# PROBABILITY DISTRIBUTION OF SPECTRAL INTEGRALS

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## INTRODUCTION

The presentation of spectra obtained from time series of finite length, together with their confidence limits or at least the number of degrees of freedom of the spectral estimate is common practice. Indeed, should the confidence intervals be omitted it renders the spectrum almost useless as there is no indication of the possible statistical variability which may be present. Quantities obtained by forming the integral of the spectrum will also be only estimates of the true value and thus possess some statistical variability. A common example of this is the calculation of the significant wave height from the integral of the surface wave spectrum.

## ANALYSIS

For a stationary (ergodic) Gaussian random process $x(t)$, the variance spectrum is [Bendat and Piersol (1)]

$$G_x(f) = 2 \lim_{T\to\infty} \frac{1}{T} E\left[|X(f,T)|^2\right] \quad \text{(1)}$$

where $E[\ ]$ indicates the expectation value; $T$ is the length of the time series; and $X(f,T)$ is the Fourier transform of the time series $x(t)$. An estimate of $G_x(f)$ can be obtained by simply omitting the limiting and expectation values in Eq. 1

$$\hat{G}_x(f) = \frac{2}{T}|X(f,T)|^2 \quad \text{(2)}$$

The resulting spectral estimate $\hat{G}_x(f)$ follows a chi-square probability distribution with $n = 2$ degrees of freedom [Bendat and Piersol (1)]

$$\frac{\hat{G}_x(f)}{G_x(f)} = \frac{\chi_2^2}{2} \quad \text{(3)}$$

Since $G_x(f)$ has a chi-square distribution, $\int_0^\infty \hat{G}_x(f)\,df$ or $\Sigma_{i=1}^N \hat{G}_x(f_i)\,\Delta f$ will also be chi-square distributed [Jenkins and Watts (3)]. Jenkins and Watts (3) have shown that if a random variable $y$ has a probability distribution $a\chi_\nu^2$, where "$a$" is a constant, then

$$E[y] = a\nu \quad \text{(4)}$$

$$\text{Var}[y] = 2a^2\nu \quad \text{(5)}$$

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$$\text{and} \quad \nu = \frac{2\{E[y]\}^2}{\text{Var}[y]} \qquad (6)$$

where Var [ ] indicates the variance.

For the case of the spectral integral, Eq. 6 involves the determination of the moments of the sum of a number of random variables. These moments can be simplified to [Jenkins and Watts (3)]

$$E\left[\sum_{i=1}^{N} \hat{G}_x(f_i)\right] = \sum_{i=1}^{N} E[\hat{G}_x(f_i)] = \sum_{i=1}^{N} \hat{G}_x(f_i) \qquad (7)$$

$$\text{and} \quad \text{Var}\left[\sum_{i=1}^{N} \hat{G}_x(f_i)\right] = \sum_{i=1}^{N} \text{Var}[\hat{G}_x(f_i)] \qquad (8)$$

In the preceding analysis, it has been assumed that the spectral estimate has only two degrees of freedom. This can be increased, however, by either frequency or ensemble averaging. For the more general case where $\hat{G}_x(f_i)$ is chi-square distributed with $n$ degrees of freedom [Bendat and Piersol (1)]

$$\text{Var}[\hat{G}_x(f_i)] = \frac{2}{n}[\hat{G}_x(f_i)]^2 \qquad (9)$$

Substituting Eq. 9 into Eq. 8 gives

$$\text{Var}\left[\sum_{i=1}^{N} \hat{G}_x(f_i)\right] = \frac{2}{N}\sum_{i=1}^{N} [\hat{G}_x(f_i)]^2 \qquad (10)$$

Eqs. 6, 7 and 10 yield

$$\nu = \frac{n\left[\sum_{i=1}^{N} \hat{G}_x(f_i)\right]^2}{\sum_{i=1}^{N} [\hat{G}_x(f_i)]^2} \qquad (11)$$

From Eq. 11 it can be seen that the number of degrees of freedom of the spectral integral depends on the shape of the spectrum.

Since the spectral integral $\hat{I}$ is chi-square distributed with $\nu$ degrees of freedom it follows that the 100(1 − α)% confidence limits are

$$\text{Upper confidence limit (UCL)} = \frac{\nu\hat{I}}{\chi^2_{\nu;1-\alpha/2}} \qquad (12)$$

$$\text{Lower confidence limit (LCL)} = \frac{\nu\hat{I}}{\chi^2_{\nu;\alpha/2}} \qquad (13)$$

Eqs. 12 and 13 are valid for any variable having a chi-square distribution [Bendat and Piersol (1), Jenkins and Watts (3)]. In the present context they are often used to determine the confidence limits of individual frequency bands of the spectrum $\hat{G}_x(f)$. Since $\hat{I}$ also has a chi-square distribution they can be equally applied to the spectral integral.





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For the special case of $\hat{G}_x(f)$ = constant, Eq. 11 becomes $\nu = nN$. This is the result which is often quoted [Bendat and Piersol (1), Otnes and Enochson (4)] for the increase in the number of degrees of freedom when neighboring frequency bands are averaged to form a smoothed spectral estimate. This result is, strictly, only true if the spectral values are constant over the smoothing band. In practice, however, it is a reasonable approximation.

## Practical Application

As an example of the application of Eqs. 11, 12 and 13, consider the determination of the significant wave height from a wave record whose spectrum is of the standard JONSWAP form [Hasselmann, et al. (2)]. It is assumed that the wave record consists of 2,048 points sampled every 0.5 sec. These values give a Nyquist frequency $f_N = 1\ \text{sec}^{-1}$ and a frequency interval $\Delta f = 9.8 \times 10^{-4}\ \text{sec}^{-1}$. Four spectra are considered, each defined by the standard JONSWAP parameters; $f_m = 0.2$ Hz, $\alpha = 0.01$, $\sigma_a = 0.07$, $\sigma_b = 0.09$ and $\gamma$ values of 1.0, 3.3, 4.2 and 7.0. The four spectra are shown in Fig. 1 and the results in Table 1. Each of the spectra are assumed to be unsmoothed and hence $n = 2$.

The influence which the spectral shape has on the value of $\nu$ is clearly apparent in Table 1 with the value of $\nu$ decreasing as the spectrum be-

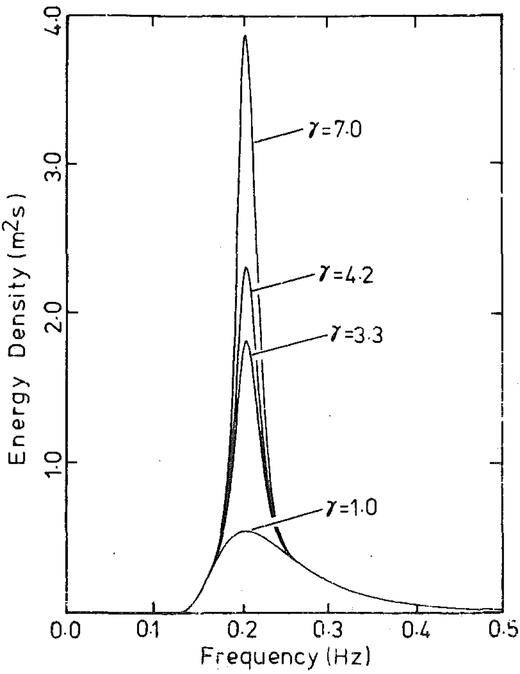


**FIG. 1.—JONSWAP Spectra with Standard Parameters $f_m = 0.2$ Hz, $\alpha = 0.01$, $\sigma_a = 0.07$, $\sigma_b = 0.09$ and Various Values of $\gamma$**

TABLE 1.—Upper and Lower Confidence Limits for Four Standard Spectral Shapes

| $\gamma$ (1) | $\nu$ (2) | $\hat{H}_s$ (m) (3) | 90% LCL as percentage of $\hat{H}_s$ (4) | 90% UCL as percentage of $\hat{H}_s$ (5) |
|---|---|---|---|---|
| 1.0 | 226 | 1.11 | 86 | 118 |
| 3.3 | 135 | 1.37 | 83 | 124 |
| 4.2 | 118 | 1.45 | 82 | 126 |
| 7.0 | 89 | 1.66 | 80 | 130 |

comes more peaked. Table 1 also shows that significant wave height estimates obtained from typical ocean wave spectra contain a potentially significant degree of statistical variability. Hence the usefulness of such estimates can be increased by the inclusion of confidence limits calculated as previously described. The results of Table 1 represent four specific cases and should not be generalized. They are included purely to demonstrate the potential error present and the influence of spectral shape.

## Appendix.—References


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