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Coastal Engineering 48 (2003) 133–135
Coastal Engineering
An International Journal for Coastal, Harbour and Offshore Engineers
www.elsevier.com/locate/coastaleng

Discussion

Discussion of "Simple and explicit solution to the wave dispersion equation" [Coastal Engineering 45 (2002) 71–74]☆

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Received 9 October 2002; accepted 10 December 2002

The author has derived a simple explicit solution to the wave dispersion equation, Eq. (1), using a logarithmic matching method of Guo (2002a). The derived explicit solution Eq. (21) is valid for any water depth with a maximum relative error less than 0.75%, and is easy to use by a hand calculator than Hunt's (1979). This kind of study is of practical importance in reducing computing time for wave models that require a large number of wavelength calculations, e.g. wave refraction programs. In fact, several other simple explicit solutions to Eq. (1) were proposed before Guo's (2002b) (e.g. Eckart, 1951; Nielsen, 1982; Fenton and Mckee, 1990). The explicit solution Eq. (21) of Guo (2002b) was not compared with the previous explicit solutions in terms of its simplicity and accuracy. In this discussion, the five explicit solutions derived by Eckart (1951), Hunt (1979), Nielsen (1982), Fenton and Mckee (1990) and Guo (2002b) will be compared with the numerical solution of Eq. (1) to assess the goodness of these explicit solutions. A simple explicit solution to the wave dispersion equation is also presented in this discussion.

Eckart (1951) developed an approximation wave theory with a corresponding dispersion relationship

$$kh = \frac{(k_0h)}{\sqrt{\tanh(k_0h)}}, \tag{30}$$

where k_0 is the wave number in deep water. The relative error caused by this approximation ranges from -5% to 5%. The relative error, Err, is calculated by Eq. (28), where the Newton–Raphson method is used to solve Eq. (1) numerically to obtain kh. An explicit solution similar to Eq. (30) but with higher accuracy can also be derived as

$$kh = \frac{(k_0h)}{\sqrt{[\tanh(k_0h)^{3/4}]^{\frac{4}{3}}}}, \tag{31}$$

by using the logarithmic matching method of Guo (2002a) but with a new matching model

$$kh = (k_0h)[\tanh(k_0h)^{\beta}]^{\alpha}, \tag{32}$$

where kh=(k_0h) at $k_0h \to \infty$. The detail of how to determine the coefficients α and β in Eq. (32) is referred to Guo (2002b). The relative error of using Eq. (32) ranges from -1.6% to 1.4% (see Fig. 4). It should be noted here that Eq. (32) derived from the logarithmic matching method is identical to that proposed by Fenton and Mckee (1990).

Hunt (1979) proposed an approximation solution to the wave dispersion equation

$$kh = (k_0h) \times \sqrt{1 + \left[(k_0h)\left(1 + \sum_{n=1}^{6} D_n(k_0h)^n\right)\right]^{-1}} \tag{33}$$

☆ PII of the original article: S0378-3839(02)00039-X.

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doi:10.1016/S0378-3839(02)00170-9

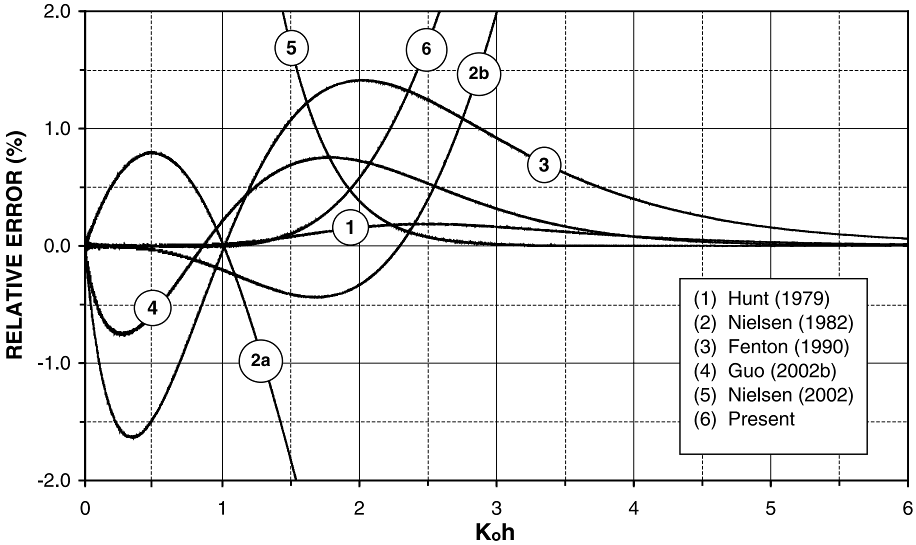

Fig. 4. The relative error of Guo's (2002b) solution compared with those of Hunt (1979), Nielsen (1982), Fenton and Mckee (1990), Nielsen (2002) and the present.

where $D_1 = 0.6666666666$; $D_2 = 0.3555555555$; $D_3 = 0.1608465608$; $D_4 = 0.0632098765$; $D_5 = 0.0217540484$; $D_6 = 0.0065407983$.

It is shown in Fig. 4 that the relative error of applying Eq. (33) is less than 0.2% for any water depth.

Nielsen (1982) also proposed two simple explicit solutions to the wave dispersion equation. The first explicit solution was derived as

$$kh = \sqrt{kh}(1 + 0.2k_0h) \qquad \text{for } k_0h < 1.57 \tag{34}$$

by fitting the data in the range of $k_0h < 1.57$. The relative error of using Eq. (34) ranges from -2.0% to 0.8% as shown in Fig. 4. The second explicit solution to the wave dispersion equation was deduced as

$$kh = \sqrt{k_0h}\left(1 + \frac{1}{6}(k_0h) + \frac{11}{360}(k_0h)^2\right) \tag{35}$$

by using the Taylor expansion approach. The relative error of applying Eq. (35) ranges from -0.5% to 0.5% at $k_0h < 2.5$ (see Fig. 4). A simple explicit solution similar to Eq. (35) but with higher accuracy is also derived here by using the Taylor expansion approach. We rewrite the wave dispersion equation, Eq. (1), as

$$\frac{1}{k_0h} = \frac{1}{(kh)}\coth(kh) \tag{36}$$

where $\coth(kh)$ can be expanded as

$$\coth(kh) = \frac{1}{(kh)} + \frac{1}{3}(kh) - \frac{1}{45}(kh)^3 + \frac{2}{945}(kh)^5 - \frac{1}{4725}(kh)^7 + \ldots \qquad \text{for } kh < \pi \tag{37}$$

which converges much faster than $\tanh(kh)$. This is why the dispersion equation, Eq. (1), is rewritten as Eq. (36). The first four terms in Eq. (37) are considered here. The truncation error is the fifth term. The explicit solution to Eq. (36) is assumed to be of the form

$$(kh)^2 = (k_0h) + \alpha(k_0h)^2 + \beta(k_0h)^3 + \gamma(k_0h)^4, \tag{38}$$

where α, β and γ are the unknown coefficients and need to be determined. Substituting Eqs. (37) and (38) into Eq. (36) leads to

$$kh = \sqrt{k_0 h} \times \left[1 + \frac{1}{3}(k_0 h) + \frac{4}{45}(k_0 h)^2 + \frac{16}{945}(k_0 h)^3\right]^{0.5}, \tag{39}$$

which has the relative error of less than 0.2% at $k_0h < 1.5$, but is not valid for deep water. In deep water, however, the explicit solution of Nielsen (2002) is applied

$$kh = (k_0 h)[1 + 2\exp(-2k_0 h)]. \tag{40}$$

The relative error of using Eq. (40) is less than 0.5% when k_0h>2.0. With the aid of Eqs. (39) and (40), kh for any water depth can be explicitly calculated as

$$kh = \begin{cases} \sqrt{k_0 h}\left(1 + \frac{1}{3}(k_0 h) + \frac{4}{45}(k_0 h)^2 + \frac{16}{945}(k_0 h)^3\right)^{0.5} & kh \le 1.94 \\ (k_0 h)[1 + 2\exp(-2k_0 h)] & kh > 1.94 \end{cases} \tag{41}$$

where $kh = 1.94$ is the joint of the two curves (5) and (6) (see Fig. 4). The relative error of using Eq. (41) is less than 0.5% for any water depth. Similarly, the explicit solution of Nielsen (1982) can be refined as

$$kh = \begin{cases} \sqrt{k_0 h}\left(1 + \frac{1}{6}(k_0 h) + \frac{11}{360}(k_0 h)^2\right) & kh \le 2.37 \\ (k_0 h)[1 + 2\exp(-2k_0 h)] & kh > 2.37 \end{cases} \tag{42}$$

where $kh = 2.37$ is the joint of the two curves (b) and (5) (see Fig. 4). The relative error of using Eq. (42) ranges from −0.5% to 0.12% for any water depth.

Table 1
The average relative error of Guo's (2002b) solution compared with those of Hunt (1979), Nielsen (1982), Fenton and Mckee (1990) and the present

Authors	Formula	Average error (%) ($k_0h=\pi$)	Average error (%) ($k_0h=6$)
Hunt (1979)	Eq. (33)	0.089	0.076
Nielsen (1982)	Eq. (42)	0.165	0.087
Fenton (1990)	Eq. (31)	1.062	0.701
Guo (2002b)	Eq. (21)	0.514	0.299
Present	Eq. (41)	0.091	0.048

The goodness of the five solutions can be also judged by the average relative error defined as

$$\overline{\mathrm{Err}} = \frac{1}{(k_0 h)} \int_0^{k_0 h} |\,\mathrm{Err}\,|\, d(k_0 h) \tag{43}$$

where Err is the relative error defined by Eq. (28) and |Err| is the absolute relative error. The upper limit of k_0h in Eq. (43) is assigned to be π and 6.0, respectively. The average errors of the five solutions are calculated from Eq. (43) and shown in Table 1. It can be seen that the present explicit solution is slightly better than or as good as Hunt's (1979), and is better than the other simple solutions.

In conclusion, Hunt's (1979) explicit solution is shown to be the most accurate one at $k_0h \le \pi$ and should be always used to explicitly calculate the wave number k in wave models that require a large number of wavelength calculations. The simple explicit solutions, Eqs. (41) and (42), are more accurate than Guo's (2002b) and can be used to give a quick calculation of k with a hand calculator.

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