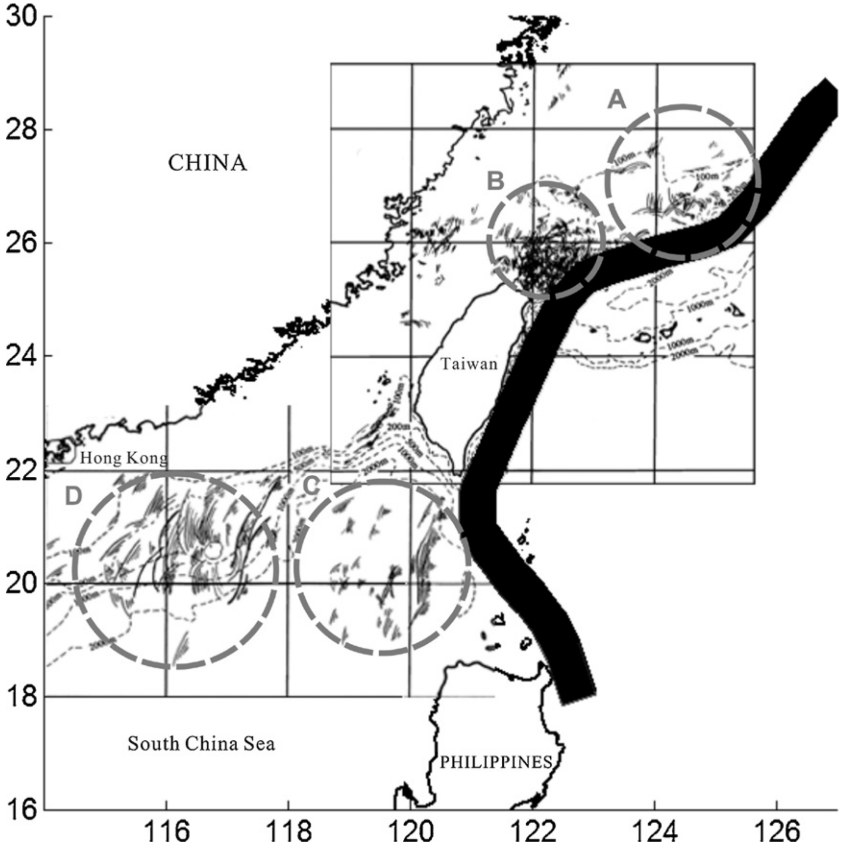

Fig. 1. The internal-wave distribution in the northern part of the China Sea. Groups of the thin solid lines represent the signature of the internal gravity waves observed in satellite SAR images by Hsu et al. (2000). Thin dashed lines depict the bottom topography. The thick solid curve represents the flow path of Kuroshio defined by the trajectories of the Argos floats (Hu et al., 2008). The dashed circles marked with A, B, C and D denote the typical regions of generation and growth of internal gravity waves, which are discussed in detail in Section 5.

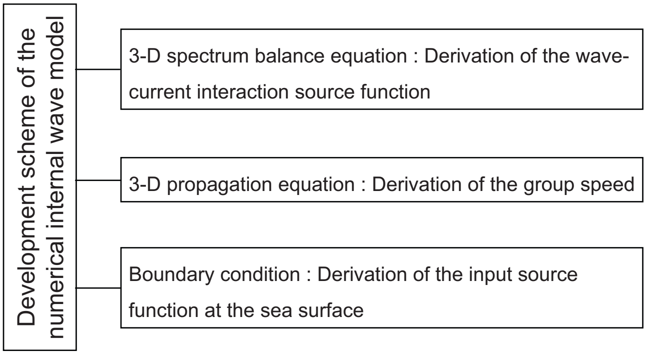

Fig. 2. A schematic diagram illustrating the crucial elements in a numerical internal-wave model aimed at computing the spectrum in three dimensions. The three components indicate the necessity of a unified wave theory for developing a numerical model.

southern part of the China Sea interpreted from synthetic aperture radar (SAR) images (Liu et al., 1998; Hsu et al., 2000). Also shown in Fig. 1 is the flow path of the Kuroshio with the thick solid curve defined by the trajectories of 323 Argos satellite-tracked drifters (Hu et al., 2008, their Fig. 5). It can be seen in Fig. 1 that the internal-wave distributions are related with many large-scale factors: the horizontal current shear, the meandering flow path, frontal zones, bathymetric features and so on. A dynamical explanation of the internal-wave characteristics exposed by the SAR images requires a wave theory based on a general background ocean (Yuan et al., 2003). Another possible application lies in the analytical estimation of the ocean fluxes transported by the wavelike perturbations. Surface wave stirring is increasingly regarded as an important mechanism of the vertical fluxes in the upper layer ocean (Babanin, 2006). An ocean flux model based on the third-generation numerical wave model and the Prandtl mixing-length theory was incorporated in a circulation model and significantly improved the model circulation (Qiao et al., 2004). Similarly, further development of circulation models requires the maturity of numerical internal-wave models and analytical estimates of the ocean fluxes transported by internal waves, as well as the development of numerical internal-wave models based on a unified wave theory as a physical basis. Fig. 2 outlines the necessity of a unified theory in constructing a numerical internal-wave model aimed at computing the three-dimensional spectrum.

An instability analysis of the wavelike perturbations under a general oceanic background is developed in Section 2, in which the model equations are presented in both physical and phase space. In Section 3, wavelike perturbations are divided into three kinds of wave motions: surface gravity waves, internal gravity waves, and inertial gravity waves according to the frequency partitions determined by vertical wave properties. In Section 4, expressions of both wavelike perturbations in the form of the Fourier integral and the complex frequency–wavenumber relations are derived using similar procedures for each kind of wave motion. In Section 5, some classical results of the surface gravity waves, the internal gravity waves and the Poincaré waves are reproduced by reducing the unified wave theory to its simplest forms under idealized conditions. Potential applications of the unified theory are demonstrated in the dynamic explanation of typical wave phenomena, the analytical estimation of the ocean fluxes transported by the surface wave stirring, and the development of the numerical internal-wave model are also presented in Section 5.

2. The instability analysis model of wavelike perturbations in a general ocean

Motivated by the dynamical explanation of the local wave characteristics, we focused on a fairly general ocean with arbitrary topography and background large-scale features such as vertical and horizontal current shear, density fronts, stratification, and a meandering flow. One such area with these typical environmental characteristics is the Kuroshio region, a vast area characterized by mesoscale eddy structures to the northeast of Taiwan, with a wide continental shelf break, and abrupt variations in topography. The characteristics of the internal gravity waves in such a region were tentatively explained with a simplified wave theory (Yuan et al., 2006). In the subsequent part of this section, a more general and complete unified wave theory is established.

2.1. Governing equations for the instability analysis model

To adapt the large-scale background motions in the general ocean, the instability analysis model adopts a local natural coordinate system. In this coordinate system, the x_2 axis is defined as the direction of large-scale mean motion, the x_3 axis points vertically upward, while the x_1 axis is determined by a right-handed orthogonal coordinate system. Thus in this coordinate frame, the adiabatic governing equations under the Boussinesq approximation can be written as (Yuan et al., 2003):

(1) The motion equations:

$$\frac{1}{R}u_1 + \frac{\partial u_\alpha}{\partial x_\alpha} + \frac{\partial u_3}{\partial x_3} = 0, \tag{1}$$

$$\frac{\partial u_1}{\partial t} + u_\alpha \frac{\partial u_1}{\partial x_\alpha} + u_3 \frac{\partial u_1}{\partial x_3} - \frac{u_2^2}{R} - fu_2 = -\frac{\partial}{\partial x_1}\left(\frac{p}{\rho_0}\right), \tag{2}$$

$$\frac{\partial u_2}{\partial t} + u_\alpha \frac{\partial u_2}{\partial x_\alpha} + u_3 \frac{\partial u_2}{\partial x_3} + \frac{u_1 u_2}{R} + fu_1 = -\frac{\partial}{\partial x_2}\left(\frac{p}{\rho_0}\right), \tag{3}$$

$$\frac{\partial u_3}{\partial t} + u_\alpha \frac{\partial u_3}{\partial x_\alpha} + u_3 \frac{\partial u_3}{\partial x_3} = -\frac{\partial}{\partial x_3}\left(\frac{p}{\rho_0}\right) - g\left(\frac{\rho}{\rho_0}\right), \tag{4}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) + u_\alpha \frac{\partial}{\partial x_\alpha}\left(\frac{\rho}{\rho_0}\right) + u_3 \frac{\partial}{\partial x_3}\left(\frac{\rho}{\rho_0}\right) = 0. \tag{5}$$

(2) The boundary conditions:

$$\left(\frac{p}{\rho_0}\right)_{x_3=\zeta} = \left(\frac{p_A}{\rho_0}\right), \tag{6}$$

$$(u_3)_{x_3=\zeta} - \left[\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x_\alpha}(u_\alpha)_{x_3=\zeta}\right] = 0, \tag{7}$$

$$(u_3)_{x_3=-H} + \frac{\partial H}{\partial x_\alpha}(u_\alpha)_{x_3=-H} = 0 \tag{8}$$

in which the indices of α are 1, 2 (representing the horizontal space). The symbols $\{u_1, u_2, u_3\}$, p and ρ represent the velocity components, pressure and density respectively; $x_3 = \zeta(x_1, x_2 ; t)$ denotes the surface elevation, $x_3 = -H(x_1, x_2)$ the bottom topography, f the Coriolis parameter, ρ_0 the basin mean water density, p_A the surface air pressure, and R the curvature radius of the mean-flow path whose sign is determined as to whether the direction of R points along or against the axis x_2.

In the local natural coordinate system, the variables (velocity components, pressure, density and surface elevation) that appear in the governing equations can be decomposed in terms of the mean motion

$$\left\{0, U_2(x_{10}, x_3), \quad 0, P(x_{10}, x_3), \quad \bar{\rho}(x_{10}, x_3), Z(x_{10})\right\}, \tag{9}$$

and perturbations

$$\left\{u_1(x_\alpha, x_3), \quad u_2(x_\alpha, x_3), \quad u_3(x_\alpha, x_3), \quad p(x_\alpha, x_3), \rho(x_\alpha, x_3), \quad h(x_\alpha)\right\}. \tag{10}$$

The variable x_{10} is defined as $x_{10} \equiv \varepsilon_1 x_1$ in which ε_1 is a non-dimensional small parameter, indicating that the spatial scale of the mean flow is much larger than that of the perturbations. The latter group of variables in (10) is in a general form that describes the three-dimensional motion in the three-dimensional space.

Substituting for the mean flow variables, Eq. (9), into Eqs. (1)–(8), and introducing the coordinate transformation $\left\{x_1' = x_1, \quad x_2' = x_2 - U_2 t, \quad x_3' = x_3\right\}$, we obtain the governing equations for the mean flow (with apostrophes omitted)

$$-\bar{F}U_2 = -\frac{\partial}{\partial x_1}\left(\frac{P}{\rho_0}\right), \tag{11}$$

$$0 = -\frac{\partial}{\partial x_3}\left(\frac{P}{\rho_0}\right) - g\left(\frac{\bar{\rho}}{\rho_0}\right), \tag{12}$$

$$\left(\frac{P}{\rho_0}\right)_{x_3=0} = \left(\frac{P_A}{\rho_0}\right) + g\left(\frac{\bar{\rho}}{\rho_0}\right)_{x_3=0} Z, \tag{13}$$

$$\frac{\partial H}{\partial x_2} = 0. \tag{14}$$

These equations indicate that the mean current flows approximately along bathymetric contours, consistent with the common understanding that the oceanic mean flow is quasi-geostrophic. In the governing equations the nominal Coriolis parameter has changed its form to include the effect of flow path curvature

$$\bar{F} \equiv \left(f + \frac{U_2}{R}\right). \tag{15}$$

Accordingly, substituting the sum of the mean and perturbation variables into Eqs. (1)–(8) and subtracting Eqs. (11)–(14) for the mean flow, we derive the governing equations for the perturbations. Under the assumption that the scale of the surface elevation of the mean flow is small, the perturbation variables are regarded as first-order small quantities, and the mean variables are zeroth-order quantities, and the governing equations for the perturbation motions can be linearized by neglecting the second- or higher-order small terms as follows:

$$\frac{u_1}{R} + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0, \tag{16}$$

$$\frac{\partial u_1}{\partial t} - F u_2 = -\frac{\partial}{\partial x_1}\left(\frac{p}{\rho_0}\right), \tag{17}$$

$$\frac{\partial u_2}{\partial t} + \left(F + \frac{\partial \bar{U}_2}{\partial x_1}\right) u_1 + \frac{\partial U_2}{\partial x_3} u_3 = -\frac{\partial}{\partial x_2}\left(\frac{p}{\rho_0}\right), \tag{18}$$

$$\frac{\partial u_3}{\partial t} = -\frac{\partial}{\partial x_3}\left(\frac{p}{\rho_0}\right) - g\left(\frac{\rho}{\rho_0}\right), \tag{19}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) + \frac{\partial}{\partial x_1}\left(\frac{\bar{\rho}}{\rho_0}\right) u_1 + \frac{\partial}{\partial x_3}\left(\frac{\bar{\rho}}{\rho_0}\right) u_3 = 0, \tag{20}$$

$$\left(\frac{p}{\rho_0}\right)_{x_3=0} = \left(\frac{p_A}{\rho_0}\right) + g\left(\frac{\bar{\rho}}{\rho_0}\right)_{x_3=0} h, \tag{21}$$

$$(u_3)_{x_3=0} - \frac{\partial h}{\partial t} = 0, \tag{22}$$

$$(u_3)_{x_3=-H} + \frac{\partial H}{\partial x_1}(u_1)_{x_3=-H} = 0 \tag{23}$$

As the result of introducing the mean-flow path curvature, the nominal Coriolis parameter and the nominal horizontal shear of the mean flow are modified as according to

$$F \equiv f + \frac{2U_2}{R}, \quad \frac{\partial \bar{U}_2}{\partial x_1} \equiv \frac{\partial U_2}{\partial x_1} - \frac{U_2}{R} \tag{24}$$

2.2. Governing equations for the wavelike perturbation motions in the phase space

In the sense of the generalized function, any arbitrary function can be written as the Fourier integral forms. Therefore, we express the perturbation variables as their Fourier integrals in the following forms:

$$u_1 = \iint_{k_1,k_2} \mu_1(x_3, k_1, k_2)\exp\{i(k_\alpha x_\alpha - \omega t)\}dk_1 dk_2, \tag{25a}$$

$$u_2 = \iint_{k_1,k_2} \mu_2(x_3, k_1, k_2)\exp\{i(k_\alpha x_\alpha - \omega t)\}dk_1 dk_2, \tag{25b}$$

$$u_3 = \iint_{k_1,k_2} \mu_3(x_3, k_1, k_2)\exp\{i(k_\alpha x_\alpha - \omega t)\}dk_1 dk_2, \tag{25c}$$

$$\left(\frac{p}{\rho_0}\right) = \iint_{k_1,k_2} \phi(x_3, k_1, k_2)\exp\{i(k_\alpha x_\alpha - \omega t)\}dk_1 dk_2, \tag{25d}$$

$$\left(\frac{\rho}{\rho_0}\right) = \iint_{k_1,k_2} \beta(x_3, k_1, k_2)\exp\left\{i(k_\alpha x_\alpha - \omega t)\right\}dk_1 dk_2, \tag{25e}$$

$$h = \iint_{k_1,k_2} \eta(k_1, k_2)\exp\{i(k_\alpha x_\alpha - \omega_0 t)\}dk_1 dk_2, \tag{25f}$$

in which $\{\mu_1, \mu_2, \mu_3, \phi, \beta, \eta\}$ are the Fourier transforms of $\{u_1, u_2, u_3, p/\rho_0, \rho/\rho_0, h\}$ respectively, and $\{k_1, k_2, \omega\}$ are the horizontal components of wave number and the complex frequency respectively. Here the real part ω_R of the complex frequency $\omega = \omega_R + i\omega_I$ is interpreted as the physical frequency. Since the vertical wavenumber is the key parameter to distinguish the different types of the wavelike perturbations, it is not explicitly included in the phase function, but incorporated with the vertical coordinate variable x_3 into the Fourier transformation functions for further special processes in the subsequent sections.

Since the spatial scale of the mean flow is much larger than that of the perturbation flow, the mean-flow related parameters are regarded as invariant when the Fourier integrals are substituted into the governing Eqs. (16)–(23). Consequently, we obtain the forms of the governing equations for the perturbation flow in the phase space as follows:

(1) Flow equations:

$$(\gamma + ik_1)\mu_1 + ik_2\mu_2 + \frac{\partial \mu_3}{\partial x_3} = 0, \tag{26}$$

$$-i\omega\mu_1 - F\mu_2 = -ik_1\phi, \tag{27}$$

$$\left(F + \frac{\partial \bar{U}_2}{\partial x_1}\right)\mu_1 - i\omega\mu_2 = -\frac{\partial U_2}{\partial x_3}\mu_3 - ik_2\phi, \tag{28}$$

$$-i\omega\mu_3 = -\frac{\partial \phi}{\partial x_3} - g\beta, \tag{29}$$

$$\beta = i\frac{1}{g\omega}(M^2\mu_1 + N^2\mu_3); \tag{30}$$

(2) Boundary conditions:

$$(\phi)_{x_3=0} = \phi_A + g\left(\frac{\bar{\rho}}{\rho_0}\right)_{x_3=0}\eta, \tag{31}$$

$$(\mu_3)_{x_3=0} = -i\omega_0\eta, \tag{32}$$

$$(\mu_3)_{x_3=-H} + \frac{\partial H}{\partial x_1}(\mu_1)_{x_3=-H} = 0, \tag{33}$$

in which

$$M^2 \equiv -g\frac{\partial}{\partial x_1}\left(\frac{\bar{\rho}}{\rho_0}\right), \quad N^2 \equiv -g\frac{\partial}{\partial x_3}\left(\frac{\bar{\rho}}{\rho_0}\right) \tag{34}$$

and $\gamma = 1/R$ denotes the curvature of the flow path.

Next we shall solve these equations for the Fourier transformation functions. From the algebraic equations (27) and (28), we obtain

$$\mu_1 = \frac{F}{\Omega^2}\frac{\partial U_2}{\partial x_3}\mu_3 + \frac{\omega k_1}{\Omega^2}I_1\phi, \tag{35}$$

$$\mu_2 = -i\frac{\omega}{\Omega^2}\frac{\partial U_2}{\partial x_3}\mu_3 - i\frac{k_1}{\Omega^2}\left(F + \frac{\partial \bar{U}_2}{\partial x_1}\right)I_2\phi, \tag{36}$$

in which

$$\Omega^2 \equiv \left[\omega^2 - F\left(F + \frac{\partial \bar{U}_2}{\partial x_1}\right)\right], \; I_1 \equiv \left(1 + i\frac{F}{\omega}\frac{k_2}{k_1}\right), \; I_2 \equiv \left[1 + i\omega\left(F + \frac{\partial \bar{U}_2}{\partial x_1}\right)^{-1}\frac{k_2}{k_1}\right] \tag{37}$$

Substituting Eqs. (35) and (36) into Eqs. (26), (29) and (30), after some manipulation, we obtain

$$k_1F\frac{\partial U_2}{\partial x_3}I_3\mu_3 + \omega k_1^2 I_4\phi - i\Omega^2\frac{\partial \mu_3}{\partial x_3} = 0, \tag{38}$$

$$(\omega^2 - (N^2)^{/})\mu_3 - \omega k_1\frac{M^2}{\Omega^2}I_1\phi + i\omega\frac{\partial \phi}{\partial x_3} = 0, \tag{39}$$

$$\beta = i\frac{1}{g\omega}\left[(N^2)^{/}\mu_3 + \omega k_1\frac{M^2}{\Omega^2}I_1\phi\right], \tag{40}$$

in which

$$(N^2)^{/} \equiv \left(N^2 + F\frac{M^2}{\Omega^2}\frac{\partial U_2}{\partial x_3}\right), \quad I_3 \equiv \left[1 - i\left(\frac{\gamma}{k_1} + \frac{\omega}{F}\frac{k_2}{k_1}\right)\right]$$
$$I_4 \equiv \left[\left(1 - i\frac{\gamma}{k_1}\right)I_1 - i\frac{1}{\omega}\left(F + \frac{\partial \bar{U}_2}{\partial x_1}\right)\frac{k_2}{k_1}I_2\right] = \left(1 - i\frac{\gamma}{k_1}\right) + \left(\frac{1}{k_1} - i\frac{1}{F}\frac{\partial \bar{U}_2}{\partial x_1}\right)\frac{F}{\omega}\left(\frac{k_2}{k_1}\right) + \left(\frac{k_2}{k_1}\right)^2 \tag{41}$$

So we have derived a system of first-order differential equations for two unknowns, μ_3 and ϕ. To obtain the boundary conditions for μ_3, we substitute Eq. (35) into the boundary condition (33) and obtain

$$(\mu_3)_{x_3=-H} + \left(1 + \frac{F}{\Omega^2}\frac{\partial U_2}{\partial x_3}\frac{\partial H}{\partial x_1}\right)^{-1}_{x_3=-H}\left(\frac{\omega k_1}{\Omega^2}\frac{\partial H}{\partial x_1}I_1\right)_{x_3=-H}(\phi)_{x_3=-H} = 0, \tag{42}$$

In order to get the homogenous boundary condition, we use the following transformation

$$\bar{\mu}_3 = \mu_3 + \delta_{-H}\phi \tag{43}$$

in which

$$\delta_{-H} \equiv \left(1 + \frac{F}{\Omega^2}\frac{\partial U_2}{\partial x_3}\frac{\partial H}{\partial x_1}\right)^{-1}_{x_3=-H}\left(\frac{\omega k_1}{\Omega^2}\frac{\partial H}{\partial x_1}I_1\right)_{x_3=-H} \tag{44}$$

Rewriting Eqs. (35), (36), (38), (39) and the boundary conditions (31)–(33) according to the variable transformation, we obtain

(1) A system of first-order differential equations for the two unknowns $(\bar{\mu}_3,\ \phi)$

$$\left[F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\bar{\mu}_3+\omega k_1(I_4)^{/}\phi-i\frac{\Omega^2}{k_1}\frac{\partial\bar{\mu}_3}{\partial x_3}=0, \tag{45}$$

$$\Omega^2(\omega^2-(N^2)^{/})\bar{\mu}_3-\omega k_1\left[M^2I_1+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\phi+i\omega\Omega^2\frac{\partial\phi}{\partial x_3}=0, \tag{46}$$

in which

$$(I_4)^{/}\equiv\left[I_4-\left(F\frac{\partial U_2}{\partial x_3}I_3-M^2I_1\right)\left(\frac{\delta_{-H}}{\omega k_1}\right)+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)^2\right] \tag{47}$$

(2) Boundary conditions for the two unknowns $(\bar{\mu}_3,\ \phi)$

$$(\phi)_{x_3=0}=g\left(\frac{\bar{\rho}}{\rho_0}\right)_{x_3=0}\eta+\phi_A, \tag{48}$$

$$(\bar{\mu}_3)_{x_3=0}=-i\varpi_0\eta+\delta_{-H}\phi_A, \tag{49}$$

$$(\bar{\mu}_3)_{x_3=-H}=0, \tag{50}$$

in which

$$\varpi_0\equiv\omega_0\left[1+i\left(\frac{\bar{\rho}}{\rho_0}\right)gk_1\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]_{x_3=0} \tag{51}$$

(3) Forms of the other variables expressed in terms of the two unknowns $(\bar{\mu}_3,\ \phi)$

$$\mu_1=\frac{F}{\Omega^2}\frac{\partial U_2}{\partial x_3}\bar{\mu}_3+\frac{\omega k_1}{\Omega^2}\left[I_1-F\frac{\partial U_2}{\partial x_3}\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\phi, \tag{52}$$

$$\mu_2=-i\frac{\omega}{\Omega^2}\frac{\partial U_2}{\partial x_3}\bar{\mu}_3-i\frac{k_1}{\Omega^2}\left[\left(F+\frac{\partial\bar{U}_2}{\partial x_1}\right)I_2-\omega^2\frac{\partial U_2}{\partial x_3}\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\phi, \tag{53}$$

$$\beta=i\frac{1}{g\omega}\left\{(N^2)^{/}\bar{\mu}_3+\frac{\omega k_1}{\Omega^2}\left[M^2I_1-\Omega^2(N^2)^{/}\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\phi\right\} \tag{54}$$

Thus the problem has been turned into one for solving for the two unknowns $\bar{\mu}_3$ and ϕ from Eqs. (45) and (46) with the boundary conditions (48)–(50), and then deriving the variables μ_1, μ_2 and β from the relations (52)–(54).

3. Classification of the gravity wave motions

3.1. The vertical component of the wave number and the phase function

Generally, the Fourier transformation functions $\bar{\mu}_3$ and ϕ can be expressed in fluctuation form as follows:

$$\bar{\mu}_3=A(\varepsilon_3x_3)\exp\left\{iX_3(x_3)\right\},\quad \phi=B(\varepsilon_3x_3)\exp\left\{iX_3(x_3)\right\}, \tag{55}$$

in which $A(\varepsilon_3x_3)$ and $B(\varepsilon_3x_3)$ denote the amplitudes, $X_3(x_3)$ the phase function of the variables or the vertical phase of u_1 and ϕ, and ε_3 is a small non-dimensional quantity reflecting the observation that the amplitudes are slow-varying in the vertical direction.

Substituting (55) into (45) and (46), we obtain the homogeneous linear equations for the amplitude functions A and B

$$\left\{k_1\left[F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]+\Omega^2\frac{\partial X_3(x_3)}{\partial x_3}\right\}A+\omega k_1^2(I_4)^{/}B=0, \tag{56}$$

$$\Omega^2(\omega^2-(N^2)^{/})A-\omega\left\{k_1\left[M^2I_1+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]+\Omega^2\frac{\partial X_3(x_3)}{\partial x_3}\right\}B=0 \tag{57}$$

With non-trivial solutions, the determinant of the coefficient matrix must be zero. As a result, we obtain the characteristic equation satisfied by the vertical component of the wave number $k_3 \equiv(\partial X_3(x_3)/\partial x_3)$ (or the vertical phase function X_3)

$$a\left[\Omega^2\frac{\partial X_3(x_3)}{\partial x_3}\right]^2+b\left[\Omega^2\frac{\partial X_3(x_3)}{\partial x_3}\right]+c=0, \tag{58}$$

in which

$$a\equiv 1,\quad b\equiv k_1\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right),\quad c\equiv k_1^2\left[FM^2\frac{\partial U_2}{\partial x_3}I_1I_3+\Omega^2(\omega^2-(N^2)^{/})I_4\right] \tag{59}$$

It is worth noting that all the three coefficients are irrelevant to the topographic factor $(\delta_{-H}/\omega k_1)$, which implies that the vertical wave number and the phase function of the perturbations are independent of the bottom topography which is reasonable.

By solving Eq. (58), we obtain expressions for the vertical wave number and phase function as follows:

$$K_3\equiv\frac{\partial X_3(x_3)}{\partial x_3}=K_{31}\pm K_{32},\quad X_3(x_3)=X_3(0)+X_{31}(x_3)\pm X_{32}(x_3), \tag{60}$$

in which

$$K_{31}=-\frac{1}{2\Omega^2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)k_1,\quad K_{32}=\frac{1}{\Omega^2}\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}k_1, \tag{61}$$

and

$$X_{31}(x_3)=-\frac{1}{2}\int_0^{x_3}\frac{1}{\Omega^2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)k_1\,dx_3,$$
$$X_{32}(x_3)=\int_0^{x_3}\frac{1}{\Omega^2}\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}k_1\,dx_3, \tag{62}$$

where

$$(N^2)^*\equiv(N^2)^{/}+\frac{1}{4\Omega^2I_4}\left(F\frac{\partial U_2}{\partial x_3}I_3-M^2I_1\right)^2 \tag{63}$$

and $X_3(0)$ represents the value of the vertical phase function at the sea surface.

3.2. Classification of the wavelike perturbations based on the vertical wave number

Clearly, the mathematical characteristics of the vertical wave number K_3 is determined by the radicand in K_{32} where the sign of the term $[\omega^2-F(F+(\partial\bar{U}_2/\partial x_1))]\cdot[(N^2)^*-\omega^2]$ naturally divides the wave frequency domain into three different parts as illustrated:

$$[Re(\omega)]_{\text{SurfaceWave}}>Re[(N^2)^*]^{1/2}>[Re(\omega)]_{\text{InternalWave}}>\left[F\left(F+\frac{\partial\bar{U}_2}{\partial x_1}\right)\right]^{1/2}$$
$$>[Re(\omega)]_{\text{InertialWave}}, \tag{64}$$

and Re denotes the real part of a complex parameter. Hence, wavelike perturbations can be defined as three types: surface gravity waves, internal gravity waves and inertial gravity waves according to the frequency partition in Eq. (64). According to the different frequency domains, the vertical wave number and the phase of three types of wavelike perturbations are expressed respectively as follows.

(1) The vertical wave number and phase of the surface gravity wave perturbations:

$$[Re(\omega)]_{\text{Surface Waves}} > Re[(N^2)^*]^{1/2} > \left[F\left(F+\frac{\partial \bar{U}_2}{\partial x_1}\right)\right]^{1/2}, \tag{65}$$

the vertical wave number and phase of the surface gravity wave perturbations can be written as

$$[K_3]_{SW} = [K_{31}]_{SW} \pm [K_{32}]_{SW}, \quad [X_3(x_3)]_{SW} = [X_3(0)]_{SW} + [X_{31}(x_3)]_{SW} \pm [X_{32}(x_3)]_{SW}, \tag{66}$$

where

$$[K_{31}]_{SW} = -\frac{1}{2}\frac{(F(\partial U_2/\partial x_3)I_3 + M^2 I_1)}{[\omega^2 - F(F+(\partial \bar{U}_2/\partial x_1))]}k_1, \quad K_{32} = i\left\{\frac{[\omega^2-(N^2)^*]}{[\omega^2 - F(F+(\partial \bar{U}_2/\partial x_1))]}I_4\right\}^{1/2} k_1, \tag{67}$$

$$\begin{aligned}[X_{31}(x_3)]_{SW} &= -\frac{1}{2}\int_0^{x_3}\frac{(F(\partial U_2/\partial x_3)I_3 + M^2 I_1)}{[\omega^2 - F(F+(\partial \bar{U}_2/\partial x_1))]}k_1\,dx_3, \\ [X_{32}(x_3)]_{SW} &= i\int_0^{x_3}\left\{\frac{[\omega^2-(N^2)^*]}{[\omega^2 - F(F+(\partial \bar{U}_2/\partial x_1))]}I_4\right\}^{1/2} k_1\,dx_3\end{aligned} \tag{68}$$

The subscript 'SW' denotes the corresponding parameters for the surface gravity wave perturbations.

(2) The vertical wave number and phase of the internal gravity wave perturbations

In the frequency domain:

$$Re[(N^2)^*]^{1/2} > [Re(\omega)]_{\text{Internal Wave}} > \left[F(F+(\partial \bar{U}_2/\partial x_1))\right]^{1/2}, \tag{69}$$

the vertical wave number and phase of the internal gravity wave motions can be written as

$$[K_3]_{IW} = [K_{31}]_{IW} \pm K_{32}, \quad [X_3(x_3)]_{IW} = [X_3(0)]_{IW} + [X_{31}(x_3)]_{IW} \pm [X_{32}(x_3)]_{IW}, \tag{70}$$

where

$$[K_{31}]_{IW} = -\frac{1}{2}\frac{(F(\partial U_2/\partial x_3)I_3 + M^2 I_1)}{[\omega^2 - F(F+(\partial \bar{U}_2/\partial x_1))]}k_1, \quad [K_{32}]_{IW} = \left\{\frac{[(N^2)^*-\omega^2]}{[\omega^2 - F(F+(\partial \bar{U}_2/\partial x_1))]}I_4\right\}^{1/2} k_1. \tag{71}$$

$$\begin{aligned}[X_{31}(x_3)]_{IW} &= -\frac{1}{2}\int_0^{x_3}\frac{(F(\partial U_2/\partial x_3)I_3 + M^2 I_1)}{\left[\omega^2 - F(F+(\partial \bar{U}_2/\partial x_1))\right]}k_1\,dx_3, \\ [X_{32}(x_3)]_{IW} &= \int_0^{x_3}\left\{\frac{\left[(N^2)^*-\omega^2\right]}{[\omega^2 - F(F+(\partial \bar{U}_2/\partial x_1))]}I_4\right\}^{1/2} k_1\,dx_3,\end{aligned} \tag{72}$$

The subscript 'IW' denotes the corresponding parameters for the internal gravity wave perturbations.

(3) The vertical wave number and phase of the inertial gravity wave perturbations

In the frequency domain:

$$Re[(N^2)^*]^{1/2} > \left[F\left(F+\frac{\partial \bar{U}_2}{\partial x_1}\right)\right]^{1/2} > [Re(\omega)]_{\text{Inertial Wave}}, \tag{73}$$

the vertical wave number and phase of the inertial gravity wave motions can be written as:

$$[K_3]_{NW} = [K_{31}]_{NW} \pm [K_{32}]_{NW}, \quad [X_3(x_3)]_{NW} = [X_3(0)]_{NW} + [X_{31}(x_3)]_{NW} \pm [X_{32}(x_3)]_{NW}, \tag{74}$$

where

$$[K_{31}]_{NW} = \frac{1}{2}\frac{(F(\partial U_2/\partial x_3)I_3 + M^2 I_1)}{[F(F+(\partial \bar{U}_2/\partial x_1)) - \omega^2]}k_1, \quad [K_{32}]_{NW} = i\left\{\frac{[(N^2)^* - \omega^2]}{[F(F+(\partial \bar{U}_2/\partial x_1)) - \omega^2]}I_4\right\}^{1/2}k_1, \quad (75)$$

$$[X_{31}(x_3)]_{NW} = \frac{1}{2}\int_0^{x_3}\frac{(F(\partial U_2/\partial x_3)I_3 + M^2 I_1)}{[F\left(F+(\partial \bar{U}_2/\partial x_1)\right) - \omega^2]}k_1 dx_3,$$

$$[X_{32}(x_3)]_{NW} = i\int_0^{x_3}\left\{\frac{[(N^2)^* - \omega^2]}{[F(F+(\partial \bar{U}_2/\partial x_1)) - \omega^2]}I_4\right\}^{1/2} k_1 dx_3 \quad (76)$$

The subscript ‘NW’ denotes the corresponding parameters for the inertial gravity wave perturbations.

However, the frequency partition is just a rough definition since frequency overlapping among various wave motions surely exists. For example, the Poincaré waves are reproduced within the frequency domain defined for internal gravity waves in Section 5.2.

4. Solution in the form of Fourier integral and the complex frequency–wavenumber relations

In this section, solutions for the five perturbation variables in the form of Fourier integrals are derived with the boundary conditions. The complex frequency–wavenumber relations are also obtained with three elements.

4.1. Solutions in the form of Fourier integral for the wavelike perturbations

Substituting the vertical wave number and phase, Eq. (60), into the first expression of (55), we obtain

$$\bar{\mu}_3 = A_1 \exp\left\{i[X_3(0) + X_{31}(x_3) + X_{32}(x_3)]\right\} + A_2 \exp\left\{i\,[X_3(0) + X_{31}(x_3) - X_{32}(x_3)]\right\}, \quad (77)$$

If neglecting the atmospheric forcing at the surface, we can obtain the coefficients in Eq. (77) by applying the boundary conditions (49) and (50)

$$A_1 = i\varpi_0\eta\frac{\exp\left\{-iX_3(0)\right\}\exp\left\{-iX_{32}(-H)\right\}}{\exp\left\{iX_{32}(-H)\right\} - \exp\left\{-iX_{32}(-H)\right\}},$$

$$A_2 = -i\varpi_0\eta\frac{\exp\left\{-iX_3(0)\right\}\exp\left\{iX_{32}(-H)\right\}}{\exp\left\{iX_{32}(-H)\right\} - \exp\left\{-iX_{32}(-H)\right\}}, \quad (78)$$

and then obtain the expression of $\bar{\mu}_3$

$$\bar{\mu}_3 = i\varpi_0\eta\,\exp\left\{iX_{31}(x_3)\right\}\frac{\sin\left\{X_{32}(x_3) - X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}} \quad (79)$$

Substituting the expression of $\bar{\mu}_3$ into Eq. (45) and employing the expression (60), we obtain the expression of ϕ

$$\phi=-\frac{\varpi_0\eta}{(I_4)^{\prime}}\exp\left\{iX_{31}(x_3)\right\}\frac{\Omega^2}{\omega k_1}\left\{\begin{aligned}&\frac{K_{32}}{k_1}\frac{\cos\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\\&+i\left[\frac{K_{31}}{k_1}+\frac{F}{\Omega^2}\frac{\partial U_2}{\partial x_3}I_3-(\omega^2-(N^2)^{\prime})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\frac{\sin\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\end{aligned}\right\} \tag{80}$$

Substituting the expressions (79) and (80) of $\bar{\mu}_3$ and ϕ into the variable transformation relation (43), we derive the expression for μ_3

$$\mu_3=i\frac{\varpi_0\eta}{(I_4)^{\prime}}\exp\left\{iX_{31}(x_3)\right\}\left\{\begin{aligned}&\left[I_4+\left(\Omega^2\frac{K_{31}}{k_1}+M^2I_1\right)\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\frac{\sin\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\\&-i\Omega^2\frac{K_{32}}{k_1}\left(\frac{\delta_{-H}}{\omega k_1}\right)\frac{\cos\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\end{aligned}\right\} \tag{81}$$

Furthermore, substituting the expressions (81) and (80) of μ_3 and ϕ into the relations (35), (36) and (40), we obtain the expressions for μ_1, μ_2 and β respectively

$$\mu_1=-\frac{\varpi_0\eta}{(I_4)^{\prime}}\exp\left\{iX_{31}(x_3)\right\}\left\{\begin{aligned}&+\frac{K_{32}}{k_1}\left[I_1-F\frac{\partial U_2}{\partial x_3}\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\frac{\cos\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\\&-i\left\{\begin{aligned}&\frac{F}{\Omega^2}\frac{\partial U_2}{\partial x_3}(I_4-I_1I_3)-\frac{K_{31}}{k_1}I_1\\&+\left[F\frac{\partial U_2}{\partial x_3}\left(\frac{K_{31}}{k_1}+\frac{M^2}{\Omega^2}I_1\right)+(\omega^2-(N^2)^{\prime})\right]\left(\frac{\delta_{-H}}{\omega k_1}\right)\end{aligned}\right\}\\&\times\frac{\sin\left\{X_{32}(x_3)-X_{32})(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\end{aligned}\right\}, \tag{82}$$

$$\mu_2=\frac{\varpi_0\eta}{(I_4)^{\prime}}\exp\left\{iX_{31}(x_3)\right\}\left\{\begin{aligned}&\left\{\begin{aligned}&\frac{\omega}{\Omega^2}\frac{\partial U_2}{\partial x_3}I_4-\left(\frac{K_{31}}{k_1}-\frac{F}{\Omega^2}\frac{\partial U_2}{\partial x_3}I_3\right)\frac{\left(F+\dfrac{\partial\bar{U}_2}{\partial x_1}\right)}{\omega}I_2\\&+\left[\left(\frac{K_{31}}{k_1}+\frac{M^2}{\Omega^2}I_1\right)\omega\frac{\partial U_2}{\partial x_3}+(\omega^2-(N^2)^{\prime})\frac{\left(F+\dfrac{\partial\bar{U}_2}{\partial x_1}\right)}{\omega}I_2\right]\left(\frac{\delta_{-H}}{\omega k_1}\right)\end{aligned}\right\}\\&\times\frac{\sin\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\\&+i\frac{K_{32}}{k_1}\left[\frac{\left(F+\dfrac{\partial\bar{U}_2}{\partial x_1}\right)}{\omega}I_2-\omega\frac{\partial U_2}{\partial x_3}\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\frac{\cos\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\end{aligned}\right\}, \tag{83}$$

$$\begin{aligned}\beta=&-\frac{\varpi_0\eta}{(I_4)^{\prime}}\exp\left\{iX_{31}(x_3)\right\}\frac{1}{g\omega}\left\{\left\{(N^2)^{\prime}I_4-M^2\left(\frac{K_{31}}{k_1}-\frac{F}{\Omega^2}\frac{\partial U_2}{\partial x_3}I_3\right)I_1+\Omega^2\left[(N^2)^{\prime}\frac{K_{31}}{k_1}+\omega^2\frac{M^2}{\Omega^2}I_1\right]\left(\frac{\delta_{-H}}{\omega k_1}\right)\right\}\right.\\&\left.\times\frac{\sin\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}+i\Omega^2\frac{K_{32}}{k_1}\left[\frac{M^2}{\Omega^2}I_1-(N^2)^{\prime}\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\frac{\cos\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\right\}\end{aligned} \tag{84}$$

Finally, by substituting the expressions (80)–(83) into each Fourier integral in Eq. (25), we obtain solutions for $\left\{u_1, u_2, u_3, (p/\rho_0), (\rho/\rho_0)\right\}$ respectively.

Here it is worth noting that the unknown variables all depend on the surface elevation, η. Since this variable is related with the wave spectrum, which can be provided by the numerical wave models, all the unknown variables, $\left\{u_1, u_2, u_3, (p/\rho_0), (\rho/\rho_0)\right\}$, can be computed numerically with the numerical wave models.

4.2. The complex frequency–wavenumber relation

The complete complex frequency–wavenumber relation derived in this section includes three ingredients: a dispersion relation at the sea surface, vertical invariance of the complex frequency, and the vertical wavenumber (or phase).

4.2.1. The complex frequency–wavenumber relation at the sea surface

Substituting (80) for ϕ into the boundary condition (48) and considering that

$$X_{31}(0) = 0, \quad X_{32}(0) = 0, \tag{85}$$

we obtain

$$\left(\frac{\bar{\rho}}{\rho_0}\right)_0 \left(\frac{\omega_0}{\varpi_0}\right)(I_4)'_0 g(-k_1) = \left\{\Omega_0^2 \frac{(K_{32})_0}{k_1} \frac{\cos\left\{X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}} - i\left[\Omega_0^2 \frac{(K_{31})_0}{k_1} + F\frac{\partial U_2}{\partial x_3} I_{30} - \Omega_0^2(\omega_0^2 - (N^2)'_0)\left(\frac{\delta_{-H}}{\omega_0 k_1}\right)\right]\right\}$$

or

$$\Omega_0^2 \frac{(K_{32})_0}{k_1} = \left\{\left(\frac{\bar{\rho}}{\rho_0}\right)_0 \left(\frac{\omega_0}{\varpi_0}\right)(I_4)'_0 g(-k_1) + i\left[\Omega_0^2 \frac{(K_{31})_0}{k_1} + F\frac{\partial U_2}{\partial x_3} I_{30} + \Omega_0^2((N^2)'_0 - \omega_0^2)\left(\frac{\delta_{-H}}{\omega_0 k_1}\right)\right]\right\} \frac{\sin\left\{X_{32}(-H)\right\}}{\cos\left\{X_{32}(-H)\right\}}, \tag{86}$$

in which the subscript '0' represents the value at the sea surface. Eq. (86) represents the sense of the complex dispersion relation at the sea surface.

4.2.2. Vertical invariance of the complex frequency

In the preceding derivations, we have implicitly assumed the vertical invariance of the complex frequency. In order to test this assumption, we have to check it with Eq. (46) which is not fully used yet. Rewriting Eq. (46) in detail, we acquire

$$\left\{\Omega^2(\omega^2 - (N^2)')\bar{\mu}_3 - \omega k_1\left[M^2 I_1 + \Omega^2(\omega^2 - (N^2)')\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\phi + i\omega\Omega^2\frac{\partial(\phi)_\omega}{\partial x_3}\right\} + i\omega\Omega^2\frac{\partial\phi}{\partial\omega}\frac{\partial\omega}{\partial x_3} = 0, \tag{87}$$

It can be proved that the sum of the terms in first braces are zero (see Appendix A). Considering $\omega\Omega^2(\partial\phi/\partial\omega) \neq 0$, we obtain

$$\frac{\partial\omega}{\partial x_3} = 0 \tag{88}$$

which implies the vertical invariance of the complex frequency.

4.2.3. Expressions of the vertical wavenumber and phase function

Since the rigid-lid assumption has been abandoned, we exclude the vertical modal decomposition, but adopt a vertical wavenumber and phase with the physical meaning of wave motions. Expressions for the vertical wavenumber and phase function have already been derived as Eqs. (61) and (62).

The dispersion relation at the sea surface (86), the vertical invariance of the complex frequency (88) and the expression of the vertical wavenumber comprise the complete frequency–wavenumber relation for the wavelike perturbation flow.

Finally, substituting the expressions of vertical wavenumber and phase function, (67) and (68) for the surface gravity waves, (71) and (72) for the internal gravity waves, (75) and (76) for the inertial gravity waves into Eqs. (80)–(84), (86), (88), (61) and (62), respectively, we can obtain expressions for the motions in the form of Fourier integrals and the frequency–wavenumber relations for the three kinds of wavelike perturbations.

5. Summary and discussion

5.1. A unified theory of wavelike perturbations was presented

In this study, we have utilized the governing equations in dynamic balance with five variables $\left\{u_1, u_2, u_3, p, \rho\right\}$ to consistently describe the surface gravity waves, the internal gravity waves and the inertial gravity waves all as perturbations. A unified wavelike perturbation theory excluding the conventional approximations was thus established.

We have derived the analytical solutions of the perturbation motions with the linear theory in a general ocean in which many background factors, such as vertical and horizontal current shear, density stratification and frontal zone, flow path curvature and bottom topography may be present. The theoretical solutions derived consist of expressions for the perturbation motions in the form of Fourier integrals and their complete complex frequency–wavenumber relations for three kinds of wave motions.

It is worthwhile to note that the inertial gravity waves studied in this article are those restored by gravity instead of the Coriolis force as the time derivative $\partial u_3/\partial t$ in the vertical motion equation is not excluded in our model, i.e. the dynamical balance is retained.

5.2. Solutions under the idealized oceanic conditions

The derived equations for the unified wave theory appear complex in form. To compare with the results of classical wave theory, we consider wavelike perturbations in idealized ocean with homogeneous terrain, where the large-scale background oceanic circulation and the horizontal gradients of water density are neglected. This condition requires that the flow path curvature and the horizontal current shear both vanish, so we obtain

$$M^2 = 0, \quad \delta_{-H} = 0, \quad \frac{\partial U_2}{\partial x_1} = 0, \quad U_2 = 0, \quad \frac{1}{R} = 0 \tag{89}$$

Since the differences in scales result in distinct characteristics of the wavelike motions, this places further limitations on the three kinds of wavelike motions respectively.

For surface gravity waves in frequency domain defined by (65), we further neglect density stratification and the Coriolis effect, and obtain

$$N^2 = 0, \quad f = 0 \tag{90}$$

Substituting (89) and (90) into (67) and (68), we obtain, after some algebraic manipulations:

$$\mu_1 = \omega\eta\left(\frac{k_1}{k_H}\right)\frac{\cos h\,[k_H(x_3+H)]}{\sin h(k_H H)}, \tag{91}$$

$$\mu_2 = \omega\eta\left(\frac{k_2}{k_H}\right)\frac{\cos h\,[k_H(x_3+k_H H)]}{\sin h(k_H H)}, \tag{92}$$

$$\mu_3 = -i\omega\eta\frac{\sin h\,[k_H(x_3+H)]}{\sin h(k_H H)} \tag{93}$$

where $k_H = \sqrt{k_1^2 + k_2^2}$ denotes the total horizontal wave number. Substituting (89)–(90) into the Fourier integral, Eq. (25), and retaining the real part, we obtain the particle velocity

$$u_1 = \iint\limits_{k_1,k_2} \omega\eta(k_1, k_2)\left(\frac{k_1}{k_H}\right)\frac{\cosh[k_H(x_3 + k_H H)]}{\sinh(k_H H)}\cos[(k_1x_1 + k_2x_2 - \omega t)]dk_1dk_2, \tag{94}$$

$$u_2 = \iint\limits_{k_1,k_2} \omega\eta(k_1, k_2)\left(\frac{k_2}{k_H}\right)\frac{\cos h[k_H(x_3 + k_H H)]}{\sin h(k_H H)}\cos[(k_1x_1 + k_2x_2 - \omega t)]dk_1dk_2, \tag{95}$$

$$u_3 = \iint\limits_{k_1,k_2} \omega\eta(k_1, k_2)\frac{\sin h[k_H(x_3 + H)]}{\sin h(k_H H)}\sin[(k_1x_1 + k_2x_2 - \omega t)]dk_1dk_2 \tag{96}$$

These expressions for the particle velocity are exactly the same as those derived by the classical linear theory of surface gravity waves (Kinsman, 1965).

After substituting Eqs. (89), (90) and Eq. (61) into the complex frequency–wavenumber relation at the sea surface, (86), we obtain

$$\omega^2 = gk_H \tan h(k_H H) \tag{97}$$

which agrees with the dispersion relation of the classical irrotational surface gravity waves of finite depth, and the vertical wave number

$$k_3 = \pm ik_H \tag{98}$$

So we obtain $\exp\left\{iX_3(x_3)\right\} = \exp(\pm k_H x_3)$, implying that surface gravity waves behave as exponential attenuation with depth in the vertical direction, and the exponential attenuation rate equals to the horizontal wave number k_H (we reject the unphysical negative sign since the vertical axis x_3 is positive upward). It also agrees with the classical linear theory of surface gravity waves.

For internal gravity waves, we discard (90) according to the frequency domain of the internal gravity wave defined in (69). If assuming horizontal propagation parallel to the x_1 axis, we obtain

$$u_1 = \int_{k_1} \omega\eta(k_1)\mu_{IW}\frac{\cos[\mu_{IW}k_1(x_3 + H)]}{\sin(\mu_{IW}k_1H)}\cos(k_1x_1 - \omega t)dk_1, \tag{99}$$

$$u_2 = \int_{k_1} f\eta(k_1)\mu_{IW}\frac{\cos[\mu_{IW}k_1(x_3 + H)]}{\sin(\mu_{IW}k_1H)}\sin(k_1x_1 - \omega t)dk_1, \tag{100}$$

$$u_3 = \int_{k_1} \omega\eta(k_1)\mu_{IW}\frac{\sin[\mu_{IW}k_1(x_3 + H)]}{\sin(\mu_{IW}k_1H)}\sin(k_1x_1 - \omega t)dk_1 \tag{101}$$

where

$$\mu_{IW} = \sqrt{\frac{N^2 - \omega^2}{\omega^2 - f^2}} \tag{102}$$

After substituting (89) and (67) into the complex frequency–wavenumber relation at the sea surface, (86), we obtain

$$\omega^2 = \frac{N^2k_H^2 + f^2k_3^2}{k_3^2 + k_H^2} \tag{103}$$

which agrees with the dispersion relation of the classical internal gravity wave theory (Gerkema and Zimmerman, 2008).

For inertial gravity wave, neither density stratification nor Coriolis effect can be neglected. Under the conditions (89) and the frequency domain defined by (73), the particle velocity and the dispersion relation can be derived similarly

$$u_1 = \int_{k_1} \omega\eta(k_1)\mu_{NW} \frac{\cos h\,[\mu_{NW}k_1(x_3+H)]}{\sin h(\mu_{NW}k_1 H)} \cos(k_1x_1-\omega t)dk_1, \tag{104}$$

$$u_2 = -\int_{k_1} f\eta(k_1)\mu_{NW} \frac{\cos h[\mu_{NW}k_1(x_3+H)]}{\sin h(\mu_{NW}k_1 H)} \cos(k_1x_1-\omega t)dk_1, \tag{105}$$

$$u_3 = \int_{k_1} \omega\eta(k_1)\mu_{NW} \frac{\sin h\,[\mu_{NW}k_1(x_3+H)]}{\sin h(\mu_{NW}k_1 H)} \sin(k_1x_1-\omega t)dk_1 \tag{106}$$

where

$$\mu_{NW} = \sqrt{\frac{N^2-\omega^2}{f^2-\omega^2}} \tag{107}$$

and the waves are also assumed to propagate along the x_1 axis. By substituting (89) and (75) into (86), the dispersion relation is given by

$$\omega^2 = f^2 - \frac{1}{\mu_{NW}} gk_H \tan h(\mu_{NW}k_H H) \tag{108}$$

which represents the dispersion relation between wave frequency and horizontal wave number. Similarly, the vertical wave number can be obtained as

$$k_3 = -i\mu_{NW}k_H \tag{109}$$

from which we obtain $\exp\left\{iX_3(x_3)\right\} = \exp(\mu_{NW}k_Hx_3)$, indicating that inertial gravity waves are exponentially attenuated in the vertical at a rate equals $\mu_{NW}k_H$. The results derived for inertial gravity waves have yet to be validated.

Actually, a more familiar form of wave motions can be recognized in the unified wave theory by imposing specific assumptions. When the wave region is bounded in the x_2 direction between $x_2 = 0$ and $x_2 = L$, and the density stratification vanishes, the vertical wavenumber in (61) can be rewritten as:

$$K_{32} = i\frac{\omega}{\sqrt{\omega^2-f^2}}k_H, \quad \text{if} \quad \omega \geq f \tag{110}$$

$$K_{32} = -\frac{\omega}{\sqrt{f^2-\omega^2}}k_H, \quad \text{if} \quad \omega < f \tag{111}$$

Focusing on the solution whose frequency is greater than the inertial frequency, we obtain a dispersion relation by substituting (110) into (86), which yields

$$\omega\sqrt{\omega^2-f^2} = gk_H \tan h\left(\frac{\omega}{\sqrt{\omega^2-f^2}}k_H H\right) \tag{112}$$

With the long-wave approximation (or shallow-water approximation), the dispersion relation becomes

$$\omega^2 - f^2 = gHk_H^2 \tag{113}$$

Since the velocity u_2 expressed in the Fourier integral in (25b) must satisfy the non-penetrating constraint at both boundaries, it can be easily proved that the wavenumber k_2 satisfies:

$$\sin(k_2L) = 0 \tag{114}$$

so that

$$k_2 = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \tag{115}$$

Combined with Eq. (113), this yields

$$\omega^2 = f^2 + gH\left(k_1^2 + \frac{n^2\pi^2}{L^2}\right), \quad n = 1, 2, 3, \ldots \tag{116}$$

which matches with the dispersion relation of Poincaré waves (Pedlosky, 1987), indicating that inertial gravity waves can also be observed in the frequency domain approximately defined for internal gravity waves.

From the applications of the unified wave theory in the idealized oceanic conditions, we could reproduce the three different kinds of oceanic wave motions by designating distinct frequency domains with the unified wave theory. The reduced theoretical results are consistent with classical wave theories.

5.3. Application in the dynamical explanation of the regional characteristics of the internal waves

Fig. 1 presents the satellite SAR images, in which four typical areas where internal gravity wave clusters are marked with symbols A–D which are explained in details below:

Region A: At the shelf break of the East China Sea and the left side of the straight section of the Kuroshio flow.
Region B: Northeast to Taiwan and at the west side of the curved section of Kuroshio flow.
Region C: At the seamount break and the west side of Kuroshio flow around the Luzon Strait.
Region D: The broad sea area with internal-wave breaking in the north part of the South China Sea and moderate hydrologic structures.

The propagation and growth characteristics shown in these areas are primarily linear and can be explained by the derived complex frequency–wavenumber relation. In our previous work (Yuan et al., 2006), we have used a simplified wavelike perturbation theory to dynamically explain the generation mechanism of internal wave by the wave–current interaction in the Luzon Strait, which indicates the importance of the unified wavelike perturbation theory in the dynamical explanation of the main regional characteristics of internal gravity waves in all the typical sea areas.

5.4. Theoretical basis of the analytical estimation of the ocean fluxes transported by the wavelike perturbations

In our previous work, we have derived the expression of the ocean flux transported by the surface gravity waves based on the semi-empirical Prandtl's mixing-length theory (Yuan et al., 1999; Qiao et al., 2004) and obtained fairly good simulations as the expressions were incorporated into the numerical ocean circulation model (Qiao et al., 2004).

Using the results obtained in this article, one could obtain a new expression of the ocean flux to a second-order approximation by applying the Reynolds averaging,

$$\left\langle u_{W3}^{S}\frac{\rho_W^S}{\rho_0}\right\rangle_{\text{Theory}} = \left\{\frac{1}{2}\iint_{k_\beta}\omega E_W(k_1,k_2)\,\Delta_\rho\left[\frac{sh\left\{\chi_{32}(x_3)-\chi_{32}(-H)\right\}}{sh\left\{\chi_{32}(-H)\right\}}\right]^2 dk_1 dk_2\right\}\frac{\partial}{\partial x_3}\left(\frac{\bar{\rho}}{\rho_0}\right) \tag{117}$$

in which u_{W3}^S and (ρ_W^S/ρ_0) denote the vertical velocity and the normalized density, and $E_W(k_1, k_2)$ the wavenumber spectrum of the surface gravity waves. The factor Δ_ρ has the form as follows:

$$\Delta_\rho \equiv \frac{M^2}{N^2}\left[\left(\frac{k_1}{k}\right)+i\frac{F}{\omega}\left(\frac{k_2}{k}\right)\right]\sqrt{\frac{(\omega^2-N^2)}{\Omega^2}}\frac{ch\left\{\chi_{32}(x_3)-\chi_{32}(-H)\right\}}{sh\left\{\chi_{32}(x_3)-\chi_{32}(-H)\right\}} \quad (118)$$

Obviously, the Reynolds averaging does not equal to zero, which is true for the irrotational wave situation. This implies that the surface gravity waves do impact the vertical ocean fluxes instead of making no contributions as stated by the classical linear wave theory with the irrotationality assumption.

However, the physical validity of the stirring effect by the surface gravity waves still needs to be examined experimentally and theoretically (Dai et al., 2010).

5.5. *Theoretical basis of numerical internal-wave model*

The mechanisms of the ocean fluxes include not only the surface waves stirring in the upper layer ocean but also the internal waves taking effect in the deeper layers. There are two aspects on this topic. One is to derive the expressions of the ocean fluxes transported by the internal-wave stirring using the Prandtl's mixing-length theory. The other is to develop the numerical internal-wave model capable of computing the three-dimensional wave spectrum.

The unified linear theory of the wavelike perturbation could provide the numerical internal-wave model with theoretical basis at least in the following two aspects.

(1) Establishing the characteristic equations

In order to establish the characteristic equations as follows:

$$\frac{\partial x_\alpha}{\partial t} = (U_\alpha + c_{g\alpha}), \quad \frac{\partial x_3}{\partial t} = c_{g3} \quad (119)$$

in which U_α, c_g represent the current velocity and group velocity. We have to derive the expression of the group velocity from the dispersion relation, i.e. the real part of the complex frequency–wavenumber relation $\omega_R \equiv Re\{\omega\}$,

$$c_{g\alpha} = \frac{\partial \omega_R}{\partial k_\alpha}, \quad c_{g3} = \frac{\partial \omega_R}{\partial k_3} = \frac{\partial \omega_R}{\partial k_\alpha}\left(\frac{\partial k_3}{\partial k_\alpha}\right)^{-1} \quad (120)$$

(2) Deriving the source functions

The exponential growth rate as the imaginary part of the complex frequency–wave number relation $\omega_I \equiv Im\{\omega\}$ could serve as the theoretical basis to derive the wave-current source function of the spectral balance equation in three dimensions. For the other source functions such as the surface input and the wave–wave interaction, some other relevant theories need to be developed.

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Appendix A.

Substituting Eqs. (79) and (80) into (46) and considering the expressions:

$$
\begin{aligned}
K_{31} &= \frac{\partial X_{31}(x_3)}{\partial x_3} = -\frac{1}{2\Omega^2}\left(F\frac{\partial U_2}{\partial x_3}I_3 + M^2 I_1\right)k_1,\\
K_{32} &= \frac{\partial X_{32}(x_3)}{\partial x_3} = \frac{1}{\Omega^2}\left[\Omega^2((N^2)^* - \omega^2)I_4\right]^{1/2}k_1,
\end{aligned}
\tag{A1}
$$

we obtain

$$
\left(
\begin{array}{l}
\left\{\Omega^2(\omega^2-(N^2)^{/})\bar{\mu}_3 - \omega k_1\left[M^2 I_1+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\phi + i\omega\Omega^2\frac{\partial(\phi)_\omega}{\partial x_3}\right\}\\
= \frac{\varpi_0\eta}{(I_4)^{/}}\exp\left\{iX_{31}(x_3)\right\}
\left\{
\begin{array}{l}
+\left\{\left[M^2I_1+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}\right.\\
\times -\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2} + \left[-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)\right.\\
\left.\left.\times + F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}\right\}\\
\frac{\cos\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}\\
+i\left\{
\begin{array}{l}
\Omega^2(\omega^2-(N^2)^{/})(I_4)^{/}\\
+\left[M^2I_1+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\left[-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)\right.\\
\left.+F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)\\
\times\left[-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)+F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\\
+\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}
\end{array}
\right\}\\
\frac{\sin\left\{X_{32}(x_3)-X_{32}(-H)\right\}}{\sin\left\{X_{32}(-H)\right\}}
\end{array}
\right\}
\end{array}
\right)
\tag{A2}
$$

Furthermore, if considering the expressions (47) and (63) we obtain

$$
\begin{aligned}
&\left\{\left[M^2I_1+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}\right.\\
&\quad\left.+\left[-\frac{1}{2}\left(F\frac{\partial_2}{\partial x_3}I_3+M^2I_1\right)+F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}\right\}\\
&=\left\{M^2I_1-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)+F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right.\\
&\quad\left.+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right\}\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}=0
\end{aligned}
,
$$

and

$$\left(\begin{array}{l} \Omega^2(\omega^2-(N^2)^{/})(I_4)^{/}+\left[M^2I_1+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\left[-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)\right. \\ \left. +F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right] \\ -\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)\left[-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)+F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right] \\ +\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2}\left[\Omega^2((N^2)^*-\omega^2)I_4\right]^{1/2} \\ =\left\{ \begin{array}{l} \Omega^2(\omega^2-(N^2)^{/})\left[I_4-\left(F\frac{\partial U_2}{\partial x_3}I_3-M^2I_1\right)\left(\frac{\delta_{-H}}{\omega k_1}\right)+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)^2\right] \\ +\left[M^2I_1+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\left[-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)\right. \\ \left. +F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right] \\ -\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)\left[-\frac{1}{2}\left(F\frac{\partial U_2}{\partial x_3}I_3+M^2I_1\right)+F\frac{\partial U_2}{\partial x_3}I_3-\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right] \\ +\Omega^2\left[-(\omega^2-(N^2)^{/})+\frac{1}{4\Omega^2I_4}\left(F\frac{\partial U_2}{\partial x_3}I_3-M^2I_1\right)^2\right]I_4 \end{array} \right\}=0 \end{array} \right)$$

Finally, we obtain

$$\left\{\Omega^2(\omega^2-(N^2)^{/})\bar{\mu}_3-\omega k_1\left[M^2I_1+\Omega^2(\omega^2-(N^2)^{/})\left(\frac{\delta_{-H}}{\omega k_1}\right)\right]\phi+i\omega\Omega^2\frac{\partial(\phi)_\omega}{\partial x_3}\right\}=0 \qquad \text{(A3)}$$

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