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A hybrid spectral/DG method for solving the phase-averaged ocean wave equation: Algorithm and validation

B. Yildirim^a, George Em Karniadakis^{b,*}

^a School of Engineering, Brown University, Providence, RI 02912, USA ^b Division of Applied Mathematics, Brown University, Providence, RI 02912, USA

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ABSTRACT

We develop a new high-order hybrid discretization of the phased-averaged (action balance) equation to simulate ocean waves. We employ discontinuous Galerkin (DG) discretization on an unstructured grid in geophysical space and Fourier-collocation along the directional and frequency coordinates. The original action balance equation is modified to facilitate absorbing boundary conditions along the frequency direction; this modification enforces periodicity at the frequency boundaries so that the fast convergence of Fourier-collocation holds. In addition, a mapping along the directional coordinate is introduced to cluster the collocation points around steep directional spectra. Time-discretization is accomplished by the TVD Runge–Kutta scheme. The overall convergence of the scheme is exponential (spectral). We successfully verified and validated the method against several analytical solutions, observational data, and experimental results.

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1. Introduction

Wave prediction models fall into two categories: phased-averaged and phase-resolving. The phased-averaged wave models solve for the energy density (or action density) to obtain important parameters, such as the significant wave height, through integration of the energy density. On the other hand, the phase-resolving models simulate directly the wave height in space and time.

Phase-averaged models are well suited for slowly varying wave fields such as ocean waves in deep water and they are based on the spectral representation of ocean waves. The first numerical spectral wave model was proposed by Gelci et al. [1]. The idea of spectral representation is to superimpose many different linear harmonic waves with different directions and frequencies in order to simulate realistic ocean waves. In this model, the basic quantity is energy density $(E(t, \mathbf{X}, \theta, \sigma))$, which is a function of time (t), physical space (\mathbf{X}), and the direction (θ) and frequency (σ) of the waves. In water waves, the energy density can be shown to relate to energy flux (F) through the group velocity (C) as F = CE [2]. Replacing the flux term in the energy conservation equation ($\partial_t E + \nabla \cdot F = 0$) will yield an energy density equation ($\partial_t E + \nabla \cdot (CE) = 0$). The conserved quantity, action density (Ef), is employed instead of the energy density, which is not conserved in the presence of current field gradients. This equation is the ancestor of phase-averaged models, which were later modified to take into account shoaling and refraction in shallow water and physical sources (wind generation) and sinks (white-capping, depth-induced breaking, etc.) as well as conservative wave transformation mechanisms (triad, quadruplet interactions), which neither add nor extract energy from the system but only distribute the energy among different frequencies; we will refer

^{*} Corresponding author. Address: Division of Applied Mathematics, Brown University, 182 George St., Box F, Providence, RI 02912, USA. Tel.: +1 401 863 1217; fax: +1 401 863 3369.

E-mail addresses: george_karniadakis@brown.edu, gk@cfm.brown.edu (G.E. Karniadakis).

to these as *source terms*. The source terms have been continuously modified over the years either by including a new mechanism or improving the existing terms. These modifications over the years have been named first, second, and third generation of models; recent attempts have been towards a newer wind-wave model with an optimized source function [3,4]. The third generation model, established by the WAMDI group [5], is still today the most widely accepted model in operational ocean codes. Phase-averaged models have been widely used from large oceans to coastal regions with great success. The most well-known are from the European Center for Medium-Range Weather Forecast (ECMWF) [6], Simulating Waves Near-shore (SWAN) from Delft University [7], and NOAA's WAVEWATCH [8] models.

Phase-resolving models, unlike phase-averaged, have diversified into many different equations based on depth characteristics, strength of nonlinearities, and bottom slopes such as Boundary Integral models; mild-slope equation models, Boussinesq models, and shallow water equations. The phase-resolving model has advantages over the pase-averaged ones in that it contains inherently nonlinear wave-interaction (triads) and better resolves the wave height in a strongly varying wave field. However, its prohibitive computation cost (much more expensive than phase-averaged models per unit area) limits it to only small regions (such as near-field of wave-structure interactions). We refer the interested readers to Battjes' [9] for a comprehensive review of both the phase-averaged and phase-resolving models.

Application of high-order methods to phase-resolving methods (fully nonlinear water waves, Boussinesq, shallow water equations, etc.) have recently received attention with DG schemes proposed and implemented in [10–15]. In particular, phase-averaged models have adopted finite difference methods for spatial discretization in the past. Community codes such as SWAN, WAVEWATCH mix first-, second- and third-order schemes for discretization of physical and spectral space derivatives together with the first-order Euler scheme in time. Although these codes are now mature, the numerical ocean wave community is aware of the indisputable advantages of unstructured grid technology, which simplifies mesh generation and offers flexibility for controlling mesh resolution around a sharp region without increasing the grid resolution on the smooth region. Finite element and finite volume schemes using unstructured grids have recently been applied to phase-averaged models [16–18] with great success.

In this paper, we present a high-order scheme in both geophysical and spectral spaces. We are not aware of any attempt to use high-order schemes on a general unstructured grid (in physical space) in the literature for phase-averaged equations. Higher-order methods can be very effective in deep ocean water simulations, which involve waves propagating over longer distances. Ocean models using a first-order scheme for the spatial advection term are limited to coastal simulations where waves propagate over relatively short distances such that the numerical solution is still not damped significantly. On the other hand, unlike geophysical space, spectral space has no tolerance for numerical diffusion of the first-order scheme; those terms are exclusively treated with second-order discretization [19]. Tolman [20] and Bender [21] reported improving results using second- and third-order schemes for the propagation term in physical spaces.

The paper is organized as follows. We first present the action balance equation (phased-averaged equation) in Section 2, and then propose a high-order DG discretization in physical space after evaluating the frequency and directional derivatives with Fourier-collocation in Section 3. In the same section, we use a mapped Fourier-collocation for local refinement in a directional coordinate and introduce a modified action balance equation for proper treatment of the absorbing boundary condition along the frequency direction. Following this section, we present verification results to show the exponential convergence of the scheme in both physical and spectral spaces and establish the convergence order of time discretization scheme (TVD Runge–Kutta). In Section 5, we validate the new method against available analytical solutions, observational data, and experimental results. Finally, we present a brief summary in Section 6. In the appendices we include technical details on the formulation, the numerical implementation, and on the type of source terms employed, so that the interested readers can reproduce our results and follow the method more closely.

2. Action balance equation

The action balance equation [2,19] for ocean waves in the Eulerian framework can be written as

$$\frac{\partial N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial t} + \frac{\partial c_{\mathbf{x}}N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial \mathbf{x}} + \frac{\partial c_{\mathbf{y}}N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial \mathbf{y}} + \frac{\partial c_{\theta}N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial \theta} + \frac{\partial c_{\sigma}N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial \sigma} = \frac{S(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\sigma}, \quad (1)$$

where $N(\theta, \sigma, x, y, t)$ is the action density defined as the ratio of energy $E(\theta, \sigma, x, y, t)$ to relative frequency σ ($N = E/\sigma$), c_x and c_y are the propagation velocities of wave energy in physical (x - y) space whereas c_θ and c_σ are the propagation velocities in spectral ($\theta \in [-\pi, \pi], \sigma \in [0, \infty]$) space (see Fig. 1). The source term $S(\theta, \sigma; x, y, t)$ takes into account wave generation, dissipation, and interaction mechanisms. The energy density is not conserved but the action density is conserved in the presence of a spatially variable current. The action density propagation velocities can be derived by differentiating the dispersion relation

$$\sigma^2 = g|\vec{k}| \tanh(|\vec{k}|d), \quad \text{where } d = d(x, y) \text{ is depth}, \quad \vec{k} = (k_x, k_y)$$
(2)

with respect to wave number k, giving

$$\frac{d(\vec{\mathbf{x}} = (x, y))}{dt} = (c_x, c_y) = \vec{c_g} + \vec{u} = \frac{1}{2} \left(1 + \frac{2|\vec{k}|d}{\sinh(2|\vec{k}|d)} \right) \frac{\sigma\vec{k}}{|\vec{k}|^2} + \vec{u}.$$
(3)



Fig. 1. (a) An illustration of geophysical and spectral spaces. The frequency direction ($\sigma = 2\pi f$) is truncated from $[0,\infty]$ to [0,2.5] Hz and the directional coordinate (θ) covers the full circle $[0,2\pi]$. (b) An illustration of coordinates *m* and *s*.

The propagation velocities in spectral space are

$$\frac{d\theta}{dt} = c_{\theta} = -\frac{1}{k} \left(\frac{\partial \sigma}{\partial d} \frac{\partial d}{\partial m} + \vec{k} \cdot \frac{\partial \vec{u}}{\partial m} \right), \quad \text{and}$$

$$\frac{d\sigma}{d\sigma} = \frac{\partial \sigma}{\partial \sigma} \left(\frac{\partial d}{\partial m} - \vec{k} \cdot \vec{n} \cdot \vec{n} \right), \quad \vec{k} = \frac{\partial \vec{u}}{\partial \vec{u}}$$
(4a)

$$\frac{d\sigma}{dt} = c_{\sigma} = \frac{\partial\sigma}{\partial d} \left(\frac{\partial u}{\partial t} + \vec{u} \cdot \nabla d \right) - c_g \vec{k} \cdot \frac{\partial u}{\partial s},\tag{4b}$$

where *s* is the space coordinate in the wave propagation direction of θ , *m* is a coordinate perpendicular to *s*, sketched in Fig. 1, and \vec{u} is a current velocity vector ($\vec{u} = (u_x, u_y)$). Note that the propagation of action density in frequency space occurs only if there is a current in the domain (assuming that the bathymetry does not change with time). The propagation velocities in spectral space are more useful to obtain in terms of Cartesian coordinates for the implementation (see Appendix A).

2.1. Modeling the source terms

Phase-averaged models represent wind generation, white-capping, nonlinear wave interaction, depth-induced breaking, and bottom friction mechanisms as source terms in the action balance equation (1). Wind generation, nonlinear wave interaction (quadruplet), and white-capping (dissipation) are wave mechanisms in deep water. The third generation of wind input is based on Miles theory [22]. The WAMDI group adopted the parametrization of the wind source term [23] given explicitly in Appendix C. The energy growth by wind becomes unstable, leading to waves which eventually break. This version of breaking other than depth-induced is called white-capping. The third generation model proposed a source function [5], which is consistent with the pulse-based model of Hasselmann [24]. The parametrization of white-capping can be found in Appendix C. Alternative formulations and their performances can be found in [25]. The interaction among ocean waves was first described by Hasselmann [26] through perturbation analysis under the assumption that waves are weakly nonlinear. Hasselmann's [26] formulation was a six-dimensional Boltzmann equation, which is prohibitively expensive to compute. Alternatively, the third generation model employs discrete interaction approximation (DIA) [27] for evaluating the Boltzmann integral. The DIA computation is described in Appendix C. This study used Laplacian filters in spectral space, which is the only difference from the original DIA computation. Triads only occur in shallow water but not in deep water [28]. Eldeberky has proposed the Lumped-Triad-Approximation (LTA) [29] to model triad interactions. The LTA parametrization for biphase used spectral energy and water depth to match the Beji & Battjes' laboratory experiments [30]. However, we found that LTA generates too much dissipation in depth-induced breaking validation cases and might be better if turned off in some cases. Appendix C has a full description of LTA due to the triad interactions for phase-average modeling. In very shallow water, depth-induced breaking is dominant among other dissipation mechanisms. Wave breaking (depth-induced) is a random process in wind-generated waves and is little understood. We have followed the modeling of depth-induced breaking based on the work of Battjes and Janssen [31]. They modeled the first single breaking wave and extended this result to random fields by applying the probability of breaking waves for a given depth [31,32]. Eldeberky [29] extended the bore model to spectral modeling after Beji and Battjes' experiments [30], which observed the evolution of a spectral shape to be almost the same for breaking and non-breaking waves. They concluded that breaking waves do not interfere with other mechanisms (such as triad interactions), and that the total energy dissipation does not influence the spectral shape but only the magnitude of the total energy [29]. This led them to model depth-induced wave breaking for spectral wave modeling (see Appendix C). The role of each of the physical source mechanisms differs according to the field of application such as shelf sea, near-shore, or harbor. We refer the interested readers to Battjes' review [9] on physical processes for a broad class of applications.

3. Numerical discretization

We will use Fourier-collocation for the spectral space and discontinuous Galerkin (DG) spectral element discretization for the geographical space with Strong Stability Preserving Runge–Kutta (SSP-RK) time discretization. Absorbing boundary conditions are considered for both coastal and frequency domain boundaries. The choice of upwind flux will eliminate the need for any special treatment for absorbing boundary condition in geographical space. Using the absorbing boundary layer (ABL) approach [33] in the frequency direction results in a solution leaving the truncated frequency domain with negligible reflections. We will present the ABL formulation for the action balance equation and we will also demonstrate its effectiveness. The directional spreading might be so narrow that evenly distributed Fourier-collocation points might be wasteful for a slowly changing region in the spectral direction. Using a mapping technique allows for local refinement in the directional axis; the arc-tan mapping [34] is used here.

3.1. Space discretizations

To achieve a hybrid spectral/DG scheme for spectral and geographical spaces, we must deal with effectively fourdimensional basis vectors which may render the computational cost prohibitive. At this point, it is useful to distinguish between the spectral and geographical spaces. Even if we have a four-dimensional problem, spectral ocean wave modeling simply superimposes many waves of different directions and frequencies in the geographical space. It is helpful to think of a fixed geographic location having a corresponding spectral space (such that $f(\theta, \sigma, X_0)$) or a fixed point in spectral space with associated geographical space (e.g. $f(\theta_0, \sigma_0, X)$) (see Fig. 1). The geographical space demands a complicated boundary in most practical applications (e.g., coastal lines) whereas the spectral space requires a very regular grid. In spectral spaces, the directional domain extends from $-\pi$ to π and the frequency from 0 to ∞ . A frequency domain is usually truncated to ($\sigma_{\min}, \sigma_{\max}$). Here, we propose a hybrid scheme with Fourier-collocation for spectral space derivatives and DG discretization for the geographical space. In this way, we can exploit the flexibility of unstructured spatial grid dealing with complex coastal lines using two-dimensional Jacobi polynomials (tensor-product expansion in a triangle [35]) and the efficiency of Fouriercollocation in spectral space.

Let us now define the following fluxes with the transformation of the frequency coordinate $\sigma \rightarrow \zeta = \pi \left[\frac{2(\sigma - \sigma_{\min})}{\sigma_{\max} - \sigma_{\min}}\right]$:

$$E = c_x N(\zeta, \theta, x, y, t), \quad F = c_y N(\zeta, \theta, x, y, t), \quad G = c_\theta N(\zeta, \theta, x, y, t), \quad H = c_\zeta N(\zeta, \theta, x, y, t).$$

With the above definitions, the action balance equation is written in flux form, i.e.,

$$\frac{\partial N}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial \theta} + J_{\zeta\sigma} \frac{\partial H}{\partial \zeta} = \frac{S}{\sigma},\tag{5}$$

where $J_{\zeta\sigma} = 2\pi/(\sigma_{max} - \sigma_{min})$. We use Fourier-collocation in a directional domain extending from $-\pi$ to π due to periodicity. However, a straightforward Fourier-collocation for the frequency direction is not possible since we are considering a truncated domain, which is non-periodic. The truncation ends should not reflect the solution back to a computational domain. Even if initial conditions are in general periodic (almost zero at $\sigma = \sigma_{min}$ and σ_{max}), the converged solutions will not necessarily be periodic due to source terms and propagation velocities. Fortunately, using the absorbing boundary layer approach [33] still allows us to employ Fourier-collocation in a frequency direction (see Section 3.4) without degrading the exponential spectral accuracy. Let us discretize the wave direction θ as

$$\begin{aligned} \theta_k &= -\pi + \frac{2\pi}{N}k \quad \forall k \in [0, 1, \dots, N-1], \\ \zeta_l &= -\pi + \frac{2\pi}{N}l \quad \forall l \in [0, 1, \dots, N-1]. \end{aligned}$$

Then using Lagrangian interpolation [36,37], we can compute

$$\left. \frac{\partial G}{\partial \theta} \right|_{\theta = \theta_k} = \sum_{j=0}^{N-1} D_{kj} G(\zeta, \theta_j, \mathbf{x}, \mathbf{y}, t), \tag{6}$$

where D_{kj} is the differentiation matrix for Fourier-collocation. Similarly, we obtain

$$\frac{\partial H}{\partial \zeta}\Big|_{\zeta=\zeta_l} = \sum_{j=0}^{N-1} D_{lj} H(\zeta_j, \theta, \mathbf{x}, \mathbf{y}, t).$$
⁽⁷⁾

Note that the derivative of both fluxes *G* and *F* can be computed by either Fast Fourier Transform (FFT) ($N\log(N)$) or by the matrix–vector product (N^2). We have implemented both methods for flux differentiations; however, we will observe a faster

code if we use the matrix method until the number of collocations is above 256. We have found that the matrix method still works faster than FFT in the range of a practical resolution. This study used the matrix method to obtain spectral space derivatives for all computations throughout the paper. The differentiation matrix [37] for an even number of collocation points (Lagrangian polynomial if q = 0) reads in a dummy coordinate z (either θ or ζ to obtain the above flux derivatives)

$$D_{ij}^{q} = \frac{2}{N} \sum_{n=0}^{N/2} \frac{1}{c_n} \begin{cases} (in)^{q} \cos[n(z_i - z_j)], \ q & \text{even}, \\ (in)^{q} \sin[n(z_i - z_j)], \ q & \text{odd}, \end{cases}$$
(8)

where $c_0 = c_{N/2} = 2$ and $c_n = 1$, otherwise. For an odd number of collocation points, the same differentiation matrix can be used by setting $c_n = 1$ for all values of *n*.

Inserting the collocation differentiation Eqs. (6) and (7) into the flux form of action Eq. (5), we obtain

$$\frac{\partial N}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \mathcal{L}(\theta_k, \zeta_l, x, y, t) = \mathbf{0},\tag{9}$$

where \mathcal{L} is

$$\mathcal{L}(\theta_k,\zeta_l,\mathbf{x},\mathbf{y},t) = \frac{\partial G}{\partial \theta}\Big|_{\theta=\theta_k} + J_{\zeta\sigma} \frac{\partial H}{\partial \zeta}\Big|_{\zeta=\zeta_l} - S(\sigma_k,\theta_l,\mathbf{x},\mathbf{y},t).$$
(10)

The discontinuous Galerkin statement now reads: Find an approximate solution N_h in the piecewise polynomial space $V_h(\Omega)$, defined as $(V_h(\Omega) \in P^k(\Omega))$ where $P^k(\Omega)$ denotes the collection of polynomials of degree up to k in the element Ω such that

$$\int_{\Omega} \frac{\partial N_h}{\partial t} \nu_h d\Omega + \int_{\Omega} \frac{\partial E_h}{\partial x} \nu_h d\Omega + \int_{\Omega} \frac{\partial F_h}{\partial y} \nu_h d\Omega + \int_{\Omega} \mathcal{L}(\theta_k, \zeta_l, x, y, t) \nu_h d\Omega = 0, \quad \forall \nu_h \in V_h,$$
(11)

where Ω is an arbitrary triangle element, and \mathcal{L} is sum of spectral space derivatives and physical source terms which is defined above. Applying integration by parts, we obtain

$$\int_{\Omega} \frac{\partial N_h}{\partial t} \nu_h d\Omega - \int_{\Omega} E_h \frac{\partial \nu_h}{\partial x} d\Omega - \int_{\Omega} F_h \frac{\partial \nu_h}{\partial y} d\Omega + \int_{\partial \Omega} E_h \nu_h \cdot n dS + \int_{\partial \Omega} F_h \nu_h \cdot n dS + \int_{\Omega} \mathcal{L}(\theta_k, \zeta_l, x, y, t) \nu_h d\Omega = 0.$$
(12)

The high-order expansions in triangular elements are given in detail Appendix B.1. Inserting the approximate solution $N_h(N_h \in V_h)$ and test function v_h in Eq. (B.9b) using Jacobi polynomials ψ_p and ψ_{pq} defined in Eq. (B.10) into Eq. (B.8), we obtain (B.11). The boundary flux term in Eq. (12) can be replaced by various numerical fluxes [38]; here the upwind flux is employed for simplicity but other choices will be explored in future work.

The discretized equations are now reduced to an Ordinary Differential Equation (ODE) for the expansion coefficients u_{pq}

$$\frac{du_{pq}(t)}{dt} = \frac{1}{J_{x\xi}} \mathbf{M}_{pq,pq}^{-1} \big(\mathbf{S}_{pq} + \mathbf{T}_{pq} - \mathbf{E}_{pq} - \mathbf{F}_{pq} - \mathbf{L}_{pq} \big), \tag{13}$$

where the mass matrix **M**, the stiffness components **S** and **T**, the edge fluxes **E** and **F**, and numerical source term **L** are, respectively, the result of integrated terms in Eq. (B.11). Each individual term is explicitly given in Appendix B.3. In summary, we use mixed Gauss–Lobatto/Gauss–Radau quadratures to compute each integration term.

3.2. Time discretization

We employ explicit Strong Stability Preserving Runge–Kutta (SSP-RK) time discretization to discretize equation (13) in time. This study employed the second and third order explicit SSP-RK methods proposed in [39], and the fourth order SSP-RK (5-stages) given in [40]. The SSP-RK schemes used in this study are presented in Section B.4.

3.3. Mapping in the spectral direction

The directional spectrum is not known a priori for many cases and it is usually proposed in a specific form. The form suggested in [41] is

$$D(\theta) = \begin{cases} A_1 \cos^m(\theta) & \text{for } |\theta| \leq 90^\circ, \\ 0 & \text{for } |\theta| > 90^\circ, \end{cases}$$
(14)

where the amplitude A_1 is defined in terms of Gamma functions $A_1 = \Gamma(\frac{1}{2}m+1)/[\Gamma(\frac{1}{2}(m+1))\sqrt{\pi}]$ and the parameter '*m*' is the width of the directional spectrum; the larger (smaller) values of the width parameter *m* generate a narrower (wider) width. The parameter '*m*' might vary from 2 to 800 in practical applications. In this work, we use the narrow directional spectrum $\cos^{500}(\theta)$ to describe incoming waves for test cases in the verification and validation Section 5.

The spectral width can be defined by an analogy to the standard deviation definition

$$\sigma_{\theta}^{2} = \int_{-\pi}^{\pi} \theta^{2} D(\theta) d\theta \tag{15}$$



Fig. 2. (a) The derivative of a $\cos^{500}(\theta)$ directional spreading ($A_1 = 1$) where the clustered region is confined to the interval $[-7.5^\circ, 7.5^\circ]$ which only makes up 4% of the whole domain 2π . (b) pointwise errors for standard Fourier-collocation, (c) mapped Fourier-collocation with ($L = 2\sigma_{\theta}$), and (d) with ($L = 4\sigma_{\theta}$) (d) for various resolutions. The spectral width for $\cos^{500}(\theta)$ is 2.5°.

and is used as a guide to determine the value of the directional width. For example, if the width parameter m = 500 then the spectral width $\sigma_{\theta} = 2.5^{\circ}$. For this specific example, the region $[-7.5^{\circ}, 7.5^{\circ}]$ would cover 99% of the total energy in Fig. 2(a). It is obvious that using the Fourier-collocation points on regions $[-180^{\circ}, -7.5^{\circ}]$ and $(7.5^{\circ}, 180^{\circ}]$ would not improve the solution significantly. However, we have no choice if we are using standard Fourier expansions, which require evenly distributed collocation points in the domain $[-\pi, \pi]$. The mapping technique is an implicit way of modifying the expansion functions and by allowing a collocation grid clustered in a specific region. The *atan mapping* [34,42] can be defined in $[-\pi, \pi]$ for the spectral direction

$$\theta = g(\chi, L) = 2 \arctan[L \tan(0.5\chi)] + \chi_0, \tag{16}$$

where χ denotes computational space which permits the standard Fourier-collocation method for differentiation, and the constant parameter *L* (in general *L*(*t*)) controls the amount of clustering around the peak center $\chi_0(\chi_0(t))$. Using a change of derivative coordinate transformation for the mapping relation in Eq. (16), we obtain the transformation between the physical θ and computational coordinates χ as follows:

$$g' = \frac{d\theta}{d\chi} = \frac{L\left[1 + \tan^2\left(\frac{\chi}{2}\right)\right]}{1 + L^2 \tan^2\left(\frac{\chi}{2}\right)} \tag{17}$$

and the directional differentiation of flux is

$$\frac{\partial G}{\partial \theta} = \frac{d\chi}{d\theta} \frac{\partial G}{\partial \chi} = \frac{1}{g'} \frac{\partial G}{\partial \chi},\tag{18}$$

where the differentiation matrix in Eq. (8) can be used to differentiate *G* with respect to χ . If we define the diagonal matrix M as

$$\mathcal{M}_{jj} = \frac{1}{g'},\tag{19}$$

then, the flux derivative $(\partial G/\partial \theta)$ in which the collocation grids are clustered is now computed by a new differentiation matrix, \mathcal{D}^1 :

$$\mathcal{D}^1 = \mathcal{M} \mathcal{D}^1. \tag{20}$$

The clustering collocation grids in θ , and the transformation metrics, g', may be computed by using a user-specified clustering parameter *L* and standard Fourier-collocation grids (equi-spaced distribution, $-\pi + \frac{2\pi}{N}k$, k = [0, ..., N - 1]). The mapped differentiation matrix is evaluated once and it is stored for all computations in fixed collocation grids (non-adaptive cases).

We specified the directional spreading of incoming waves as $\cos^{500}(\theta)$ (see Fig. 2(a)) for a verification test later in this paper. In this section, we will present derivative results of this specific directional spreading form. The standard Fourier expansion wastes collocation points in a very a smooth region of this example, such that very smooth region will require about 96% of all the collocation points. In Fig. 2, we compare equi-spaced standard Fourier differentiation with the modified Fourier differentiation with mapping. It is worth mentioning that the standard method also shows Gibbs-like oscillations if the high-gradient region is poorly resolved for this specific example. We have tried uniform collocation grids with N = 64, 128, 256, 512 and examine the pointwise errors ($u - u_N = \frac{dD(\theta)}{d\theta} - \frac{dD(\theta)_N}{d\theta}$, see Fig. 2(b)). This clearly shows that once the solution is resolved, the Fourier method achieves excellent spectral convergence behavior. Fig. 2(c) and (d) present pointwise errors for two different constant parameters, e.g., $L = 2\sigma_{\theta}$ and $L = 4\sigma_{\theta}$. We see that the pointwise errors are decreasing exponentially for both cases. These comparisons reveal that the precise value of mapping parameter *L* does not seem to be playing an important role (see N = 16 resolutions from Fig. 2(c) and (d)) for the coarse collocation grids but this might become important as the grid resolution increases. The choice of the parameter *L* should still be based on the spectral width σ_{θ} ; we suggest that at least $L = 3\sigma_{\theta}$. The differentiation with mappings have achieved *comparable error levels* of the standard method with N = 128 (see Fig. 2(b), (c) and (d)) in the region where a high-gradient solution occurs and *much smaller errors* away from this region, with an *eightfold* reduction in resolution.

3.4. Absorbing boundary treatment in frequency direction

Our present problem has a semi-infinite $([0,\infty])$ domain, and solutions decay exponentially fast as $\sigma \to \infty$ and $\sigma \to 0$. The initial conditions at the truncated boundaries $(N(\theta, \sigma_{\min}, x, y)$ and $N(\theta, \sigma_{\max}, x, y))$ for real applications after the truncation is made are very small $(N(\sigma_{\min}) \neq N(\sigma_{\max}) \ll 1)$. The periodicity of a function is in the literal sense violated, and ideally we should use for this case Chebyshev method instead of Fourier-collocation. However, Boyd's work [43] has shown that Fourier-collocation is more efficient than the Chebyshev method even in truncated domains in the case of very small domain truncation errors if there is exponential decay of the solution. Ocean wave solutions truly satisfy both conditions such that there are very small truncation errors and exponentially decaying solutions. This is the main reason to still use Fourier-collocation rather than Chebyshev methods. The true boundary conditions for a truncated domain would be non-reflecting boundary conditions permitting incoming waves without any reflection back to the computational domain. We have used the Absorbing Boundary Layer (ABL) approach [33] to satisfy non-reflective boundary conditions in the frequency direction. This approach will *enforce periodicity* for even *a non-periodic developing solution* at the truncated domain boundaries. The combination of Fourier-collocation and ABL is preferable for imposing non-reflective boundary condition along the frequency direction.

In order to impose the ABL approach, we add the term $g(\sigma) N(\theta; x, y, t)$ to the original equation Eq. (1), and hence the new modified action balance equation is

$$\frac{\partial N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial t} + \frac{\partial c_{g,\mathbf{x}}N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial \mathbf{x}} + \frac{\partial c_{g,\mathbf{y}}N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial \mathbf{y}} + \frac{\partial c_{\theta}N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial \theta} + \frac{\partial c_{\sigma}N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}{\partial \sigma} + \underbrace{g(\sigma)*N(\theta,\sigma;\mathbf{x},\mathbf{y},t)}_{\text{modification term}}$$

$$= \frac{S(\sigma,\theta;\mathbf{x},\mathbf{y},t)}{\sigma}, \qquad (21)$$

where $\mathbf{g}(\sigma)$ is the absorbing function, achieving zero values inside the domain and non-zero values in the absorbing layers, hence damping the numerical solution in these regions. The absorbing function can be defined in the computational domain ζ in $[-\pi, \pi]$ as follows:

$$g(\zeta) = \gamma_L (1 - \tanh[\alpha_L(\zeta + \pi)]) + \gamma_R (1 + \tanh[\alpha_R(\zeta - \pi)]),$$
(22)

where γ controls the magnitude of the absorbing functions, α determines the effective width of the absorbing layers, and *L* and *R* denote, respectively, the left and right layers. The term \mathcal{L} in Eq. (10) is now modified as

$$\mathcal{L}(\theta_k,\zeta_l,\mathbf{x},\mathbf{y},t) = \frac{\partial G}{\partial \theta}\Big|_{\theta=\theta_k} + J_{\zeta\sigma} \frac{\partial H}{\partial \zeta}\Big|_{\zeta=\zeta_l} + g(\sigma_l)N(\theta_k,\sigma_l,\mathbf{x},\mathbf{y},t) - S(\theta_l,\sigma_k,\mathbf{x},\mathbf{y},t).$$
(23)

The parameter couples α_L and α_R control the ABL widths. The computation of controlling parameters is possible by introducing an arbitrary cut-off values ϵ . For certain cut-off values, we can relate the α_L values to the absorbing boundary layer width $\Delta\zeta_L$ as

$$\alpha_L = \frac{\operatorname{atanh}(1 - \epsilon/\gamma_L)}{\Delta\zeta_L}.$$
(24)

In a similar way we have for α_R

$$\alpha_R = \frac{\operatorname{atanh}(1 - \epsilon/\gamma_R)}{\Delta\zeta_R},\tag{25}$$

where $\Delta \zeta_L$ and $\Delta \zeta_R$ are user-defined and represent the left and right-absorbing boundary layer width.

Next we will demonstrate propagation of the JONSWAP spectrum (enforced as initial condition) [44] at a truncated domain. The boundary conditions will be simulated by the ABL method. The propagation velocities c_x , c_y , c_θ are set to zero, and c_σ is set to 2π . The physical domain (now 1D frequency direction) extends from 0.2 Hz to 2.0 Hz. The left and right absorbing



Fig. 3. Evolution of the JONSWAP spectrum for non-reflecting boundary conditions. The analytical solution (-) and numerical solution (--) are compared at t = 0, 1.5, 1.625, 1.75, 1.8, 1.9.

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4. Verification of temporal and spatial accuracy

We will present (accumulative) errors but for a fixed domain and fixed time for all cases in the norms L_2 and L_{∞} defined here as follows:

$$\|\boldsymbol{e}\|_{L_2}^2 = \int_t \int_{\theta} \int_{\sigma} \int_{\Omega} (\boldsymbol{u} - \hat{\boldsymbol{u}})^2 d\Omega d\sigma d\theta dt,$$
(26)

where u and \hat{u} are, respectively, the exact and numerical solutions. The maximum norm L_{∞} is defined as

$$\|\boldsymbol{e}\|_{L_{\infty}} = \int_{t} \max |\boldsymbol{u} - \hat{\boldsymbol{u}}| dt.$$
(27)

4.1. Temporal accuracy verification

We simplify the equation by dropping the source term and setting the propagation velocities $(c_x, c_y, c_{\theta}, c_{\sigma})$ to one in Eq. (1). We propose the analytical function

$$N(\sigma, \theta; x, y, t) = \sin(x - t)\cos(y - t)\sin(\theta - t)\cos(\sigma - t),$$
(28)

which satisfies the action balance equation after the above simplifications. We have generated boundary and initial conditions from the manufactured solution Eq. (28). The geographical space discretizations using the mesh (see Fig. 4(a)) employs



Fig. 4. (a) Meshes used in Runge–Kutta (RK) discretization verifications: (b), (c) and (d) The sequence of meshes used in verification of space discretizations, respectively, (h,h/2,h/4) where *h* denotes the side of triangle. Each triangle has collocation points (ot shown here) and corresponding degrees of freedom.

L2 error for RK2, RK3, and RK4 at time = 1.0. 12th order Jacobi polynomials (in geographical space) and 8 Fourier-collocation points (in spectral space) are used.								
Δt	Error in RK2	Rate	Error in RK3	Rate	Error in RK4	Rate		
0.01	8.13787e-02	2.0	2.48741e-03	3.02	4.92365e-05	4.13		
0.005	2.03336e-02	2.0	3.07691e-04	3.01	2.82155e-06	3.87		
0.0025	5.08308e-03	-	3.83024e-05	-	1.92636e-07	-		

12th order of Jacobi polynomials to eliminate the domination of spatial discretization errors with eight collocation grid points in both spectral directions (θ and σ). Our simulation runs up to time unit 1.0 for second, third, and fourth-order SSP-RK schemes. The numerical tests, see Table 1, confirm the order of accuracy expected by SSP-RK time discretizations.

4.2. Spatial accuracy verification

The spatial errors have contributions from both the Fourier-collocation and DG spectral element discretization. We will show that the proposed scheme has spectral accuracy in both spaces. We also want to investigate the effective convergence



Fig. 5. L_2 errors for the sequence of meshes (h,h/2,h/4) for the domain $[-\pi,\pi] \times [-\pi,\pi]$. Frequency (σ) and Directional (σ) collocation resolutions 4, 8, 16, 24 are tested against polynomial order *P*. The errors are integrated over simulation time $t \in [0, 1]$ with the SSP-RK4 scheme (Δt = 0.001).

Table 2

 L_2 and L_∞ errors and convergence rates ($N_\theta = 24$, $N_\sigma = 24$, $\Delta t = 0.001$) between meshes (h/2) and (h/4). The errors are integrated over simulation time $t \in [0,1]$ using the RK4 scheme. The rates * hit either/both time-related or/and spectral space-related discretization.

р	L_{∞}			L_2		
	h/2	h/4	Rate	h/2	h/4	Rate
1	1.18308e+01	4.36525e+00	1.44	3.00660e+00	9.70100e-01	1.63
3	8.80192e-01	7.60808e-02	3.53	1.79565e-01	1.35042e-02	3.73
5	4.08564e-02	8.71108e-04	5.55	8.20653e-03	1.23181e-04	6.06
7	1.71219e-03	7.40512e-06	7.85	2.68193e-04	9.08277e-07	8.21
9	6.23888e-05	7.24080e-08	9.75	7.36509e-06	7.72374e-09	9.90
11	1.98009e-06	2.13654e-08	6.53*	1.79989e-07	3.15963e-09	5.83*

order in spectral space. The manufactured solution proposed before in Eq. (28) will converge too rapidly to very low levels of errors as we increase the number of collocations in spectral space. Hence, we introduce another manufactured solution to better quantify the exponential convergence in spectral space, i.e.,

$$N(\theta,\sigma;\mathbf{x},\mathbf{y},t) = e^{\sin(x-t)} + e^{\cos(y-t)} + e^{\sin(\theta-t)} + e^{\cos(\sigma-t)}.$$
(29)

We run the code for a sequence of meshes presented in Fig. 4(b), (c) and (d) for a varied order of Jacobi polynomials (*p*-convergence test) with the SSP-RK4 scheme in a simulation using time step size $\Delta t = 0.001$ up to final time t = 1. The fourthorder time scheme yields the L_2 error of time discretization about 1e–9. Each curve in Fig. 5 represents the L_2 error for a specific resolution of Fourier-collocation and mesh with a varying order of Jacobi polynomials. We made the time discretization errors sufficiently low that we can expect exponential convergence in the geographical scheme by increasing the order of Jacobi polynomials, provided that we have sufficiently low Fourier-collocation discretization errors in spectral space. By inspecting Fig. 5(a), we observe exponential convergence of the DG scheme for a specific spectral space discretization $(N_{\theta} = 24, N_{\sigma} = 24)$. All other figures have consistently confirmed the exponential accuracy of the DG related discretization scheme if the number of spectral space discretization errors is kept low. We see that the L_2 error curves eventually hit a plateau in Fig. 5. This indicates another error barrier due to Fourier-collocation discretizations. The DG discretization errors are still decreasing but they are comparable to the Fourier-collocation errors. The exponential accuracy for Fourier-collocation (spectral space discretization) can be examined on these plateau regions. The errors of these plateau levels can be used to quantify the convergence behavior of spectral space discretizations (Fourier-collocation).

Another way to quantify convergence is to study *h*-convergence. The convergence rate of the DG scheme can be established by evaluating the rate of error with respect to the refined meshes (see Fig. 4(b), (c) and (d)). We run simulations up to final time 1.0 with the time step Δt = 0.001 by using the SSP-RK4 time scheme and 24 Fourier-collocations in spectral space (N_{θ} = 24, N_{σ} = 24). The convergence rates given in Table 2 confirm the accuracy (about (p + 1)th order) of a DG scheme up to polynomial order 9. It is also clear that we reach either time or Fourier-collocation discretization errors for the Jacobi polynomial of order 11 (p = 11).

5. Verification and validation cases

5.1. Case I: numerical diffusion in deep water (S = 0)

We adopt the deep water diffusion test from Booij et al. [19] and compare our scheme to several other approaches using first- and third-order finite difference, finite element (FEM), and finite volume (FV) schemes. Incoming long-crested waves through the bottom-left corner with a gap $150\sqrt{2}$ m have been prescribed at an angle of 45° in a domain $10 \text{ km} \times 10 \text{ km}$ (see Fig. 6(a)). The mean wave direction $\bar{\theta}$ is 45° for this specific test problem. The incoming wave is assumed to be a Gaussian-shape in frequency direction and directional spectra form as $\cos^{m}\theta$. The two-dimensional energy spectrum prescribed at the corner of the bottom-left boundary is

$$E(\theta,\sigma) = A_1 \cos^m(\theta - \bar{\theta}) \frac{E_{\text{tot}}}{16\sqrt{2\pi}\sigma_0} \exp\left[-\frac{(\sigma - \sigma_{pk})^2}{2\sigma_0^2}\right],\tag{30}$$

where we specified the standard deviation σ_0 as 0.01 Hz and the directional parameter *m* as 500 (the resultant spectral width $\sigma_0 = 2.5^\circ$) for significant wave height $H_s \ 1 \ m (E_{tot} = 4\sqrt{H_s})$ at a peak frequency $\sigma_{pk} = 0.1$ Hz.

We used two different resolution meshes, as shown in Fig. 6, with a varying order of Jacobi polynomials over the elements (p = 2, 4, 6, 8). We employed fifteen collocation points in $[-\pi, \pi]$ clustered around the directional interval $(\bar{\theta} - 3\sigma_{\theta})$ and $(\bar{\theta} + 3\sigma_{\theta})$. For a frequency domain in [0.07–0.13] Hz (with Gaussian distribution, $\sigma_{pk} \pm 3\sigma_0$), we used 16 collocation points.

This same study was conducted for a test of first- and third-order SWAN finite-difference ([19]), second-order finite element method (FEM) [16], and second-order finite volume (FV) [17] in the past. We present here comparisons from Qi et al.'s work [17] in Fig. 7. Among them, it is clear that the first-order SWAN code is the most diffusive scheme for even a relatively short distance (10 km). The other schemes' relatively higher-order than the first-order SWAN scheme generated much less



Fig. 6. Case I: diffusion test meshes.



Fig. 7. Significant wave heights (a) SWAN first order scheme (FD scheme, 10,000 grid points) (b) SWAN third order scheme (FD scheme, 10,000 pts) (c) FE-WAVE (about 20,000 triangles) (d) FVCOM-SWAVE (about 20,000 triangles) (reproduced with permission from [17]).

diffusion, outperforming the first-order scheme (see Fig. 7). We note that this comparison may not be fair since it used the about same grid resolution but different order methods without taking into account the computational cost.

Our simulation results are given for the coarse and fine meshes in Figs. 8 and 9. We have obtained a steady-state solution using the integration scheme SSP-RK2. The relative computational work load can be approximately computed between two schemes A and B by the relation $\eta_{AB} = (P_A + 1)^2/(P_B + 1)^2$, where P denotes the polynomial order of a scheme used on an

element. Our cases, either in coarse or fine mesh, performed better than all the other schemes in Fig. 7. The computational work of our scheme in the runs is either comparable or much less (up to 5 times cheaper in spatial space) than other schemes.

Among the results in Fig. 7, the third-order scheme is considered superior to all other schemes since it is the highest-order accuracy scheme in Qi et al.'s work [17]. The third-order scheme will be much less dissipative and more accurate than the



Fig. 8. Case I: significant heights (Hs) for a mesh (122 triangles) for two different polynomial orders.



Fig. 9. Case I: significant height (Hs) for a mesh (966 triangles) for three different polynomial orders.

first or second-order schemes. None of the contour patterns in Fig. 7 exactly matched each other (by focusing on a specific contour). However, looking at a specific contour level (0.4), our present results closely agree with those of the third-order SWAN scheme. As for the FV scheme, the corresponding contour level is slightly higher than 0.4.

5.2. Case II: current-induced shoaling and retractions in deep water (S = 0)

These cases are indented to test the scheme for waves of variable currents for deep water. We specified Gaussian shape energy spectra (30) as incoming waves (with $T_{\text{peak}} = 10$ s, $H_s = 1$ m) on the west boundary of a rectangular domain 4 × 10 km (see Fig. 11). A current-induced shoaling will occur if the incoming waves encounter a current going in the same or opposite directions but not a refraction (mean wave direction $\bar{\theta} = 0$), which is schematically shown in Fig. 10(a). A current-induced refraction additionally happens with rotating both incoming waves with ±30° and the current vectors with ±90° (up and down) with respect to positive coordinate *x*, which is also illustrated in Fig. 10(b). The computational grid given in Fig. 11 used 70 triangles and fourth-order Jacobi polynomials (p = 4) over the elements along with 16 directions in [-15°, 15°] and 16 frequencies in [0.05, 0.25] Hz. Our results will be directly compared to the analytical results obtained from kinematic conservation of a wave number ($\sigma + k U = const$) [45] in the case of no refraction (cases (a) and (b)):



Fig. 10. Case II: a sketch of wave-current interaction setups for four test cases. The current velocity vector V_c has a linear distribution from 0 to 2 m/s.



Fig. 11. Case II: the computational mesh used for wave–current interaction tests. Fourth-order Jacobi polynomials (*p* = 4) are employed in each element for all simulations; the corresponding collocation points are not shown here.

$$\frac{c}{c_0} = \frac{1}{2} + \frac{1}{2} \left(1 + \frac{4U}{c_0} \right)^{1/2},$$
(31a)
$$\frac{H}{H_0} = \frac{c_0}{[c(a+2U)^{1/2}]},$$
(31b)

 $H_0 = [c(c+2U)]^{1/2}$, where *c* and *U* denote deep water propagation and current velocities, respectively. The quantities subscripted with 0

represent the incoming waves. The current velocity profiles shown in Fig. 10(a) and (b) vary linearly from 0 to 2 m/s for all cases. Longuet-Higgins and Stewart [46] extended the above analytical results for waves with a shearing current (cases (c) and (d)) as

• •

$$\sin \theta = \frac{\sin \theta_0}{\left[1 - \frac{U}{c_0} \sin \theta_0\right]^2},$$

$$\frac{H}{H_0} = \left[\frac{\sin 2\theta_0}{\sin 2\theta}\right]^{1/2}.$$
(32a)
(32b)

We present significant wave heights H_s for all four cases in Fig. 12. The numerical results for the significant wave heights are in good agreement with the above analytical results. The mean wave direction changes along the coordinate x only for cases (c) and (d) but not for the other cases ($\bar{\theta}(x) = 0^\circ$). Both the analytical and numerical results of mean wave direction are plotted in Fig. 13 and match very well along the coordinate x.

Fig. 12. Case II: comparison of significant wave height *H_s* versus distance from analytical solution and numerical simulations for all four cases of depth-induced shoaling and current-induced refraction (see the illustration of all cases in Fig. 10(a) and (b)).

 $\bar{\theta}^{(o)}$ $\bar{\theta}^{(o)}$ 20 3000 3500 1000 2000 2500 3000 3500 4000 4000 x(m)x(m)(a) case c (b) case d

Fig. 13. Case II: analytical and numerical results are compared along the coordinate x for cases (c) and (d) (see the illustration of the cases in Fig. 10(b)).





Fig. 14. Case III: depth-induced shoaling and refraction test case.



Fig. 15. Case III: the significant wave height (H_s) for case (a) $\bar{\theta}_0 = 0^\circ$ and case (b) $\bar{\theta}_0 = 30^\circ$.

5.3. Case III: depth-induced shoaling and refractions in shallow water (S = 0)

Shoaling and refractions may occur due to depth changes in coastal waters; we model these phenomena next. We impose the incoming waves specified in the previous section on the west boundary of a rectangular domain 4×10 km which has a linearly varying depth (from 20 m to 0.5 m) perpendicular to the coastal line. We consider two cases: (a) incident waves moving perpendicular to the coastal line, and (b) turning incident wave direction to 30° with respect to positive coordinate *x*; both test cases are sketched in Fig. 14. The same computational grid (both spatial, as shown in Fig. 11, and spectral space resolutions) as in the previous section is used but expanded with the Jacobi polynomial of seventh-order over the triangles. The numerical simulation results are compared to the solution of the linear wave theory for wave height as

$$\frac{H}{H_0} = \frac{c_0}{c} \frac{\cos \theta_0}{\cos \theta},\tag{33}$$

where the wave direction can be calculated by using Snell's law

$$\theta = \arcsin[c/c_0 \sin \theta_0]. \tag{34}$$

The significant wave heights are accurately predicted for incident waves moving perpendicular to the coastal line (case a) and turning incident waves to 30° with positive *x* (case b) in Fig. 15. The incoming wave directions perpendicular to the coastal lines (case a) are not refracted since there is no variation in coordinate *y*. The numerical prediction presented in Fig. 16 showed no refraction happening for case (a), which is consistent with intuition. The depth-induced refraction happens in case (b) where the incident wave turns over 30° . In the same figure, the mean wave direction deflecting toward the coastal lines is also predicted accurately by the numerical simulation.

5.4. Case IV: duration-limited growth in deep water ($S = S_{in} + S_{nl4} + S_{wc}$)

Duration-limited wave growth is simulated next over a single physical point at which a constant wind blows for a very long time. In deep water we have to include three source terms: wind generation, white-capping, and nonlinear wave-wave interactions, all of which contribute to the evolution of the wave. The wave energy is generated and sustained by wind and



Fig. 16. Case III: the mean wave direction $(\bar{\theta}^{(0)})$ for case (a) $\bar{\theta}_0 = 0^\circ$ and case (b) $\bar{\theta}_0 = 30^\circ$.

 Table 3

 Case IV: activated source parameters in duration-limited growth (left) and fetch-limited growth (right) test cases.

		Duration-limited	Fetch-limited
S _{in}	<i>u</i> 10	18.45 m/s	20 m/s
	Α	0.8	0.8
	В	0.065	0.065
Swc	Cwc	3.33e-5	2.36e-5
	n	2.0	0.0
	р	4.0	4.0
	Ĩ <i>₽</i> ₽М	0.0676	0.0549
S _{nl4}	λ	0.25	0.25
	C_{nl4}	3e7	3e7
	C _{sh1}	5.5	5.5
	C _{sh2}	0.85714	0.85714
	C _{sh3}	-1.25	-1.25

dissipated by white-capping and is redistributed energy among the frequencies by nonlinear wave-wave interactions. The physical space propagation velocities (c_x and c_y) are set to zero to mimic our solver as if it is behaving as a single point solver. Spectral space propagation velocities would be computed as zero since the spatial derivatives of depth and current are zero (there is no current in this case).

The duration-limited test case adopted from Jannsen [47] assumes initial sea conditions as the JONSWAP spectrum, which has a peak frequency of $f_p = 0.34$ Hz, the scale parameter $\alpha = 0.025$, and shape parameters $\gamma = 3$, σ_a and σ_b as 0.1. The directional spreading of $\cos^2\theta$ is specified with a directional resolution of 10°. The frequency distribution is specified by a geometric growth factor 0.05 ($f_i = 0.05f_{i+1}$) with an initial frequency of 0.0418 Hz (f_0) and a final frequency of 1.0 Hz (f_N). Note that the frequency domain collocation points are not evenly distributed because Fourier-collocation is not active for this problem (there is no need to differentiate among the spectral space derivatives). The wind speed (U) over the single point is chosen as 18.45 m/s.

We will compare the simulation results of our implementation to those of a single-grid-point version of ECWAM (European Center Wave Model) [6]. In order to do this, we use only one element with zero order of Jacobi polynomial (there is no need to resolve any spatial variation) and set the spatial propagation velocities as zero, which makes the code a single-point-grid solver if we interpret the solution at any point inside the single element (the solution will be constant over the element). The action balance equation is integrated from time t = 0.0 to 10^6 s using a SSP-RK4 integration scheme with time step size $\Delta t = 100$ s. The parameters of source terms wind generation S_{in} , white-cap dissipation S_{wc} , and nonlinear-wave interaction S_{nl4} used in this case are given in Table 3.

The non-dimensionalized energy growth is compared to the results of ECWAM, and the empirical data fit for durationlimited growth from the North Sea [48] in Fig. 17 along the non-dimensionalized time. Energy and time are non-dimensionalized with respect to gravity g and friction velocity u_* . Wave age non-dimensionalization also uses peak-propagation velocity cp(t). In Fig. 17, our numerical simulation has resolved the initial growth region much better because we have used a fourth-order accurate scheme with a much smaller time step size. The ECWAM code has an implicit integration capability allowing to perform time integration with a time step of 15 min.



Fig. 17. Case IV: dimensionless energy growth ($E^* = g^2 E_{tot}/u_*^4$) and time ($t^* = gt/u_*$) plot for the duration-limited wave growth. ECWAM (WAM modeling) and our numerical results are compared to the empirical fit (JKdV) by Jannsen et al. [48].



Fig. 18. Case IV: Dimensionless energy ($E^* = g^2 E_{tot}/u_*^4$) growth curve versus wave age ($\chi^* = cp(t)/u_*$) are presented for duration-limited growth. Wave age from left to right showing the young sea transition to old sea. The present numerical results [\blacksquare \blacksquare] are compared to ECWAM results [$\bullet \bullet \bullet \bullet$] and the empirical data (JKdV) [$__$] compiled by Jannsen et al. [48].

Toba [49,50] derived the 3/2 power law between significant wave height H_s and significant wave period T_p , which led to the dimensionless energy being proportional to the third power of the wave age. The dimensionless energy and wave age results are plotted in Fig. 18. Both numerical results from our implementation and ECWAM code are in agreement with empirical fits obtained by Jansen et al. [48].

5.5. Case V: fetch-limited growth in deep water ($S = S_{in} + S_{nl4} + S_{wc}$)

A steady wind blows perpendicular to the coastal line generating waves growing gradually over distance. The energy steadily grows until it reaches a balance between atmospheric wind input and dissipation through white-capping and energy redistribution through nonlinear wave-wave interactions. We impose specific spectral waves on the west boundary of mesh given in Fig. 19. Note that Fig. 19 is in logarithmic axis; the first element edge (in coordinate *x*) is 1 km and then the next edge grows as multiples of tenth of the initial edge (10 km, 100 km, etc.), which results in a triangulation mesh of a total of 10 elements. The directional spectra spread out between -90° and 90° with the frequency domain over [0.04-1] Hz, which has a frequency resolution due to geometrical growth factor 0.1 ($f_i = 0.1f_{i+1}$). The initial state of the domain is assumed to



Fig. 19. Case V: computational mesh plotted in logarithmic axis used for the fetch-limited wave growth test case.

be a calm sea (zero energy). We have a coarse resolution in the spectral domain due to the absence of current in deep water ($c_{\theta} = 0$ and $c_{\sigma} = 0$). The wind input S_{in} , nonlinear wave-wave interaction S_{nl4} , and white-capping dissipation S_{wc} parameters of this study are presented in Table 3.

The non-dimensional energy growth $E^* = g^2 E_{tot}/u_*^4$ over fetch $x^* = gx/u_*^2$ is plotted against the generalized energy growth formulas (B&H) [51] obtained from data sets over Lake George, Australia [52] and the data sets of the SWAMP Group [53] in Fig. 20. In the present simulations we ran different cases using second, fourth, and eighth-orders of Jacobi polynomials over the elements. The energy growth is captured reasonably well and agrees with both the observed data set (B&H) [51] and the (SWAMP) envelope results of various numerical ocean wave modeling codes [53].

Using a higher polynomial clearly resolved the wave growth for smaller fetches, as seen in Fig. 20. We used a constant polynomial order over elements which vary greatly from one to another in size. It would be more efficient to use variable polynomial orders based on the size of the element to obtain optimal errors. It is worth mentioning that the SWAN code



Fig. 20. Case V: energy growth curve versus fetch. $E^* = g^2 E_{tot}/u_*^4$ and $x^* = gx/u_*^2$. B&H (2006) empirical data fit and SWAMP Group (1985) and u_* is friction velocity.

[19], FVCOM-SWAVE [17] and FEM SWAN [16] used about three to four orders of magnitude finer meshes than the mesh of the current simulation in order to predict accurately the fetch-limited energy growth.

5.6. Case VI: depth-induced wave breaking $(S = S_{br})$

Depth-induced wave breaking is a dominant dissipation mechanism in shallow water. Here, we have used the modeling by Battjes and Janssen [31]. This study will compare the results from both SWAN and our simulation to those of the laboratory experiments conducted by Battjes and Janssen [31].

Incoming incident waves are generated on the left boundary of the experimental setup. The schematic representation is given in Fig. 21(b). We have assumed a JONSWAP spectrum with significant wave height (H_s) 0.202 m and a peak frequency (f_p) 0.53 Hz for the incident wave. The directional spreading is $\cos^{500}(\theta)$ narrow spectra distribution. A relatively coarse mesh (see Fig. 21(a) with the collocation points superimposed) with seventh-order Jacobi polynomials is used in geographical space; the directional and frequency derivatives, respectively, employed 32 collocation points in the domains [-15° , 15°] and [0.25, 3.125]Hz. The bathymetry is resolved accurately by interpolating the given data on all collocation points as seen in Fig. 21(a). In our implementation we only activated the depth-induced wave breaking source term S_{br} ; however, the SWAN code has additionally activated triad-wave interactions, which apparently led to slightly more dissipation, which will be



(a) mesh (24 elements) with collocation points represented by blue circles.





Fig. 21. Case VI: significant height variation in the Battjes and Janssen [31] experiment. SWAN and the present simulation used $\gamma = 0.73$ and $\alpha_{BJ} = 1$ in the depth-induced wave breaking source term.

investigated in the next test case. We used the breaker parameter γ as 0.73, which corresponds to an average value for different types of bathymetry [54].

We have simulated this case and compared the significant wave height H_s with SWAN and the experiment along coordinate x in Fig. 21(c). The significant wave height is slightly underpredicted for the first quarter of the submerged bar (0–5 m) due to the deactivated triad-wave interactions in our simulation (triad dissipations are not homogeneous). The remaining part of the experimental results is in good agreement with both SWAN and our simulation. The significant wave heights have not changed much in the trough region (9.92–17 m) where almost no breaking happens in the experiment (observed visually). Towards the end of the region, the experimental results show again an increase in significant wave height, which clearly shows that the rate of dissipation due to wave-breaking is slower than that of the potential energy (results in shoaling) of the waves [31] in very shallow water. Both numerical simulations using the same wave-breaking modeling introduce more dissipation toward the end of the experiment, unlike the results of the experiment in which they started rising again. In the limit of shallow water, the fraction of breaking waves Q_b will approach one and the wave breaking dissipation will decrease proportionally to h^2 [31]); such a dissipation decrease will pave the way for shoaling again.

5.7. Case VII: flume test case $(S = S_{br} + S_{nl3})$

Next we consider another experimental work by Wood et al. [55] and compare our method along with the SWAN code in case that the triad term is activated. The triad effects in depth-induced wave breaking are briefly mentioned in [56]. According to Ris' work [56], the activation of the triad term tends to slightly decrease the significant wave heights. Our results suggest that a decrease of significant wave height (more specifically, energy) is not homogeneous in the domain. We observed that the total energy has been more dissipated when the triads are active. However, we have additionally observed that the dissipation is not uniform along the coordinate *x* such that it might increase the significant wave height locally even though it introduced more dissipation in total.

The triad source term modeled by lumped-triad-approximation (LTA) is recommended not to be activated in the SWAN model in Wood et al.'s work [55] due to additional dissipation (other than depth-induced breaking) in the prediction of waves in a flume. We will simulate waves of Wood et al.'s experimental setup, whose side view is given in Fig. 22(a), where crosses (x) are the measurement gauges with three different bottom slopes, respectively, 1:12, 1:20, and 1:30. We use an extremely coarse mesh with only two triangular elements (see Fig. 22(b), mesh triangles with superimposed collocation points); Jacobi polynomials (p = 18) are employed in each element. The bathymetry is resolved typically by the vertices of the elements in the computational mesh in low-order methods. Hence, for a very coarse mesh as the one shown in Fig. 22(b), the bathymetry cannot be resolved accurately with low-order methods. In our approach, this is done accurately by employing interpolation of the bathymetry data at all collocation points. Therefore, the geometry is represented isoparametrically, i.e., with the same-order of accuracy as the field variables. In the presence of sharp gradients in the bathymetry, we use smaller elements (*h*-refinement) around this area, which together with high order polynomials (*h*-*p* refinement) is



(b) computational mesh (twoelements) with collocation points represented by blue circles.

Fig. 22. Case VII: the lateral view of the experimental setup (top) and computational mesh (below) is presented. The crosses (x) are experimental measurement locations. The mesh has only two elements and 18th-order of Jacobi polynomials were used over the elements.



(c) Significant Wave Heights (Hs)

Fig. 23. Case VII: significant height variations in the Wood et al. [55] experiment; the SWAN and present code results are compared. Here $\gamma = 0.73$ and $\alpha_{BJ} = 1$ in the depth-induced wave breaking source term. The triad proportionality coefficient α_{EB} 0.05 and 1.0 are used if the triad source term is activated.

more effective than simply p-refinement. The spectral space resolution has 32 collocation points for directional $[-15^{\circ}, 15^{\circ}]$ and frequency [0.2, 2.0] Hz domains.

The energy frequency distribution extracted from experimental data is noisy (the blue¹ dots in Fig. 23(a)), hence it needs to be smoothed through filtering. This data filtered is enforced at measurement location 1 as a boundary condition for the SWAN code in Wood et al. [55]. We specified a boundary spectrum different from Wood et al.'s work (the original data is not

¹ For interpretation of colour in Fig. 23, the reader is referred to the web version of this article.

available to us) such that the JONSWAP spectrum is enforced on the west boundary with a peak frequency 0.6 Hz, scale parameter α 0.0137, and shape parameters γ , σ_a and σ_b 3.3, 0.7 and 0.09, respectively. This way of imposing boundary conditions resulted in smooth energy spectra unlike those of SWAN in Fig. 23(a) and (b).

The energy spectra at wave gauges 1 and 12 are presented in Fig. 23(a) and (b). The energy spectra of both numerical simulations (the SWAN and our code) agreed best with the spectra of the experiment in Fig. 23(b) if the triads source terms (LTA) are turned off. It seems also that the experimental data does not show any first ($f = 2f_p = 1.2 \text{ Hz}$) or second ($f = 4f_p = 2.4 \text{ Hz}$) super-harmonics. However, the LTA model generates super-harmonics, which are apparent in our numerical and SWAN results in Fig. 23(b). Energy spectra for activated triad interactions in Fig. 23(b) have shown to be underestimating the energy spectra by both our simulation and the SWAN codes. The triad interactions transform energy distribution by extracting energy from the primary peak frequency and adding to its super-harmonic [41,57]. Our code simulates the lumped-triad approximation (LTA) of Eldeberky [57] (see blue or clay line results in Fig. 23(b)). The coefficients of LTA (α_{EB}) used in the SWAN input are not given explicitly in Wood et al. [55]. We used two different values of (α_{EB}) 0.05 and 1.0. The energy spectra showed no significant difference between the two different values of (α_{EB}) in Fig. 23(b). The larger values of coefficient α_{EB} are apparently generating slightly more dissipation.

Fig. 23c presents comparisons of significant wave heights for both numerical simulation and experimental measurements. The SWAN simulation *without triad interactions* matched better with experimental results than our simulation does after the point of the flume where the slope becomes zero. It seems that our depth-induced wave breaking introduced more dissipation after that point. However, once the triads are activated, the SWAN results are *more dissipative* in the ramp region than our results. As for the triad coefficients, the case of the largest value of (α_{EB}) 1.0 has shown increasing significant wave height from the bottom of the ramp to the middle of the slope. The rise of significant wave height in this portion of the experiment will cause further wave-breaking dissipation. We see that this case is more dissipated toward the end of the flume. Our numerical experiments agree with Wood et al. that lumped-triad approximation (LTA) is better to be excluded in the numerical model if depth-induced wave breaking dominates; otherwise, it will introduce more dissipation than necessary.

6. Summary and discussion

We have presented a high-order numerical method to solve the phase-averaged ocean wave equations. Accuracy verification tests have shown that the proposed scheme exhibits exponential convergence in both physical and spectral spaces. The high-order scheme results in minimal dissipation in the diffusion test case (case I) even when using a much coarser mesh than other available codes. Simulation results from the current-induced shoaling/refraction (case II) and the depth-induced shoaling/refraction (case III) tests agreed well with corresponding available analytical results. Again, these simulations only used a few high-order triangular elements. The duration-limited growth example (case IV) tested our solver as a single point solver and the result of this test matched well with available observations in the literature. The fetch-limited growth test (case V) admittedly was the toughest among the test cases because so far we have only implemented an explicit time integration scheme to reach a steady-state solution with a relatively very small time step. We have been able to run tests for various orders of polynomial order, demonstrating that with higher order polynomials the accuracy of the numerical solutions improved significantly. However, we have observed an apparent instability, which we attribute it to quadruplet interactions [20]; this was easily overcome by filtering. The depth-inducing break (case VI) and flume (case VII) tests validated our implementation against laboratory tests. Comparisons are also made to SWAN codes. One should not expect this implementation to be more accurate than other ocean models in a realistic simulation because modeling errors in the source terms may dominate the overall simulation. However, in the examples of a depth-induced test (case VI) using a very coarse mesh of 24 triangular high-order elements (p = 7) and a flume test (case VII) employing an extremely coarse mesh of two high-order elements (p = 18), we captured the wave height and energy spectra as accurately as in the SWAN code, which requires the use of at least three orders of magnitude finer resolution meshes. While more testing is required to appreciate the advantages of the new method, these two examples point to the efficiency of the scheme proposed here.

Next, we recommend several improvements of our method that will make it more robust and more suitable for largescale simulations. Although we have had good success for a range of problems, we have some limitations too.

The most important issue is the so-called Garden Sprinkler effect (GSE) [58–60] since it may be particularly pronounced for spectral discretization due to the minimal inherent dissipation of the scheme. GSE manifests itself as clustering of the energy in the geophysical space. This issue has been reviewed in [60], and also by Tolman in [59], who summarized available techniques to alleviate GSE for standard low-order methods. For spectral discretizations, we can also follow similar approaches in introducing selectively artificial dissipation to control GSE. Indeed, there has been great progress on this topic in the context of spectral methods in the last twenty years [35]. In particular, one effective approach is the use of spectral vanishing viscosity, introduced by Tadmor [61], and applied to geophysics problems in [62], and to turbulence simulations in [63]. A different approach is direct filtering of the modal coefficients [64], which eliminates the problem of solving the expensive diffusion problem implicitly. This is totally equivalent to adding diffusion selectively as shown in [37]. In our present scheme, GSE is implicitly controlled by using the DG formulation, which itself employs the correct characteristic information ("upwind-like"), similar in fact to the approach suggested by Janssen in [60].

We tried to validate our implementation against the wave-blocking test case [56,65], and initially smooth runs eventually blew up due to instability. This instability seems to be due to the spontaneous change in the energy-density profile (also

observed in Lai et al.'s work [65]) along the frequency direction. This sharp profile generates Gibbs-like oscillations in the spectral space, which if not properly treated render the simulation unstable. This can be overcome by local refinement adaptively. Note that if these oscillations are a result of discontinuities, which we do not have here, filtering would be an effective way to stabilize the scheme [37]. These oscillations provide a warning of poor resolution. We currently have fixed collocation points (no local refinement and adaptivity) in frequency directions. Future work should include *adaptivity in the frequency* direction.

We used an explicit RK scheme, which inhibits us from using large time steps due to the CFL condition. In particular, the size of time step in our scheme is restricted by the size of geophysical mesh triangles as well as the degree of polynomial expansion over the triangles or the frequency space resolution in the absence of current for deep water. However, in the case of shallow water, the CFL number is additionally restricted by the directional space mesh resolution. High gradients of the current field (either in deep water or shallow water) and of the depth (shallow water) yield large propagation velocities in frequency and direction space. In those cases, the CFL is more restricted by the spectral space discretizations. The stability of Fourier-collocation is related to minimal spacings ($\Delta \sigma_{min}$ or $\Delta \theta_{min}$). Resolving a sharp profile in a frequency direction (after local refinement) would yield very dense distribution of collocation points around the clustering region, which will strictly limit the scheme to very small time steps.

To this end, we recommend the use of *implicit treatment of spectral space discretizations* [66–69] or even better the use of a semi-Lagrangian scheme. Semi-Lagrangian schemes have been used in meteorology very effectively and also for direct numerical simulations of turbulence where the temporal scale is highly over-resolved at high Reynolds number using semi-implicit schemes [70,71].

All applications here have very simple boundaries. Coastal applications will need very complicated boundaries to be gridded. This implementation currently supports only fixed polynomial order elements. In order to exploit the high-order efficiency we can use large elements with high-order polynomials. This is possible for cases with simple boundaries as already seen in case VII (simulated with only two very high-order elements). The implementation with fixed polynomial order can still use very large elements on a very complex boundary by using curved boundaries such that the boundary element is represented as accurately as possible. Future implementation should handle *high-order curvature* on coastal boundaries. Another way to resolve complex boundaries is to use unstructured grid technology for generating very fine mesh on coastal lines, which eventually forces us to consider *an implicit scheme in physical space* as well as *p-adaptivity*. In realistic applications we will have triangular elements having large disparity in size, which require *p*-adaptivity for enhanced efficiency.

Finally, currently our code is parallel, which limits its use for large-scale simulations. However, parallelization of highorder methods typically leads to very high efficiency, e.g. using domain decomposition methods, due to the large volumeto-area ratio, see [72,73]. Additionally, directional splitting can be pursued in this multi-dimensional problem to enhance further parallel efficiency. This important issue is left for future work.

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Appendix A. Propagation of velocities in spectral space

The derivatives with respect to wave propagation direction (s) and its perpendicular coordinate (m) are related to Cartesian coordinate derivatives as follows:

$$\begin{bmatrix} \frac{\partial(\bullet)}{\partial s} \\ \frac{\partial(\bullet)}{\partial m} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \frac{\partial(\bullet)}{\partial x} \\ \frac{\partial(\bullet)}{\partial y} \end{bmatrix}.$$
 (A.1)

The inner product of a wave vector (\vec{k}) with partial derivatives of current (\vec{u}) with respect to the coordinate (s) is:

$$\vec{k} \cdot \frac{\partial \vec{u}}{\partial s} = k_x \frac{\partial u_x}{\partial s} + k_y \frac{\partial u_y}{\partial s}.$$
(A.2)

In a similar way, the inner product for the current with a derivative of coordinate (m) can be obtained.

The dispersion relationship between frequency (σ) and wave number *k* is

$$\sigma^2 = gk \tanh(kd). \tag{A.3}$$

Differentiation of the dispersion relation Eq. (A.3) with respect to depth (d) will give us frequency change in depth as

$$\frac{\partial \sigma}{\partial d} = \frac{k\sigma}{\sinh(2kd)}.$$
(A.4)

The propagation velocity (c_{θ}) in Eq. (4b) using Eqs. (A.1), (A.2), and (A.4) will take the following form in the Cartesian coordinate:

$$c_{\theta} = \frac{\sigma}{\sinh(2kd)} \left(\sin(\theta) \frac{\partial d}{\partial x} - \cos(\theta) \frac{\partial d}{\partial y} \right) + \sin^2(\theta) \frac{\partial u_y}{\partial x} + \sin(\theta) \cos(\theta) \left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} \right) - \cos^2(\theta) \frac{\partial u_x}{\partial y}, \tag{A.5}$$

while the propagation velocity along the frequency direction is:

$$c_{\sigma} = \frac{k\sigma}{\sinh(2kd)} \left(\frac{\partial d}{\partial t} + u_x \frac{\partial d}{\partial x} + u_y \frac{\partial d}{\partial y} \right) - c_g |\vec{k}| \left[\cos^2(\theta) \frac{\partial u_x}{\partial x} + \sin(\theta) \cos(\theta) \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + \sin^2(\theta) \frac{\partial u_y}{\partial y} \right].$$
(A.6)

The derivatives of depth and current velocity with respect to Cartesian coordinates have been computed using collocation differentiation [35] to ensure high-order accuracy.

Appendix B. Numerical implementation

In this appendix we present details of the spatial and temporal discretization so that the interested reader can follow our implementation more closely.

B.1. Expansions in unstructured domains

Eq. (12) can be transformed into the standard triangle region $\Omega_S(\mathcal{T}^2 = \{(\xi_1, \xi_2) | -1 \leq \xi_1, \xi_2; \xi_1 + \xi_2 \leq 0\})$ by using barycentric coordinates to readily evaluate the above integration terms. The transformation from (*x*, *y*) can be expressed in terms of the standard coordinate (ξ_1, ξ_2):

$$\binom{x}{y} = -\frac{\xi_1 + \xi_2}{2} \binom{x_1}{y_1} + \frac{1 + \xi_1}{2} \binom{x_2}{y_2} + \frac{1 + \xi_2}{2} \binom{x_3}{y_3},$$
(B.1)

where the subscripts 1,2,3 denote triangle vertices. The Jacobian of transformation is

$$\left[\mathcal{J}_{\mathbf{x}\xi}\right] = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \xi_1} & \frac{\partial \mathbf{y}}{\partial \xi_1} \\ \frac{\partial \mathbf{x}}{\partial \xi_2} & \frac{\partial \mathbf{y}}{\partial \xi_2} \end{bmatrix}.$$
 (B.2)

The spatial derivatives can be computed in terms of transformed ones such that

$$\begin{bmatrix} \frac{\partial(\bullet)}{\partial x} \\ \frac{\partial(\bullet)}{\partial y} \end{bmatrix} = \frac{1}{|\mathcal{J}_{x\xi}|} \begin{bmatrix} \frac{\partial y}{\partial \xi_2} & -\frac{\partial y}{\partial \xi_1} \\ -\frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_1} \end{bmatrix} \begin{bmatrix} \frac{\partial(\bullet)}{\partial \xi_1} \\ \frac{\partial(\bullet)}{\partial \xi_2} \end{bmatrix}.$$
(B.3)

The discontinuous Galerkin formulation in the standard region $\Omega_{\rm S}$ now reads

$$\int_{\Omega_{S}} \frac{\partial N_{h}}{\partial t} v_{h} |\mathcal{J}_{x\xi}| d\Omega_{S} - \int_{\Omega_{S}} E_{h} \left[\frac{\partial y}{\partial \xi_{2}} \frac{\partial v_{h}}{\partial \xi_{1}} - \frac{\partial y}{\partial \xi_{1}} \frac{\partial v_{h}}{\partial \xi_{2}} \right] d\Omega_{S} - \int_{\Omega_{S}} F_{h} \left[-\frac{\partial x}{\partial \xi_{2}} \frac{\partial v_{h}}{\partial \xi_{1}} + \frac{\partial x}{\partial \xi_{1}} \frac{\partial v_{h}}{\partial \xi_{2}} \right] d\Omega_{S} + \int_{\partial \Omega_{S}} E_{h} v_{h} \cdot ndS + \int_{\partial \Omega_{S}} F_{h} v_{h} \cdot ndS + \int_{\Omega_{S}} \mathcal{L}(\theta_{k}, \theta_{l}, x, y, t) v_{h} |\mathcal{J}_{x\xi}| d\Omega_{S} = 0,$$
(B.4)

where the standard region area is equal to the product of differential sides of the element as $d\Omega_5 = d\xi_1 d\xi_2$. The construction of two-dimensional expansion functions are obtained readily by the tensorial product of two one-dimensional expansion functions in a standard structured domain ($Q^2 = \{-1 \le \xi_1, \xi_2 \le 1\}$). A new coordinate system will be necessary for a triangle where the local coordinates are not orthogonal. Introducing a collapsed coordinate transformation allows us to construct two-dimensional expansion functions for a triangle region in a similar fashion as in a structured region. The collapsed transformation is defined as [35]

$$\eta_1 = \frac{2(1+\xi_1)}{1-\xi_2} - 1; \quad \eta_2 = \xi_2, \tag{B.5}$$



Fig. 24. Transformation from the standard triangle to rectangular region.

where η_1 and η_2 are new coordinates $-1 \leq \eta_1$, $\eta_2 \leq 1$, see Fig. 24, hence removing the coordinate dependence of bounds in a triangle domain $(-1 \leq \xi_1 + \xi_2 \leq 1)$.

The Jacobian of transformation $J_{\xi\eta}$ and its partial derivatives on a standard triangular coordinate (ξ_1, ξ_2) can now be written in terms of a new coordinate system (η_1, η_2)

$$\mathcal{J}_{\xi\eta} = \frac{1 - \eta_2}{2}, \tag{B.6}$$
$$\begin{bmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \end{bmatrix} = \frac{1}{\mathcal{J}_{\xi\eta}} \begin{bmatrix} \frac{\partial}{\partial \eta_1} \\ \frac{1 + \eta_1}{2} \frac{\partial}{\partial \eta_1} + \frac{1 - \eta_2}{2} \frac{\partial}{\partial \eta_2} \end{bmatrix}. \tag{B.7}$$

The discontinuous Galerkin formulation in Eq. (B.4) becomes in the new coordinate system:

$$\int_{\Omega_{c}} \frac{\partial N_{h}}{\partial t} \nu_{h} \mathcal{J}_{x\xi} \mathcal{J}_{\xi\eta} d\eta_{1} d\eta_{2} - \int_{\Omega_{c}} E_{h} \left[\left(\frac{\partial y}{\partial \xi_{2}} - \frac{\partial y}{\partial \xi_{1}} \frac{1 + \eta_{1}}{2} \right) \frac{\partial \nu_{h}}{\partial \eta_{1}} - \frac{\partial y}{\partial \xi_{1}} \frac{1 - \eta_{2}}{2} \frac{\partial \nu_{h}}{\partial \eta_{2}} \right] d\eta_{1} d\eta_{2} \\ - \int_{\Omega_{c}} F_{h} \left[\left(-\frac{\partial x}{\partial \xi_{2}} + \frac{\partial x}{\partial \xi_{1}} \frac{1 + \eta_{1}}{2} \right) \frac{\partial \nu_{h}}{\partial \eta_{1}} + \frac{\partial x}{\partial \xi_{1}} \frac{1 - \eta_{2}}{2} \frac{\partial \nu_{h}}{\partial \eta_{2}} \right] d\eta_{1} d\eta_{2} + \int_{\partial \Omega_{c}} E_{h} \nu_{h} \cdot ndS + \int_{\partial \Omega_{c}} F_{h} \nu_{h} \cdot ndS \\ + \int_{\Omega_{c}} \mathcal{L}(\theta_{k}, \theta_{l}, \mathbf{x}, \mathbf{y}, t) \nu_{h} \mathcal{J}_{x\xi} \mathcal{J}_{\xi\eta} d\eta_{1} d\eta_{2} = \mathbf{0}, \tag{B.8}$$

where a standard triangle region is defined in terms of new coordinates (η_1, η_2) as

$$\mathcal{T}^2\{(\eta_1,\eta_2)|-1\leqslant\eta_1,\eta_2\leqslant1\}.$$

We seek approximate solution N_h expanded by orthogonal expansional functions and choose test function v_h as the same orthogonal expansion functions (Galerkin projection), respectively,

$$N_h = \sum_{p=0}^{P_1} \sum_{q=0}^{P_2} u_{pq}(t) \psi_p(\eta_1) \psi_{pq}(\eta_2) \quad \text{and}$$
(B.9a)

$$\boldsymbol{\nu}_h = \boldsymbol{\psi}_m(\boldsymbol{\eta}_1)\boldsymbol{\psi}_{mn}(\boldsymbol{\eta}_2). \tag{B.9b}$$

The complete polynomial space for a triangular region will be $\{(pq)-0 \le p,q; p \le P1; p+q \le P_2; P_1 \le P_2\}$. The orthogonal expansion functions in terms of Jacobi polynomials are [35]:

$$\begin{split} \psi_{p}(\eta_{1}) &= P_{p}^{0,0}(\eta_{1}); \quad \psi_{m}(\eta_{1}) = P_{m}^{0,0}(\eta_{1}), \\ \psi_{pq}(\eta_{2}) &= \left(\frac{1-\eta_{2}}{2}\right)^{p} P_{q}^{2p+1,0}(\eta_{2}); \quad \psi_{mn}(\eta_{2}) = \left(\frac{1-\eta_{2}}{2}\right)^{m} P_{n}^{2m+1,0}(\eta_{2}) \end{split}$$
(B.10)

and hence:

$$\begin{aligned} \mathcal{J}_{x\xi} \sum_{p=0}^{P_{1}} \sum_{q=0}^{P_{2}} \frac{du_{pq}(t)}{dt} \int_{-1}^{1} \int_{-1}^{1} \psi_{p}(\eta_{1})\psi_{m}(\eta_{1})\psi_{pq}(\eta_{2})\psi_{mn}(\eta_{2})\mathcal{J}_{\xi\eta}d\eta_{1}d\eta_{2} \\ &- \int_{-1}^{1} \int_{-1}^{1} E(\theta_{k},\sigma_{l};\eta_{1},\eta_{2},t) * \left\{ \left(\frac{\partial y}{\partial\xi_{2}} - \frac{\partial y}{\partial\xi_{1}} \frac{1+\eta_{1}}{2} \right)\psi_{mn}(\eta_{2}) \frac{d\psi_{m}(\eta_{1})}{d\eta_{1}} - \frac{\partial y}{\partial\xi_{1}} \frac{1-\eta_{2}}{2}\psi_{m}(\eta_{1}) \frac{d\psi_{mn}(\eta_{2})}{d\eta_{2}} \right\} d\eta_{1}d\eta_{2} \\ &- \int_{-1}^{1} \int_{-1}^{1} F(\theta_{k},\sigma_{l};\eta_{1},\eta_{2},t)) * \left\{ \left(-\frac{\partial x}{\partial\xi_{2}} + \frac{\partial x}{\partial\xi_{1}} \frac{1+\eta_{1}}{2} \right)\psi_{mn}(\eta_{2}) \frac{d\psi_{m}(\eta_{1})}{d\eta_{1}} + \frac{\partial x}{\partial\xi_{1}} \frac{1-\eta_{2}}{2}\psi_{m}(\eta_{1}) \frac{d\psi_{mn}(\eta_{2})}{d\eta_{2}} \right\} d\eta_{1}d\eta_{2} \\ &+ \int_{\partial\Omega} E_{h} \upsilon_{h} \cdot ndS + \int_{\partial\Omega} F_{h} \upsilon_{h} \cdot ndS + \mathcal{J}_{x\xi} \int_{-1}^{1} \int_{-1}^{1} \mathcal{L}(\theta_{k},\theta_{l},\eta_{1},\eta_{2},t)\psi_{m}(\eta_{1})\psi_{mn}(\eta_{2})\mathcal{J}_{\xi\eta}d\eta_{1}d\eta_{2} = 0, \end{aligned}$$
(B.11)

where u_{pq} are the basis coefficients to be determined.

B.2. Forward and backward transformations

The initial condition for equation in Eq. (13) (only available in physical space) need to be transformed from physical to modal space, using a *forward transformation*. Similarly, the modal coefficients $u_{pq}(t)$ need to be transformed back to the physical variable of interest at various computation stages, using a *backward transformation*.

The approximate solution given in Eq. (B.9a) is expanded in terms of modal coefficients. We define the Vandermonde transformation matrix as

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$$V = \begin{bmatrix} \psi_{0}(\eta_{1}^{0})\psi_{00}(\eta_{2}^{0}) & \psi_{0}(\eta_{1}^{0})\psi_{01}(\eta_{2}^{0}) & \cdots & \psi_{1}(\eta_{1}^{0})\psi_{10}(\eta_{2}^{0}) & \cdots & \psi_{p}(\eta_{1}^{0})\psi_{pq}(\eta_{2}^{0}) \\ \psi_{0}(\eta_{1}^{0})\psi_{00}(\eta_{2}^{1}) & \psi_{0}(\eta_{1}^{0})\psi_{01}(\eta_{2}^{1}) & \cdots & \psi_{1}(\eta_{1}^{0})\psi_{10}(\eta_{2}^{1}) & \cdots & \psi_{p}(\eta_{1}^{0})\psi_{pq}(\eta_{2}^{1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_{0}(\eta_{1}^{1})\psi_{00}(\eta_{2}^{0}) & \psi_{0}(\eta_{1}^{1})\psi_{01}(\eta_{2}^{0}) & \cdots & \psi_{1}(\eta_{1}^{1})\psi_{10}(\eta_{2}^{0}) & \cdots & \psi_{p}(\eta_{1}^{1})\psi_{pq}(\eta_{2}^{0}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_{0}(\eta_{1}^{0})\psi_{00}(\eta_{2}^{0}) & \psi_{0}(\eta_{1}^{0})\psi_{01}(\eta_{2}^{0}) & \cdots & \psi_{1}(\eta_{1}^{0})\psi_{10}(\eta_{2}^{0}) & \cdots & \psi_{p}(\eta_{1}^{0})\psi_{pq}(\eta_{2}^{0}) \\ \end{bmatrix}_{[(ij),(pq)]},$$
(B.12)

where Q_1 and Q_2 are, respectively, the total quadrature points in the direction of η_1 and η_2 . In the triangle region, the complete space can be defined by only $N_p = \frac{(P_1+1)(P_2+2)}{2}$ polynomials rather than $N_P = (P_1 + 1)(P_2 + 2)$ as in the quadrilateral region, and the total number of quadrature points is a product of Q_1 and Q_2 as $N_Q = Q_1 * Q_2$. As mentioned in Appendix B.3, Q_1 and Q_2 should be chosen $(P_1 + 1)$ and $(P_2 + 1)$ for exact integration of the terms, respectively. The above Vandermonde matrix has row and column dimensions of N_Q and N_P and has a rectangular shape (not square matrix, $N_Q \neq N_P$) for the triangular region. The computation of the Vandermonde matrix is convenient to compute once and store it at the initial stage provided that the quadrature points have been fixed (*p*-adaptivity certainly will force us to adjust the quadrature points during the computation) for the entire simulation.

The backward transformation can now be represented as matrix-vector multiplication:

$$N_h = V u_{pq}. \tag{B.13}$$

The *forward transformation* (evaluating $\mathbf{u_{pq}}$ from $\mathbf{N_h}$) is not obvious from Eq. (B.13) because the Vandermonde matrix is not a square matrix and hence is not invertible in a traditional way. Using Singular Value Decomposition (SVD), the Vandermonde matrix in Eq. (B.13) is factorized as

$$\mathbf{V} = \mathcal{U}\mathbf{S}\mathcal{V}^{\mathrm{T}},\tag{B.14}$$

where \mathcal{U} is an $N_Q \times N_P$ unitary matrix, **S** is a min (N_Q, N_P) diagonal matrix, and \mathcal{V}^T is a transpose of $N_P \times N_P$ unitary matrix. We invert the above equation by using a property of unitary matrices ($\mathbf{A}^{-1} = \mathbf{A}^T$), and get

$$\mathbf{V}^{-1} = \mathbf{\mathcal{V}}\mathbf{S}^{-1}\mathbf{\mathcal{U}}^{T}.$$
(B.15)

Inversion of the diagonal matrix **S** and transpose of unitary matrix \mathcal{U} is a trivial task after computation of SVD; in this study, we have used the linear algebra library LAPACK [74]. The *forward transformation* can be now computed by using the inverse Vandermonde matrix as follows:

$$\mathbf{u}_{pq} = \mathbf{V}^{-1} \mathbf{N}_{\mathbf{h}}.$$
 (B.16)

B.3. Gauss quadrature integrations

The mass matrix **M** is the result of the following quadrature integration from Eq. (B.11) by using the orthogonality of Jacobi polynomials $(\int P_m(x)P_n(x)dx = 0, m \neq n)$:

$$\begin{split} \mathbf{M}_{mn,pq} &= \int_{-1}^{1} \int_{-1}^{1} P_{p}^{0,0} P_{m}^{0,0} \left(\frac{1-\eta_{2}}{2}\right)^{p} P_{q}^{2p+1,0} \left(\frac{1-\eta_{2}}{2}\right)^{m} P_{n}^{2m+1,0} \frac{1-\eta_{2}}{2} d\eta_{1} d\eta_{2}, \\ \mathbf{M}_{pn,pq} &= \frac{2}{1+2p} \int_{-1}^{1} \left(\frac{1-\eta_{2}}{2}\right)^{2p+1} P_{q}^{2p+1,0} P_{n}^{2p+1,0} d\eta_{2}, \\ \mathbf{M}_{pq,pq} &= \frac{2}{(1+2p)(1+p+q)}. \end{split}$$
(B.17)

The derivative of expansion functions Eq. (B.10) with respect to η_1 and η_2 directions are

$$\frac{d\psi_m(\eta_1)}{d\eta_1} = \frac{dP_m^{0,0}(\eta_1)}{d\eta_1},$$

$$\frac{d\psi_{mn}(\eta_2)}{d\eta_2} = -\frac{m}{2} \left(\frac{1-\eta_2}{2}\right)^{m-1} P_n^{2m+1,0}(\eta_2) + \left(\frac{1-\eta_2}{2}\right)^m \frac{dP_n^{2m+1,0}(\eta_2)}{d\eta_2}.$$
(B.18)

The stiffness components **S** and **T** are the result of integration-by-parts, and can be computed by using the relation (B.18) as follows:

$$\mathbf{S}_{pq} = \sum_{i=0}^{Q_{1}-1} w_{i} \left(\frac{\partial y}{\partial \xi_{2}} - \frac{\partial y}{\partial \xi_{1}} \frac{1+\eta_{1i}}{2} \right) \frac{d\psi_{p}(\eta_{1i})}{d\eta_{1}} \sum_{j=0}^{Q_{2}-1} w_{j} E(\theta_{k}, \sigma_{l}, \eta_{1i}, \eta_{2j}, t) \psi_{pq}(\eta_{2j}) \\ - \sum_{i=0}^{Q_{1}-1} w_{i} \frac{\partial y}{\partial \xi_{1}} \psi_{p}(\eta_{1i}) \sum_{j=0}^{Q_{2}-1} w_{j} E(\theta_{k}, \sigma_{l}, \eta_{1i}, \eta_{2j}, t) \frac{1-\eta_{2j}}{2} \frac{d\psi_{pq}(\eta_{2j})}{d\eta_{2}},$$
(B.19a)

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$$\begin{aligned} \mathbf{T}_{pq} &= \sum_{i=0}^{Q_{1}-1} w_{i} \left(-\frac{\partial \mathbf{x}}{\partial \xi_{2}} + \frac{\partial \mathbf{x}}{\partial \xi_{1}} \frac{1+\eta_{1i}}{2} \right) \frac{d\psi_{p}(\eta_{1i})}{d\eta_{1}} \sum_{j=0}^{Q_{2}-1} w_{j} F(\theta_{k}, \sigma_{l}, \eta_{1i}, \eta_{2j}, t) \psi_{pq}(\eta_{2j}) \\ &+ \sum_{i=0}^{Q_{1}-1} w_{i} \frac{\partial \mathbf{x}}{\partial \xi_{1}} \psi_{p}(\eta_{1i}) \sum_{j=0}^{Q_{2}-1} w_{j} F(\theta_{k}, \sigma_{l}, \eta_{1i}, \eta_{2j}, t) \frac{1-\eta_{2j}}{2} \frac{d\psi_{pq}(\eta_{2j})}{d\eta_{2}}, \end{aligned}$$
(B.19b)

where η_{1i} , η_{2j} are the quadrature points in the η_1 and η_2 directions and, whose corresponding weights are, respectively, w_i and w_j . We have still not mentioned which quadrature rule will be used in the above integrations. We have only one restriction which is not to choose quadratures as Gauss–Lobatto points in the one end of η_2 direction due to singularity; otherwise, we are free to choose any quadrature points. The integration terms in Eq. (B.19b) containing polynomial \mathbf{P}_{2p+1} are numerically exact if the quadrature points Q will be chosen Q = p + 1 for Gauss–Lobatto, Q = p + 1 (total quadrature points Q must be integer and Q = p + 1 is exact for \mathbf{P}_{2p+2}) for Gauss–Radau, and Q = p for Gauss–Jacobi quadratures. The Gauss–Lobatto quadratures are inside the interior domain (-1,1). It is obvious that choosing Gauss–Jacobi quadrature points is better if accuracy is the only consideration; namely, for a given polynomial order p, Gauss–Jacobi would use the least number of total quadrature points Q among the others. However, for ease in applying boundary conditions, it is advantageous to employ Gauss–Lobatto in η_1 and Gauss–Radau quadratures (excluding a singular point introduced by the collapsed coordinate transformation) in η_2 directions at the expense of a slightly increased computation cost. Gauss–Lobatto and Gauss–Radau quadratures are used for numerical integration of the terms in this study.

The integration by parts yields two boundary terms, the "edge flux terms" in Eq. (B.11), which need to be integrated over the edge of the elements. The numerical solutions are allowed to be discontinuous across elements in the domain so that multiple physical solutions occur at the element boundaries. This will be resolved by replacing the fluxes E_h and F_h with numerical fluxes such as H_h and G_h , respectively. The fluxes in a normal direction can be replaced as

$$E_h \cdot n_e = H_e \left(N_h^{int}, N_h^{ext} \right) \quad \text{and} \tag{B.20a}$$

$$F_h \cdot n_f = G_e \left(N_h^{int}, N_h^{ext} \right), \tag{B.20b}$$

where interior "*int*" and exterior "*ext*" will be interpreted differently according to a specific edge of element and subscript "*e*" denotes an element. We have chosen upwind fluxes with the justification that it is the simplest to implement and there is no need of any special treatment in the code for absorbing boundary condition treatment in coastal boundaries; more advanced flux treatments will be considered in a future study. The computation of edge integrals is approximated as

$$\int_{e} H_{e} \nu_{h} dS \approx \mathbf{E}_{pq} = \frac{1}{2} \sum_{i=0}^{Q-1} w_{i} H_{e}(x_{ei}, y_{ei}, t) \nu_{h}(x_{ei}, y_{ei}) |de|,$$
(B.20c)

$$\int_{e} G_{e} v_{h} dS \approx \mathbf{F}_{pq} = \frac{1}{2} \sum_{i=0}^{Q-1} w_{i} G_{e}(x_{ei}, y_{ei}, t), v_{h}(x_{ei}, y_{ei}) |de|,$$
(B.20d)

where the letter "*e*" denotes an arbitrary element. Remember that the trial function was defined as $v_h = \psi_p(\eta_1)\psi_{pq}(\eta_2)$. The numerical source term integration is

$$\mathbf{L}_{pq} = \mathcal{J}_{x\xi} \sum_{i=0}^{Q_1-1} w_i \psi_p(\eta_{1i}) \sum_{j=0}^{Q_2-1} w_j \mathcal{L}(\theta_k, \theta_l, \eta_{1i}, \eta_{2j}, t) \psi_{pq}(\eta_{2j}) \frac{1-\eta_{2j}}{2}.$$
(B.22)

B.4. Strong Stability Preserving Runge-Kutta (SSP-RK) methods

Eq. (B.11) can be rewritten as

$$\frac{du(t)}{dt} = r(u,t),\tag{B.23}$$

where the unknown u is alias to u_{pq} to shorten the notation.

The 2nd and 3rd order SSP-RK schemes are, respectively, [39,75]

$$\nu^{(1)} = u^n + \Delta tr(u^n, t^n),$$

$$u^{n+1} = \frac{1}{2} \left(u^n + \nu^{(1)} + \Delta t \ r(\nu^{(1)}, t^n + \Delta t) \right)$$
(B.24)

and

$$+ 0.93501063100924\Delta t$$
). (B.26)

Appendix C. Source terms

C.1. Wind input (S_{in})

The

The third generation wind input can be written as

$$S_{in}(\theta, \sigma) = \mathbf{A} + \mathbf{B} \ \mathbf{E}(\theta, \sigma), \tag{C.1}$$

where A is linear growth term which reads

$$A = \frac{1.5 \times 10^{-3}}{2\pi g^2} (U_* \max[0, \cos(\theta - \theta_w)])^4 H,$$
(C.2)

 θ_w is wind direction, and *H* is an exponential filter (exp($-(\sigma/\sigma^*)^{-4}$). The peak frequency of Pierson and Moskowithz (σ_{PM}^*) [76] may be computed by using friction velocity $U_*^2 = C_D U_{10}^2$ as

$$\sigma_{PM}^* = \frac{0.13g}{28U_*} 2\pi \tag{C.3}$$

with drag coefficient [77] defined as

$$C_D(U_{10}) = \begin{cases} 1.2875 \times 10^{-3}, & \text{for } U_{10} < 7.5 \text{ m/s} \\ (0.8 + 0.065s/m \times U_{10}) \times 10^{-3}, & \text{for } U_{10} < 7.5 \text{ m/s}. \end{cases}$$
(C.4)

The exponential growth by wind is formulated by WAMDI [5] using results from Snyder et al. [78] and the friction velocity definition of Komen [23] in the third generation model as follows:

$$B = \max\left[0, 0.25 \frac{\rho_a}{\rho_b} \left(28 \frac{U_*}{c_{ph}} \cos(\theta - \theta_w) - 1\right)\right] \sigma.$$
(C.5)

C.2. Dissipation by white-capping (S_{wc})

The white-capping in the third generation model [5,24,23,41] is modeled as

$$S_{wc}(\theta,\sigma) = -\mu k E(\sigma,\theta), \tag{C.6}$$

where μ is

$$\mu = C_{wc} \left((1-n) + n \frac{k}{\tilde{k}} \right) \left(\frac{\tilde{s}}{\tilde{s}_{PM}} \right)^p \frac{\tilde{\sigma}}{\tilde{k}}$$
(C.7)

and tilde \sim refers to a mean quantity of variables. The mean wave steepness \tilde{s} is defined as $\tilde{s} = \tilde{k} \sqrt{E_{tot}}$ and it is normalized by the Pierson–Moskowitz spectrum [76] value $\tilde{s}_{PM} = \sqrt{3.02 \times 10^{-3}}$. The remaining parameters C_{wc} , n and p are tunable coefficients. The mean frequency and wave number are, respectively,

$$\tilde{\sigma} = \left[E_{tot}^{-1} \int_{0}^{2\pi} \int_{0}^{\infty} \sigma^{-1} E(\theta, \sigma) d\theta d\sigma \right]^{-1},$$

$$\tilde{k} = \left[E_{tot}^{-1} \int_{0}^{2\pi} \int_{0}^{\infty} k^{-1/2} E(\theta, \sigma) d\theta d\sigma \right]^{-2}.$$
(C.8a)
(C.8b)

C.3. Quadruplet wave-wave interactions (S_{nl4})

Hasselmann [26] has derived a six-dimensional Boltzmann integral to describe nonlinear wave-wave interaction. The computation of the Boltzmann equation is prohibitively expensive to use in operational sea models. Hasselmann [27] has proposed various approximations to the Boltzmann equation. Among them, Discrete Interaction Approximation (DIA), which considers only two the most significant configurations, is adapted in third generation models. The first quadruplet configuration is

$$\sigma_1 = \sigma_2 = \sigma,$$

 $\sigma^+ = \sigma(1 + \lambda),$
 $\sigma^- = \sigma(1 - \lambda),$

(2)

The angle (θ^+) between wave numbers \vec{k} and $\vec{k^+}$ is -11.5° ; the angle (θ^-) between wave numbers \vec{k} and $\vec{k^-}$ is 33.6°. The second configuration is only a mirror of the above configuration $(k \angle k^+ = 11.5^\circ \text{ and } k \angle k^- = -33.6^\circ)$.

The energy contributions due to the wave interactions among the three resonant waves at σ , σ^+ and σ^- will be

$$\begin{pmatrix} \delta_{nl4} \\ \delta_{nl4}^+ \\ \delta_{nl4}^- \end{pmatrix} = R \begin{pmatrix} -2 \\ +1 \\ +1 \end{pmatrix} C_{nl4} (2\pi)^2 g^{-4} \left(\frac{\sigma}{2\pi}\right)^{11} \left[E^2(\theta, \sigma) \left(\frac{E(\theta^+, \sigma^+)}{(1+\lambda)^4} + \frac{E(\theta^-, \sigma^-)}{(1-\lambda)^4}\right) - \frac{E(\theta, \sigma)E(\theta^+, \sigma^+)E(\theta^-, \sigma^-)}{(1-\lambda)^4} \right],$$
(C.9)

where *R* is a depth scaling parameter [32] for shallow water and C_{nl4} is a constant parameter taken as 3×10^7 . Note that the summation of individual contributions would sum up to zero, so the whole process conserves the total energy. The computation of the above contribution requires energy at locations either out of the spectral domain or off-grid (such that θ^+ , σ^+). We have assumed zero energy if the directional locations are out of domain and the frequency locations are less than minimum frequency f_{min} . The high-frequency energy tail is assumed as f^{-4} to compute energy beyond cut-off frequency $f_{max}\left(E(\sigma) = E(\sigma_{max}) * \left(\frac{\sigma}{\sigma_{max}}\right)^{-4}\right)$. Inside the domain, energy requiring off-grid collocation has been interpolated from the nearest grid collocations. The contribution among low frequencies might fall very close to each other; therefore, it leads to irregular shapes in the source term. We have used a simple filter by just averaging a specified neighborhood after the DIA computation; this helped smooth the quadruplet source term. To retain numerical stability, the total change of action

density is limited at every time step for each wave component by using an action limiter proposed by Tolman [20].

C.4. Triad wave-wave interactions (S_{nl3})

The Lumped Triad Approximation (LTA) by Eldeberky [29] is the simplest expression for triad wave interactions. The approximation is

$$S_{nl3}(\theta,\sigma) = S_{nl3}^{-}(\theta,\sigma) + S_{nl3}^{+}(\theta,\sigma), \tag{C.10}$$

and the positive term contribution is computed as

$$S_{nl3}^{+} = \max\left[0, \alpha_{EB} 2\pi c c_g J^2 |\sin\beta| \left\{ E^2(\theta, \sigma/2) - 2E(\theta, \sigma/2)E(\theta, \sigma) \right\} \right], \tag{C.11}$$

where α_{EB} is a control parameter. A negative contribution of source terms can be defined in terms of the positive one as

$$S_{nl3}^{-} = -2S_{nl3}^{+}(\theta, 2\sigma). \tag{C.12}$$

The biphase parameter β is approximated by using Ursell number Ur:

$$\beta = -\frac{\pi}{2} + \frac{\pi}{2} \tanh\left(\frac{0.2}{Ur}\right) \tag{C.13}$$

and

$$Ur = \frac{g}{8\sqrt{2}\pi^2} \frac{H_s T_{m01}^2}{d^2}.$$
 (C.14)

We have not computed the interaction for very small values of Ursell numbers that occur in deep water (where no triad wave-wave interactions are possible). In a very large Ursell number which occurs in shallow water ($d \ll Hs$), triad computations will also drop out. The coefficient *J* [79] can now be defined as

$$J = \frac{k_{\sigma/2}^2 (gd + 2c_{\sigma/2}^2)}{k_{\sigma} d \left(gd + \frac{2}{15}gd^3k_{\sigma}^2 - \frac{2}{5}\sigma^2d^2\right)}.$$
(C.15)

The computation of the positive and negative terms requires frequency bins beyond the discretization bins. We have assumed zero energy if the energy is out of the spectral domain and interpolated energy inside the domain for off-grid frequency locations (σ /2). However, it will be more proper to use shallow water high-frequency tail ($\approx f^{-3}$) after cut-off frequency $f_{cut-off}$.

C.5. Depth-induced wave breaking (S_{br})

The mean rate of energy dissipation due to depth-induced breaking is:

$$D_{tot} = -\frac{1}{4} \alpha_{BJ} Q_b \left(\frac{\tilde{\sigma}}{2\pi} \right) H_{max}^2, \tag{C.16}$$

where the value of α_{BJ} is in the order of 1. The fraction of breaking waves is represented by Q_b , which is derived by assuming cumulative probability distributions of all waves [31,54] from the following relation:

$$\frac{1-Q_b}{\log Q_b} = -8\frac{H_{rms}}{H_{max}^2},\tag{C.17}$$

where H_{rms} is the root-mean-square of wave height and H_{max} is the maximum wave height for a given depth. The maximum wave height is defined as $H_{max} = \gamma d$. The breaker parameter γ is chosen as an average value of 0.73 from the work of Battjes and Stive [54]. The characteristic frequency in the original formulation has been chosen as the mean frequency ($\tilde{\sigma}$) (see Eq. (C.8b)) in the dissipation rate formula.

The fraction of breaking waves Q_b can be numerically computed through Eq. (C.17). However, the computation of Q_b follows the SWAN technical manual such that

$$Q_{b} = \begin{cases} 0, & \text{for } \beta \leq 0.2, \\ Q_{0} - \beta^{2} \frac{Q_{0} - \exp[(Q_{0} - 1)/\beta^{2}]}{\beta^{2} - \exp[(Q_{0} - 1)/\beta^{2}]}, & \text{for } 0.2 < \beta < 1, \\ 1, & \text{for } \beta \geq 1, \end{cases}$$
(C.18)

where parameter β is the ratio of root-mean-square H_{rms} height and max wave height H_{max} , and Q_0 reads

$$Q_0 = \begin{cases} 0, & \text{for } \beta \le 0.5\\ (2\beta - 1)^2, & \text{for } 0.5 < \beta \le 1 \end{cases}$$
(C.19)

The depth-induced wave breaking source term [29] can be modeled as

$$S_{br}(\theta,\sigma) = \frac{D_{tot}}{E_{tot}} E(\theta,\sigma).$$
(C.20)

Note that S_{br} will be a negative sign due to D_{tot} .

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