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A close approximation of wave dispersion relation for direct calculation of wavelength in any coastal water depth

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ABSTRACT

The most commonly used wave dispersion relation, $k_0h = kh\tanh(kh)$, has been closely approximated as an explicit formula for direct calculation of wave number k in any coastal water depth h by using two different root-finding methods. One explicit formula derived with the Newton–Raphson method has a maximum relative error of only 0.01% in calculating k in any water depth. This is the most accurate explicit one proposed so far and should be used to compute k in wave-related models that require a large of number of wavelength calculations. The other explicit formula derived with the one-point iteration method has a maximum relative error of 0.1%, but is of a simple form and can be easily applied to compute k in any water depth with a hand calculator. The validity of the wave dispersion relation under the coastal conditions is also investigated based on the comprehensive field data on wave pressure and orbital velocity.

1. Introduction

The wave number k is a fundamental parameter in modelling of wave transformation, wave hydrodynamics and coastal sediment transport, but it can't be directly calculated from the wave dispersion relation, $k_0h = kh\tanh(kh)$, where k_0 is the wave number in deep water and h is the mean local water depth. Therefore, it is of practical engineering importance to approximate the implicit wave dispersion relation as a single, explicit, and accurate formula for direct calculation of k in any coastal water depth without laborious iterations.

A number of explicit formulas have been approximated from the wave dispersion relation for calculation of k in shallow, intermediate, and deep waters. Basically, these explicit formulas were obtained by approximating the hyperbolic tangent function in the wave dispersion relation with different approaches. For example, the explicit solution of Hunt [9] was derived with a Pade approximation method, Nielsen [11] with a Taylor expansion approach, and Guo [8] with a logarithmic matching method. The explicit formula of Eckart [5] was derived with a different wave theory. The explicit solutions derived with the Taylor expansion are quite accurate in shallow water, less accurate in intermediate water, and become invalid in deep water. You [16] discussed and compared several commonly used explicit solutions, and concluded that the explicit solution of Hunt [9] is the most accurate one and should be used to directly calculate k in wave-related models that require a large number of wavelength calculations. However, it may not be convenient for engineers to apply Hunt's explicit solution to give a quick calculation of k with a hand calculator.

The wave dispersion relation is derived from linear wave theory under the assumptions of irrotational flow, linear waves, constant water depth or flat seabed, and no current. In coastal zones, however, many of these assumptions become invalid. For example, the slope of the seabed always exists even though it is quite small and subsequently the mean water depth varies from one location to another, the tidal current always co-exists with waves, and coastal waves are often nonlinear especially during coastal storms. There are few studies undertaken to investigate the validity of the wave dispersion relation under the real field conditions. As the result, it is always assumed that the wave dispersion relation derived from linear wave theory under idealized monochromatic waves is still valid for the calculation of the wavelength of irregular waves under the real field conditions.

In this study, two explicit formulas will be derived from the wave dispersion relation for direct calculation of k in any water depth by using two different root-finding methods. The validity of the dispersion relation under the real field conditions is also investigated based on the comprehensive field data on wave pressure and orbital velocity.

2. Wave dispersion relation

The relationship among wave period T, wave number k and mean water depth h is described by the most commonly used wave

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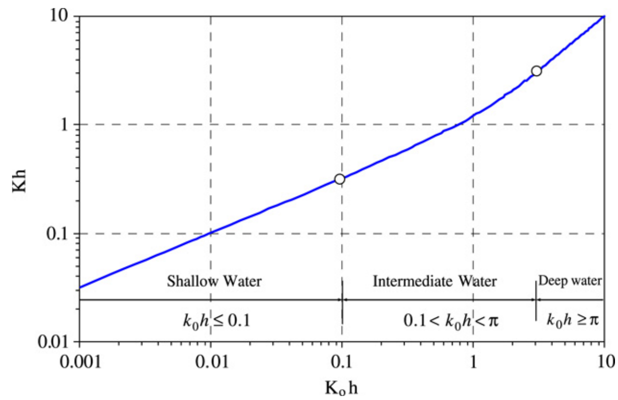

Fig. 1. The variation of kh with k_0h is explicitly calculated from Eq. (1) and the three relative water depths are defined in terms of k_0h.

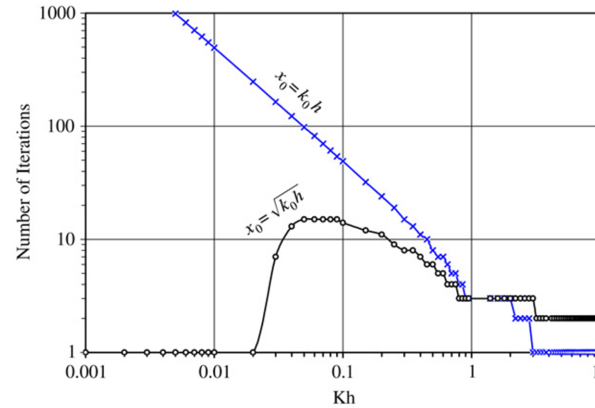

Fig. 2. The number of times required to iterate Eq. (1) for calculation of kh varies with different initial values of x_0 and kh when $\varepsilon \leq 0.01$.

dispersion relation

$$\omega^2 = gk \tanh kh \quad \text{or} \quad k_0h = kh \tanh kh \tag{1}$$

where ω the wave angular frequency and $\omega = 2\pi/T$, k_0 is the wave number in deep water, and g is the acceleration of gravity. The wave dispersion relation, Eq. (1), is derived under the assumptions of irrotational flow, linear and small waves, constant water depth, and no current. A different dispersion relation from Eq. (1) was proposed by Kirby and Dalrymple [10] to consider the effect of nonlinear waves on the wave number k. Fenton [6] and Fenton and Mckee [7] also derived a more general dispersion relation that had taken the effects of wave height and currents on k into account. Thus, the wave dispersion relation, Eq. (1), is only a simple approximation to the real problem.

The distribution of kh with k_0h can be theoretically studied by considering k_0h as the function of kh. When kh is given, the value of k_0h can be calculated directly from Eq. (1). The distribution of kh with k_0h can be then explicitly constructed by plotting the calculated values of k_0h versus the given values of kh. Fig. 1 shows the distribution of kh with k_0h. The shallow, intermediate and deep waters have been commonly defined as $kh \leq 0.1\pi$, $0.1\pi < kh < \pi$ and $kh \geq \pi$, respectively (e.g. [4,1]). Since kh is always unknown before it is calculated from Eq. (1), it may be convenient to define the shallow, intermediate and deep waters in terms of the known relative water depth k_0h rather than kh. On substituting $kh = \pi/10$ and $kh = \pi$ into Eq. (1), the shallow water is then defined as $k_0h \leq 0.1$, the intermediate water as $0.1 < k_0h < \pi$ and the deep water as $k_0h \geq \pi$ as shown in Fig. 1.

The wave dispersion relation, Eq. (1), is often approximated as $kh = \sqrt{k_0h}$ at $kh \to 0$ in shallow water, and $kh = k_0h$ at $kh > \pi$ in deep water. The true relative error of $kh = \sqrt{k_0h}$ is found to increase with k_0h and has a maximum of 1.6% in shallow water, while the use of $kh = k_0h$ in deep water results in a maximum error of only 0.4%. In intermediate water, however, Eq. (1) needs to be iterated for calculation of k. In this study, the true relative error ε_t is calculated as follows: for a given kh, k_0h is directly calculated from Eq. (1) and then used to compute kh from an approximated formula, and finally the calculated $(kh)_C$ and the given (true) $(kh)_T$ are used to calculate the true relative error as $\varepsilon = [(kh)_C/(kh)_T - 1.0] \times 100\%$. The method used to compute ε in this study is somehow different from that in previous studies. For example, Guo [8] and You [16] directly iterated Eq. (1) to obtain the "true" value of kh for a given k_0h. The iterated value of kh may be close enough to the true value, but never be exact and thus their value of ε is relative error, but not the true relative error.

3. Methodology

3.1. Newton–Raphson method

The Newton–Raphson method is one of the most popular methods for root finding and it has been most commonly used to iterate Eq. (1) for the calculation of k. With this method, an improved estimate of kh can be computed from Eq. (1) as

$$kh \approx x_0 - f(x_0)/f'(x_0) = x_0 \left[\frac{k_0h + (x_0/\cosh x_0)^2}{x_0 \tanh x_0 + (x_0/\cosh x_0)^2} \right] \tag{2}$$

where $f(x) = x \tanh x - k_0h$, $f'(x)$ is the first derivative with respect to x, $\tanh(x)$ and $\cosh(x)$ are the hyperbolic functions and x_0 is an initial guess of kh. Eq. (2) reduces to $kh = k_0h$ at $x_0 \to \infty$ in deep water. The calculated value of kh should be always much closer to the exact value of kh than the initial guess of x_0 because the Newton– Raphson method is of quadratic convergence.

There are three factors in Eq. (2), which determine the number of iterations required for the calculation of k, namely, the initial guess x_0, the relative water depth k_0h and the accuracy of approximation or relative error ε. It is illustrated in Fig. 2 that the number of iterations required for Eq. (2) is determined by kh and x_0 when $\varepsilon \leq 0.01$ is given. In shallow and intermediate waters, the number of iterations ranges from 3 to 15 when $x_0 = \sqrt{k_0h}$ is chosen, but varies so widely from 15 to more than 1000 when $x_0 = k_0h$ is used. In deep water, however, the number of iterations required for Eq. (2) varies only from 1 to 2 and is almost independent of x_0. Fig. 2 also clearly indicates that the direct iteration of Eq. (1) would take enormous computing time in wave-related models that require a large number of wavelength calculations especially when the number of computing grids is extremely large, the computing time step is quite small (e.g. less than several seconds), and the initial value x_0 could not be properly assigned.

In this study, however, we are not interested in numerically iterating Eq. (2), rather than provide an explicit and accurate formula for the calculation of kh. If we could find a solution for $x_0 = f(k_0h)$ that is quite close to the exact solution of kh in shallow and intermediate waters and also gives $x_0 \geq \pi$ in deep water, Eq. (2) might be directly used to compute kh in any water depth without laborious iterations. The requirement of $x_0 \geq \pi$ in deep water will automatically reduce Eq. (2) to $kh = k_0h$. The detail on how to determine a simple explicit solution for $x_0 = f(k_0h)$ in shallow and intermediate waters is given as follows.

A simple curve fitting method is used to derive an explicit solution for $x_0 = f(k_0h)$ based on the data that are generated in Fig. 1. The discrete data points (kh, k_oh) are generated explicitly

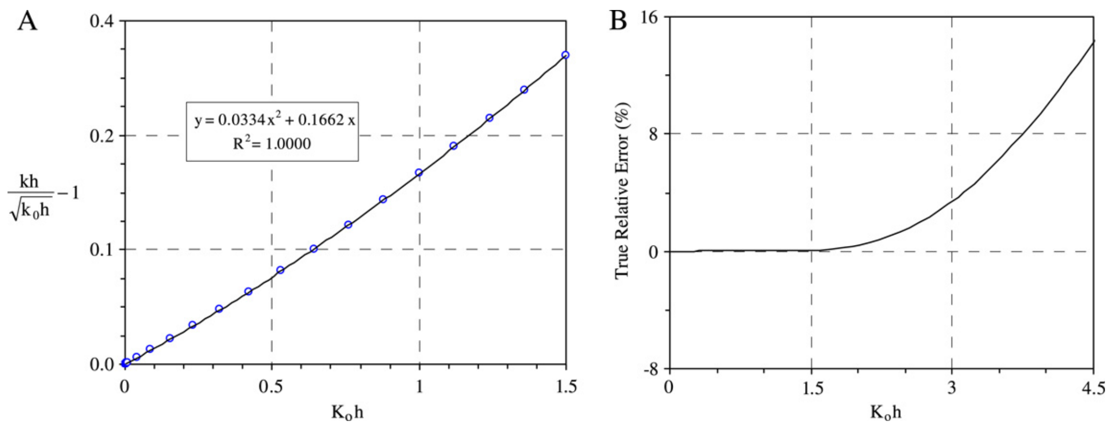

Fig. 3. [A] A simple solution of x_0 is obtained by directly fitting the data to a quadratic regression line, and [B] the variation of ε with k_0h.

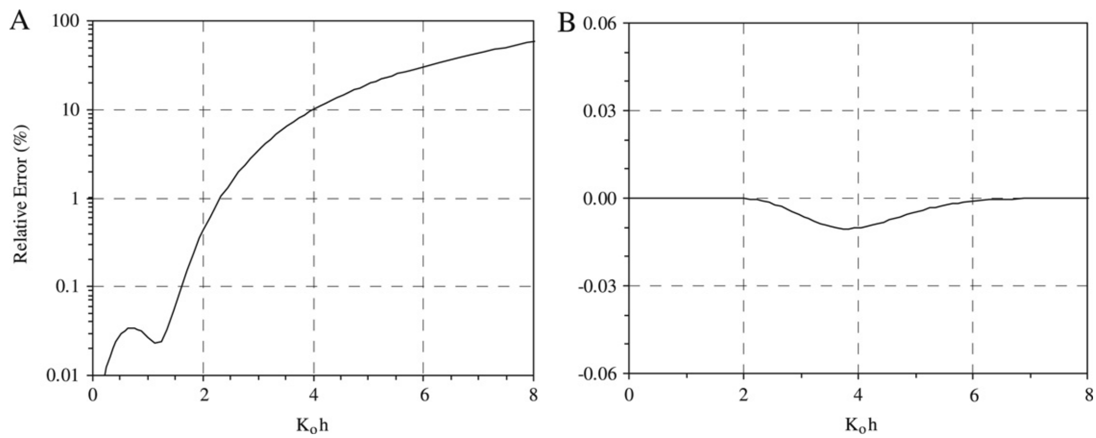

Fig. 4. [A] The relative errors calculated from Eq. (3), and [B] the relative errors computed from Eq. (2) with Eq. (3).

from Eq. (1) for given different values of kh, and then used to construct the data $(k_0h, kh/\sqrt{k_0h})$. A quadratic regression curve is fitted to the constructed data points to obtain $x_0 = f(k_0h)$ as

$$\frac{x_0}{\sqrt{k_0h}} \approx 1 + \frac{1}{6}(k_0h) + \frac{1}{30}(k_0h)^2, \tag{3}$$

where $x_0 = kh$. The use of $(kh/\sqrt{k_0h} - 1)$ as the y axis in Fig. 3[A] is to ensure that Eq. (3) will reduce to $x_0 = \sqrt{k_0h}$ in shallow water. The range of the data used in Fig. 3[A] is determined to give the best quadratic fitting to the data in shallow and intermediate waters. The true relative errors of Eq. (3) in shallow, intermediate and deep waters are shown in Fig. 3[B]. It can be seen that Eq. (3) is almost equal to the exact solution of kh in shallow water, and then becomes less accurate with increasing k_0h and has a maximum error of about 4% in intermediate water, and finally becomes invalid in deep water. Eq. (3) is also found as accurate as those of Nielsen [11] and Wu and Thornton [13] derived with different methods.

Since Eq. (3) is generally close to the exact solution of Eq. (1) in shallow and intermediate waters, and also gives $x_0 > \pi$ in deep water, Eq. (2) together with Eq. (3) may then be used for the calculation of kh in any water depth without any laborious iterations required. It is shown in Fig. 4[B] that the maximum relative error of Eq. (2) is less than 0.01% and occurs at $k_0h \approx 4$. It can be also seen from Fig. 4 that Eq. (2) with Eq. (3) is much closer to the exact solution than Eq. (3) especially when $k_0h > 2$. This is because the Newton–Raphson method is of quadratic convergence. Hunt's [9] explicit solution with a maximum relative error of 0.2% was concluded by You [16] as the most accurate explicit solution, but Eq. (2) with Eq. (3) is much simpler and more accurate than Hunt's one and should be used to compute k in wave-related numerical models that require a large number of wavelength calculations. The use of this explicit formula, Eq. (2) with Eq. (3), will significantly reduce computing time in modelling of wave transformation, wave hydrodynamics and coastal sediment transport.

3.2. One-point iteration

One-point iteration method is another root-finding method that employees a formula to approximate the root of a function of $x = g(x)$ Chapra and Canale [2]. This method is simpler, but converges much more slowly than the Newton–Raphson method. Let us rewrite the wave dispersion relation, Eq. (1), as

$$kh = \frac{k_0h}{\tanh(kh)} = \frac{k_0h}{\tanh(x_0)}, \tag{4}$$

where x_0 is an initial guess of kh. Similar to Eq. (2), Eq. (4) also reduces to $kh = k_0h$ in deep water. The number of iterations required for Eq. (4) has been found much larger than that for Eq. (2). This is because the one-point iteration method is of linear convergence, but the Newton–Raphson method is of quadratic convergence. Similarly, if we could find a simple solution for $x_0 = f(k_0h)$ that is generally close to the exact solution of kh in both shallow and intermediate waters and satisfies the condition of $x_0 > \pi$ in deep water, Eq. (4) may be also used as an explicit formula for calculation of k in any water depth without any laborious iterations.

Again, Eq. (3) is used for the calculation of x_0 in Eq. (4). The relative error of Eq. (4) is shown in Fig. 5[B], and the maximum

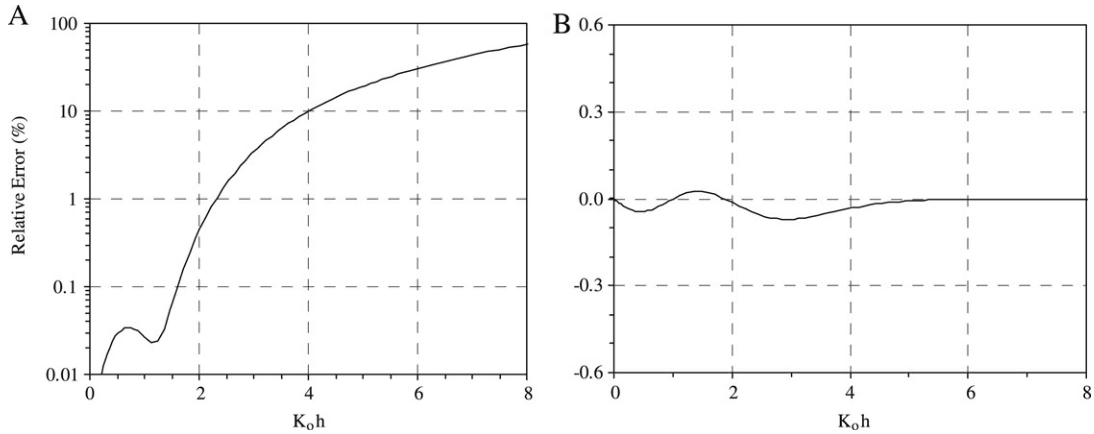

Fig. 5. [A] The relative errors calculated from Eq. (3), and [B] the relative errors computed from Eq. (4) with Eq. (3).

relative error is less than 0.1%. Fig. 5 also shows that Eq. (4) together with Eq. (3) is much closer to the exact solution of Eq. (1) than Eq. (3). Eq. (4) is similar to Eckart's [5] simple solution, $kh = k_0h/\sqrt{\tanh(k_0h)}$ with a maximum error of 5%, but much more accurate. Eckart's solution is recommended by SPM [12] and CEM [1] for direct calculation of wavelength in intermediate water depth. Thus, Eq. (4) with Eq. (3) is preferred for a quick calculation of k in any water depth with a hand calculator.

3.3. Comparison

There are several explicit formulas proposed for direct computation of k in shallow, intermediate and deep waters, but only a few of them are valid for the calculation of k in any water depth. Some of these explicit solutions have been compared in Fig. 4 of You [16]. The explicit solution of Hunt [9] is shown to be the most accurate one of these explicit formulas

$$kh = (k_0h)\sqrt{1+\left[(k_0h)\left(1+\sum_{n=1}^{6} D_n\,(k_0h)^n\right)\right]^{-1}}, \tag{5}$$

where $D_1 = 0.6666666666$, $D_2 = 0.3555555555$, $D_3 = 0.1608465608$, $D_4 = 0.0632098765$, $D_5 = 0.0217540484$, and $D_6 = 0.0065407983$. Chen and Thompson [3] refined the explicit solution of Hunt [9] by revising the six coefficients as $D_1 = 0.6522$, $D_2 = 0.4622$, $D_3 = 0$, $D_4 = 0.0864$, $D_5 = 0.0675$, and $D_6 = 0$ in Eq. (5).

The relative errors of the four explicit solutions of Hunt [9], Chen and Thompson [3], Eq. (2), and Eq. (4) are compared and shown in Fig. 6. Eq. (3) is used to compute x_0 in Eqs. (2) and (4). It can be seen that Eq. (2) is almost equal to the exact solution of Eq. (1) and is much more accurate than those of Hunt [9] and Chen and Thompson [3]. Eq. (4) is shown twice more accurate than Hunt's solution and slightly better than Chen and Thompson [3], but much simpler than the formula of Hunt or Chen and Thompson. For coastal engineering applications, Eq. (2) or Eq. (4) with Eq. (3) is preferred to the explicit formulas of Hunt [9] and Chen and Thompson [3] for the calculate of k in wave-related models that require a large number of wavelength calculations.

Fig. 7 also shows the comparison of the relative errors ε_u in calculating the maximum nearbed wave orbital velocity U_0 when the four explicit formulas of Hunt [9] and Chen and Thompson [3], Eqs. (2) and (4) are used to calculate k. The maximum wave orbital velocity U_0 is an essential input parameter in modelling of wave hydrodynamics and coastal sediment transport [14], and it is often calculated from linear wave theory as $U_0 = 0.5H\omega/\sinh(kh)$, where H is the wave height. Based on the linear wave theory, the

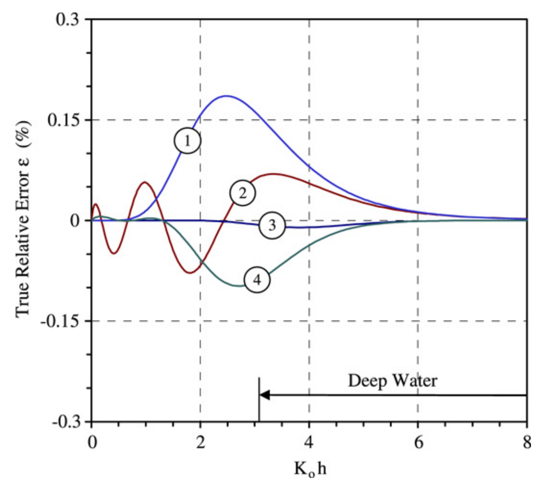

Fig. 6. Comparison of the relative errors in calculating k from four formulas: ① Hunt [9], ② Chen and Thompson [3], ③ Eq. (2), and ④ Eq. (4).

"true" value of U_t is calculated with the "exact" value of kh and the calculated value U_c is computed with the estimated values of kh from the four different explicit formulas, and then the relative error ε_u in U_0 is computed as

$$\varepsilon_u = \frac{U_c - U_t}{U_t} \times 100\% = \left[\frac{\sinh(kh)_t}{\sinh(kh)_c} - 1\right] \times 100\%, \tag{6}$$

where $(kh)_t$ and $(kh)_c$ are the true and calculated values of kh. It is again shown that Eq. (2) is most accurate in computing U_0 than the other three, and Eq. (4) is more accurate than the formulae of [9]. It can be also seen that the maximum absolute relative errors of the four formulas all occurred in deep water where U_o is close to zero. In comparing Figs. 6 and 7, it is found that a small error in estimating k will result in a larger (about twice) error in computing U_o. It should be noted here that this finding is based on the linear wave theory under regular waves, but may not be conclusive under the coastal conditions where the linear wave theory may become inaccurate for calculation of U_o. It may be postulated that an error in estimating k would result in at least the equal or larger error in the calculated U_o under the coastal conditions.

4. Dispersion relation of coastal waves

The wave dispersion relation, Eq. (1), is derived from linear wave theory under idealized monochromatic waves with constant height and period. In the coastal zone, however, waves are irregular and with different height and period. The wave-by-wave analysis

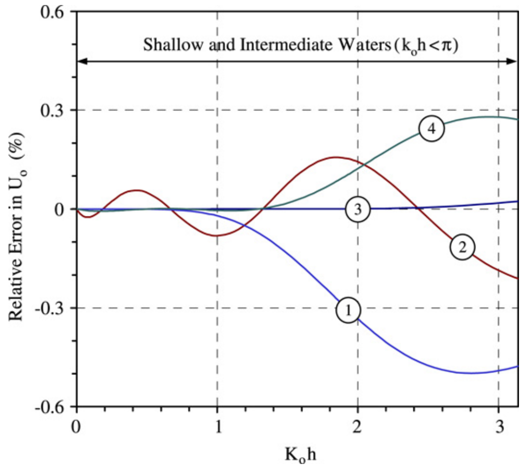

Fig. 7. Comparison of the relative errors in calculating the nearbed wave orbital velocity U_o when the four formulas, ① Hunt [9], ② Chen and Thompson [3], ③ Eq. (2), and ④ Eq. (5), are used to compute kh, respectively.

is often applied to define individual wave periods from time series of wave pressure measured in the field. The wave period data are then used to determine the probability distribution of wave period or characteristic wave periods. Fig. 8 shows the probability density distributions of wave periods measured in two coastal water depths of 12 m and 24 m by You and Hanslow [15]. Instantaneous wave orbital velocities and wave pressures were measured at 0.5 m above the seabed for 17.07 min every 1.0 h at a 2 Hz sampling rate with two Ocean ADV currents. The duration of the five field deployments ranged from 31 to 65 days. It can be seen from Fig. 8 that the measured density distribution is symmetry about the mean wave period and it is lower and spreads out farther in the 12 m water than that in the 24 m water. The measured distributions of wave period may be well described by the modified Gaussian density distribution

$$f(T) = \frac{1}{C\,T_{\mathrm{rms}}\sqrt{\pi}} \exp\left[-\left(\frac{T-\bar{T}}{C\,T_{\mathrm{rms}}}\right)^2\right] \tag{7}$$

where T_{rms} is the root-mean-square wave period, $\bar{T}$ is the mean wave period, and $C = 0.36$ in the water depth of 12 m and $C = 0.33$ in 24 m. Several useful characteristic periods are also calculated from the field data and can be generally described as

$$T_{\max} \approx T_{1/10} \approx T_{1/3} = 1.05T_{\mathrm{rms}} = 1.09\bar{T}, \tag{8}$$

where $T_{\max}$ is the maximum wave period, and $T_{1/10}$ is the average period of the wave periods that correspond to the largest 1/10 of all waves. Eq. (8) clearly shows that the probability distribution of wave period is much narrower than that of wave height, and it lies mainly in the range of $T = (0.5 - 1.5)\bar{T}$. This is termed as the narrow-banded condition in which all the wave periods are in a narrow period band about $\bar{T}$.

The wave dispersion relation, Eq. (1), may be used to compute the wavelength of individual irregular waves. It should be noted here that Eq. (1) is derived from linear wave theory under the assumptions of irrotational flow, monochromatic and linear waves, constant water depth, and non mean current. Under the coastal conditions, however, the flow is often rotational, waves are irregular and often nonlinear, the mean water depth often varies with location, and there are always mean currents co-existing with waves. These field conditions are directly conflict with the assumptions made for the derivation of Eq. (1). Now, the question is if the linear wave theory is still valid for the deviation of Eq. (1) under these field conditions. However, the validity of Eq. (1) may not be directly verified under irregular waves because the wavelength of individual irregular waves may not be directly measured in the field. An alternative approach is proposed here to verify the validity of Eq. (1) under coastal waves based on the field wave data of You and Hanslow [15].

Under idealized monochromatic waves, a simple relationship among the maximum wave orbital velocity U, wave period T, and maximum hydrodynamic pressure P at a level above the bed can be easily derived from the linear wave theory as

$$\frac{\omega P}{U} = \tanh(kh), \tag{9}$$

where k will be computed from the wave dispersion relation, Eq. (1) or Eq. (2). In this study, the wave-by-wave analysis is used to analyse individual irregular coastal waves from time series of wave pressures recorded by the ADV current meter. The wave data (U_i, P_i, T_i) of individual coastal waves can be then obtained to verify if the wave dispersion relation, Eq. (1), which is derived under monochromatic waves, is still valid under irregular coastal waves based on Eq. (9). When the value of $\omega P/U$ measured in the field generally agree with the theoretical value of Eq. (9),

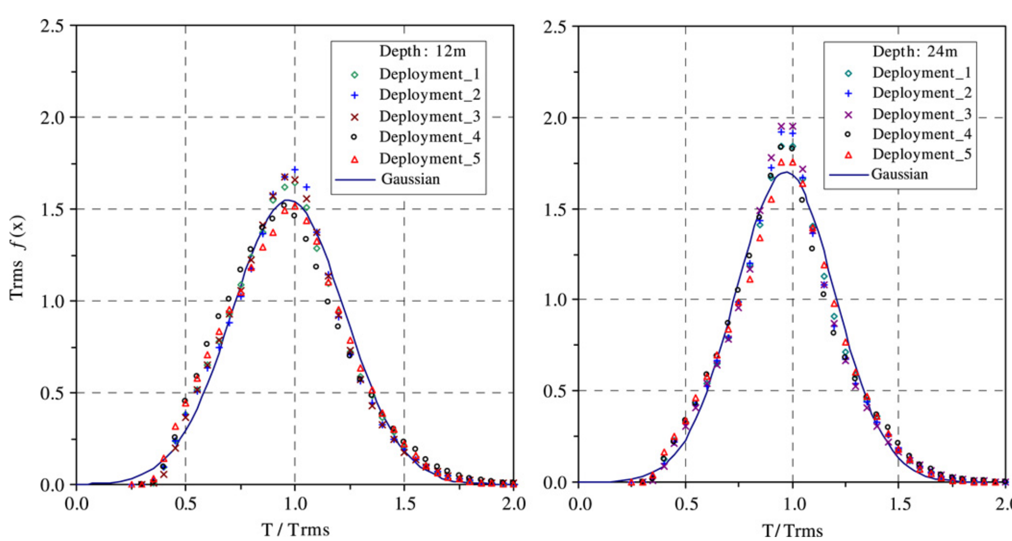

Fig. 8. Probability density distributions of individual coastal wave period T measured in water depths of 12 m and 24 m by You and Hanslow [15] are compared with Eq. (7).

the linear wave theory may be proven to be valid under the coastal conditions and then Eq. (1) may be still applicable to coastal waves. Fig. 9 shows the comparison of Eq. (9) with the field maximum wave data on $(U_{max}, P_{max}, T_{max})$. The zero-crossing method is applied to determine the values of U_{max}, P_{max} and T_{max} of individual maximum waves from time series of wave orbital velocities and wave pressures measured for 17 min every hour over the period of 65 days. There is only one data point $(U_{max}, P_{max}, T_{max})$ obtained every hour from a 17 min-timeseries of wave pressure and orbital velocity. It can be seen from Fig. 9 that the linear wave theory is generally valid for individual maximum waves measured every hour in the two water depths even though Eq. (9) slightly overestimates the measured values of $\omega P/U$. Some data points collected in the water depth of 12 m are shown to be scattering. This may be due to the fact that the ADV current meter was affected by biological growth on the sensors. In general, the wavelength of individual maximum waves may be computed from Eq. (1) or Eq. (2) with $T = T_{max}$.

For most engineering applications, however, we may be interested in the wavelength of characteristic waves, rather than in the wavelength of individual irregular waves. The characteristic of the real sea is often described by significant wave height and period. The significant wave height H_s and period T_s has been postulated to represent the characteristic of the real sea in the simple form of monochromatic waves (H_s, T_s). With this simplification, all the formulas derived under regular waves could be used to compute the kinematic and dynamics of significant waves, and Eq. (1) could be then used to compute k with $T = T_s$. Fig. 10 shows the comparison of Eq. (9) with the field data (U_s, P_s, T_s) of significant waves. The significant wave data on (U_s, P_s, T_s) are analysed from time series of instantaneous wave orbital velocities and pressures measured in individual bursts with the zero-crossing method. It can be seen from Fig. 10 that the values of $(\omega P/U)_s$ measured in the two water depths of 12 m and 24 m indeed are only the function of T as predicted theoretically, but generally overestimated by Eq. (9). The best fitting curve fitted to the field data is $(\omega P/U)_s = 0.92 \tanh(kh)_s$. This means that Eq. (9) overestimates the measured values of $(\omega P/U)_s$ by about 8%. This discrepancy may be due to several factors, e.g. the validity of the assumptions used to derive Eq. (1), the applicability of the linear wave theory to the real sea waves, and the simplicity of the dispersion relation. In comparing Figs. 9 and 10, the maximum wave data $(U_{max}, P_{max}, T_{max})$ are better predicted by Eq. (9) than the significant wave data (U_s, P_s, T_s). It may be concluded here that the wave dispersion relation, Eq. (1), may be still valid under the coastal conditions, but only an approximation to the real problem, and Eq. (2) or Eq. (4) together with Eq. (3) is accurate enough for practical engineering applications.

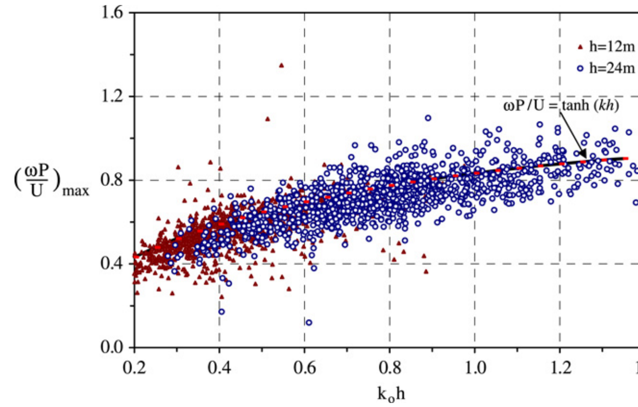

Fig. 9. The theoretical relationship, Eq. (9) [dashed line], derived from linear wave theory under monochromatic waves, is compared with the wave data on $(U_{max}, P_{max}, T_{max})$.

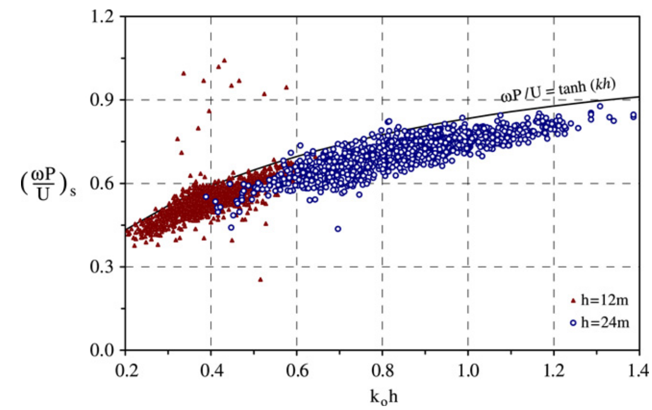

Fig. 10. The theoretical relationship, Eq. (9) [solid line], derived from linear wave theory under monochromatic waves, is compared with the significant wave data on (U_s, P_s, T_s).

5. Conclusion

The wave dispersion relation, Eq. (1), has been successfully approximated as a single and explicit formula for direct calculation of k in any coastal water depth by using the Newton–Raphson and one-point iteration methods. The shallow, intermediate and deep waters are explicitly defined in this study as $k_0h \le 0.1$, $0.1 < k_0h < \pi$ and $k_0h \ge \pi$, respectively, in terms of k_0h rather than kh. The proposed explicit formula, Eq. (2) together with Eq. (3), is derived with a maximum relative error of only 0.01% for calculation of k in any water depth. This formula is the most accurate explicit one proposed so far, and should be used to calculate k in wave-related models that require a large number of wavelength calculations. The other explicit formula, Eq. (4) together with Eq. (3), is simple and also accurate with a maximum relative error of 0.1% and can be easily used to calculate k in any water depth with a hand calculator. The newly derived relationship, Eq. (9), has enabled us to quantitatively verify the validity of the wave dispersion relation under the coastal conditions. It is found that the wave dispersion relation derived from linear wave theory under idealized monochromatic waves may be still valid for calculation of the wavelength of coastal characteristic waves, but only an approximation to the real problem. The use of the newly proposed two explicit formulas will significantly reduce computing time in wave-related numerical models.

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