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ON THE HIGHEST AND OTHER SOLITARY WAVES*

JAMES WITTING†

Abstract. For solitary waves an expansion parameter which differs from those previously employed permits calculation to extremely high order. The observed behavior of the coefficients entering the power series for the position or velocity field yields much information about the nature of singularities in the solution. The principal conclusions are, (i) the wave of maximum amplitude has a nondimensional amplitude $\cong 3\sqrt{3}/2\pi$ and a nondimensional speed (Froude number) $\cong (3\sqrt{3}/\pi)^{1/2}$. (ii) All previous theories employing an expansion parameter are incomplete. (iii) The various series relating Froude number to amplitude, recently advanced to the ninth order, are asymptotic.

For undular bores direct numerical calculations show that (i) the relationship between relative elevation and relative velocity given by long wave theory is approached for the "ahead of" and "behind" an undular bore even when the bore is generated in ways which violate the conditions of the long wave theory, (ii) the amplitude of first crest of an undular bore approaches a finite limit, approximately at an exponential rate, and (iii) the distance between the first two crests increases without bound, approximately logarithmically.

Most of this paper deals with the solitary wave, a subject which has been studied for more than one hundred years. If time permits, I shall present some preliminary results of numerical calculations of a first cousin to the solitary wave, the undular bore.

Figure 1 sets the stage for the solitary wave research. At the top of the figure, we see a schematic drawing of the profile of a solitary wave. When viewed from a coordinate system in which the fluid away from the bump is at rest, the profile is one of unchanging shape and it travels at a constant speed. The results of experiments and approximate theories all confirm the fact that many disturbances which initially have horizontal scales sufficiently larger than the fluid depth evolve to a succession of solitary waves. We can also view the solitary wave from a coordinate system in which the profile is stationary. The problem is then one of a

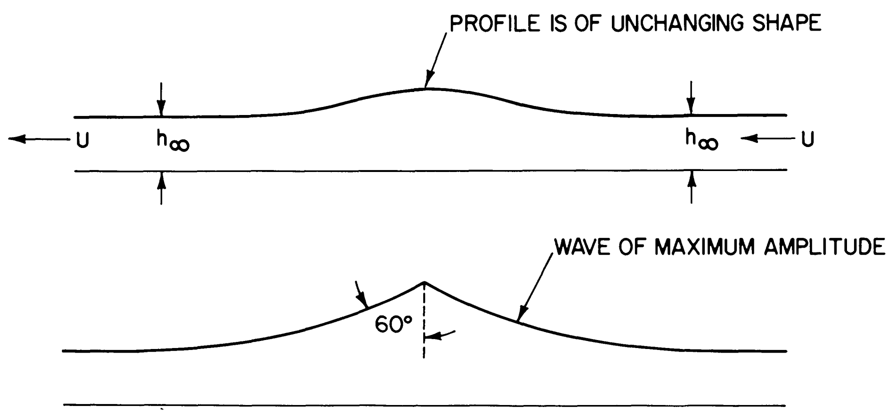

FIG. 1. *Solitary waves*

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† Ocean Sciences Division, United States Naval Research Laboratory, Washington, D.C. 20375.

two-dimensional, incompressible, irrotational steady flow. In this coordinate system, the velocity and the depth at each infinity are the same, denoted here by U for velocity and h_∞ for depth (see Fig. 1). It may be true that a solitary wave can exist all the way up to a breaking wave. By the argument of Stokes [see, e.g., Lamb (1932)] the half-angle at the crest must then be 60°. The body of theory for the solitary wave is obviously not complete unless it includes a prescription for computing accurately the shape and other properties of the solitary wave from small amplitudes to the wave of maximum amplitude. To date, there is not a complete solitary wave theory, although numerous papers have been written about solitary waves in recent and distant history, starting with the original observations of Russell in the 4th and 5th decades of the last century.

Table 1 is helpful in outlining the status of solitary wave theory before this work was started. A key piece of knowledge was provided by Friedrichs and Hyers (1954), who gave an existence proof of solitary waves. To describe their structure, most theorists have expanded the solitary wave parameters entering

TABLE 1
A schematic outline of the current body of solitary wave theory

Solitary waves of finite amplitude exist (Friedrichs and Hyers (1954))
$y = 1 + \sum_{n=1}^{\infty} a_n \operatorname{sech}^{2n} \tfrac{1}{2}\beta x \qquad [y \equiv \text{fluid depth} \div h_\infty]$ $a \equiv \sum_{n=1}^{\infty} a_n;\ F^2 = 1 + b_1 a + b_2 a^2 + \cdots + b_9 a^9 + O(a^{10})$ b_1: Russell (1844); Boussinesq (1871); Rayleigh (1876); McCowan (1891), (1894); Benjamin and Lighthill (1954); Others. b_2: Korteweg and de Vries (1895) [in principle]; Weinstein (1926) b_5; Long (1956) b_9: Fenton (1972)
$F_b^2 = 1.6568;\ a_b = 0.8284\ (\pm 0.0010)$ Yamada (1957) $F^2 = 1.4455;\ a = 0.4804\ (\pm 0.0010)$ Yamada (1958)

the solution in powers of some small parameter related to the amplitude. The expansion for the profile shown at the top of Table 1 is a typical example. The expansion parameter can be a, the ratio of the amplitude of the solitary wave to h_∞. Everything entering the free surface boundary condition is expanded in powers of a, including, for example, the square of the Froude number, F^2. In the original experiments on the solitary wave, Russell was able to relate F^2 to the amplitude with an empirical formula which, in retrospect, is correct to the first order in a. Theorists have derived the same result to the first order. Boussinesq was first, but many distinguished authors have continued to provide different insights into the structure of the solitary waves to this order, usually in a more general work. Implicit in the work of Korteweg and deVries (1895) is a solution correct to the second order in a (the explicit calculations can be found in Wenhausen and Laitone (1960)). Weinstein (1926) also computed to the second order but made an error. Following this, in 1956, Long computed the relationship between the Froude number and amplitude which is correct to the fifth order in amplitude. Finally, in a massive work, Fenton recently computed using the expansion to the ninth order in amplitude. The sampling of theories involving expansions shown in Table 1 is incomplete; many important papers are not listed there, though those shown are representative.

There are also some numerical approaches to the problem of the solitary wave which do not rely upon an expansion of the form shown at the top of Table 2. In particular, Yamada (1957), (1958) published the results of numerical studies of solitary wave structure. In 1957 he presented results for the wave of maximum amplitude. His analysis led to values of the square of the Froude number F_b^2 and

TABLE 2
Principal results of this work

$$y = 1 + \sum_{n=1}^{\infty} a_n \operatorname{sech}^{2n} \tfrac{1}{2}\beta x \times \text{More}$$

$$\left[\text{“More” might be } \prod_{m=1}^{\infty} 1 + \sum_n a_{nm} \operatorname{sech}^{2n} \tfrac{1}{2}\beta_m x\right]$$

$$F_b^2 = \frac{3\sqrt{3}}{\pi} \cong 1.6540; \qquad a_b = \frac{3\sqrt{3}}{2\pi} \cong 0.8270$$

Fenton's 9th order solution is correct in the limit $a \to 0$; i.e.,

$$F^2(0) = 0; \frac{\partial F^2(0)}{\partial a} = +1.0; \qquad \frac{\partial^2 F^2(0)}{\partial a^2} = -\frac{1}{10}; \text{ etc.}$$

the amplitude a_b given in Table 1. Later he also computed the structure of a second high-amplitude wave. Now the result of Fenton's ninth order expansion should agree with the nonmaximum amplitude wave computed by Yamada, but it does not. The two differ by some 0.02 in amplitude, an amount far greater than the cited errors in Yamada's paper and far greater than expected contributions from any term ignored by Fenton. Furthermore, Fenton extrapolated his results to obtain conditions for the wave of maximum amplitude, and these yield an amplitude of approximately $a = 0.85$. This also is not in agreement with the results of Yamada. Thus, at the present time we see there is something wrong with one or other class of theory, barring numerical errors. I think that I have found the source of this discrepancy. Figure 2 again displays a solitary wave profile. It can be shown that at the tails of the solitary wave the profile must be exponential, as shown in Fig. 2. Because in this portion of the solitary wave the disturbance from a still medium is very small, linear theory can be applied. A "dispersion relationship" emerges which is of the form $F^2 = \tan(\beta)/\beta$. This is analogous to the ordinary dispersion relation for linear sinusoidal waves, which go as $y = 1 + \varepsilon e^{ikx}$. There the dispersion relation is $F^2 = \tanh(k)/k$ in the same dimensionless units used here. Note that if we set $ik = \beta$ we arrive at the dispersion relation shown at the center of Fig. 2. This dispersion relation is sketched at the bottom of Fig. 2. Note that for a given Froude number greater than 1 there exists infinitely many values of β which satisfy the dispersion relation. In all theories of the solitary wave which rely upon an expansion, the expansion is taken about $F^2 = 1$, and

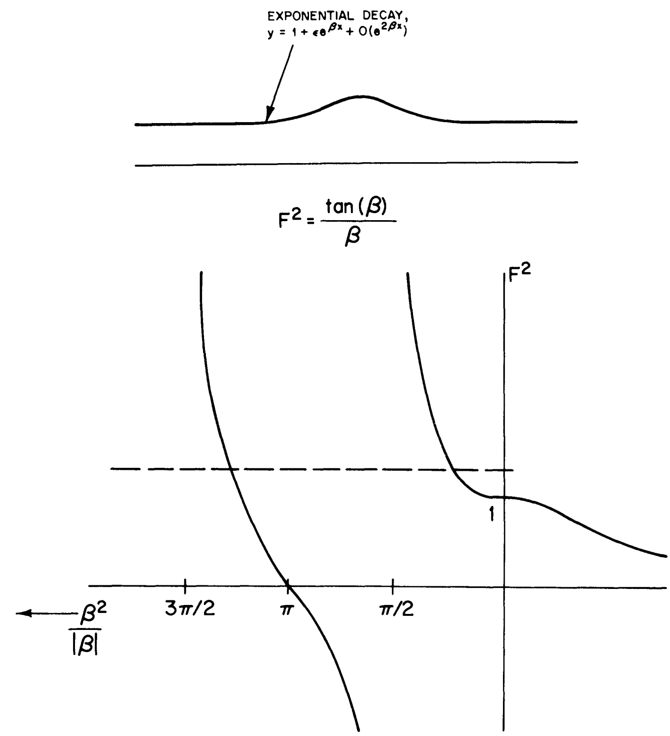

FIG. 2. *The linearized parts of solitary waves*
Uppermost: Solitary wave
Center: Dispersion relation
Lowermost: Sketch of the dispersion relation

only the lowest value of β entering the dispersion relation is used. Indeed, the expansion converges only up to a value of β of $\pi/2$, and the possibility of higher modes entering the solution are excluded. Only McCowan (1891) makes explicit reference to the fact that other values of β do satisfy this dispersion relation. Based upon the particular form of his expansion, however, he argues that only the lowest value of β can enter his solution. Unfortunately, the form of his expansion parameter is not the only one that can be used, and other forms do not rule out the inclusion of higher values of β. There seems, therefore, to be nothing fundamental about the demand that only the lowest value of β be included in the solution. Indeed, the major result of this work is to show that the higher values of β are *necessary* to obtain the solution and that expansions such as Fenton's are only asymptotic, and never converge to a correct solution.

Table 2 shows a summary of the results of the work described here. We shall demonstrate that the solution in an expansion of the form used by Fenton and others is incomplete. The form of a candidate set of functions which are necessary to complete the solutions is shown in Table 2. Furthermore, I shall present what I consider to be very strong arguments to demonstrate that the wave of maximum amplitude has an amplitude $a_b = 3\sqrt{3}/2\pi$. This result is obtained by examining the structure of the singularities in an expansion, and the numerical results agree within reasonable error bounds to the approximate results of Yamada for the wave of maximum amplitude. Finally, I argue that Fenton's ninth order solution is correct in the limit that the amplitude approaches zero, but is incorrect, i.e., it diverges, for any finite value of a. This is simply the definition of an asymptotic series, and asymptotic series are frequently quite accurate. In fact, the series computed by Fenton differs from accurately computed solitary waves by only a few percent at most.

Equations (1) and (2) show the nondimensionalization used in the theory described here:

$$\{x, y\} \equiv \{x^*, y^*\}/h_\infty, \tag{1}$$

$$\{\phi, \psi\} \equiv \{\phi^*, \psi^*\}/F\sqrt{gh_\infty^3}. \tag{2}$$

The starred variables represent, in order of their appearance, the horizontal and vertical components of position, the velocity potential, and the stream function; h_∞ denotes the still water depth (more precisely, in a coordinate system where the profile is stationary, h_∞ is the depth at infinity); F denotes the Froude number $(U/\sqrt{gh_\infty})$; g denotes the acceleration of gravity. With this nondimensionalization, the value of the stream function for the surface streamline is 1, and the value of y at infinity is 1.

The boundary conditions are that the normal velocity at the bottom vanishes, that the velocity at infinity is horizontal and unity, and that the pressure is constant along the free surface. Following Stokes' recasting of the theory of Stokes waves in 1880, we choose ϕ and ψ to be the independent variables, x and y being dependent. With this choice of independent variables, the boundary of the fluid is specified in advance, a significant advantage over letting x and y be the independent variables. The mathematical statement of the bottom, sides and upper boundary condition is given in (3)–(5), where the surface pressure is set to zero:

$$y = 0 \quad \text{for } \psi = 0, \tag{3}$$

$$\frac{\partial x}{\partial \phi} = 1 \quad \text{for } \phi \to \pm\infty; \tag{4}$$

$$y + \tfrac{1}{2}F^2(u^2 + v^2) = 1 + \tfrac{1}{2}F^2 \quad \text{for } \psi = 1. \tag{5}$$

We construct the complex variables z and w as shown in (6)–(7):

$$z = x + iy, \tag{6}$$

$$w = \phi + i\psi. \tag{7}$$

The square of the fluid speed is

$$u^2 + v^2 = \left|\frac{dw}{dz}\right|^2 = \left|\frac{dz}{dw}\right|^{-2}. \tag{8}$$

The Bernoulli relation equation (5) then becomes

$$y\left|\frac{dz}{dw}\right|^2 + \tfrac{1}{2}F^2 = (1 + \tfrac{1}{2}F^2)\left|\frac{dz}{dw}\right|^2. \tag{9}$$

The particular feature which distinguishes this theory from all other theories which expand in powers of amplitude is the choice of an expansion parameter. Here we choose to expand about the point at infinity in the form given by

$$z = w + \sum_{n=1}^{\infty} a_n q^n, \tag{10}$$

where

$$q = e^{\beta w}. \tag{11}$$

This form of expansion is chosen in order to permit the calculation of very high order terms, as will soon appear. The expansion given by (10)–(11) is an expansion about the point at $w = -\infty$. The boundary conditions at the bottom are satisfied if all of the parameters a_n are real. The boundary condition at $-\infty$ is obviously satisfied, although the boundary condition at $+\infty$ is not obviously satisfied. If a solitary wave exists for the preassigned value of β, we should be able to write down an expansion of the solitary wave about the point at $+\infty$, and use numerically-generated analytic continuation procedures to connect the two expansions. For definiteness, consider Fig. 3, which shows the mapping of the w-plane of Fig. 3a into the q plane of Fig. 3b. The region occupied by the fluid maps into a wedge of interior angle β. There may be singularities which prevent the expansion from converging. We show these as a $+$ singular point and a $-$ singular point. (It turns out that singularities must come in conjugate pairs if all a_n are real.) Analytic continuation procedures can be used anywhere except near singular points.

Now we plug (10) into (9) and equate terms in equal powers of q. At the order q^0, we have a restatement that the Bernoulli constant in (5) is correct. At the order q^1 we obtain the dispersion relation shown earlier in Fig. 2. The value of a_1 is arbitrary. At the order q^2 and higher, the value of a_n is given by

$$\begin{aligned} a_n = \frac{1}{\sin(\beta n) - \beta F^2 \cos(\beta n)} \Bigg\{ &\sum_{l=1}^{n-1} f(l, n, \beta, F^2) a_l a_{n-l} \\ &+ \sum_{l=1}^{n-2} \sum_{m=1}^{n-2-l} g(l, m, n, \beta, F^2) a_l a_m a_{n-l-m} \Bigg\}, \end{aligned} \tag{12}$$

where f and g are readily derived but complicated functions of the indices l, m, n, β and F^2. Note that each a_j appearing on the right-hand side of (12) has $j < n$. Thus, by an iterative process, one can start with $n = 2$ and compute each a_n from previously computed a_j. Computations to very large values of n can be accomplished without excessive computer time.

We should note that, although a_1 appears to be arbitrary, the resulting solution does not depend in any significant way upon the choice of a_1. We have not yet specified the horizontal coordinate of the crest in this theory. A change of a_1 merely shifts the position of ϕ_0 marking the crest. The resulting calculations, if the expansion is complete, should generate a solitary wave for any value of a_1, the only differences among solutions with different initial values of a_1 being a horizontal translation of the profile.

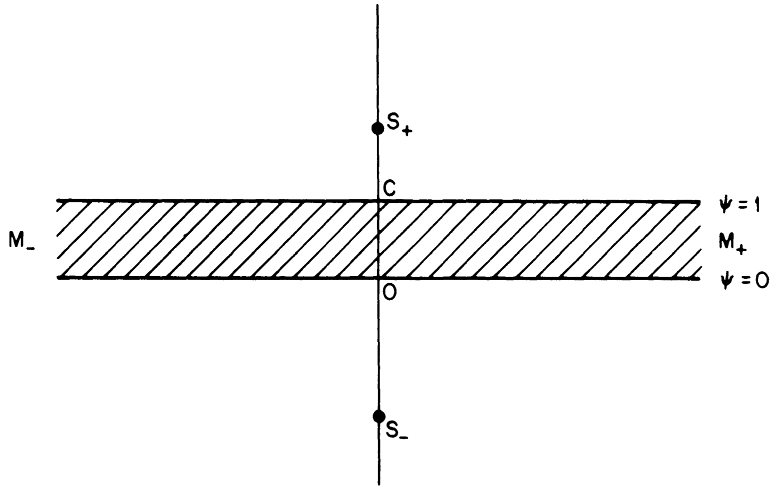

FIG. 3a

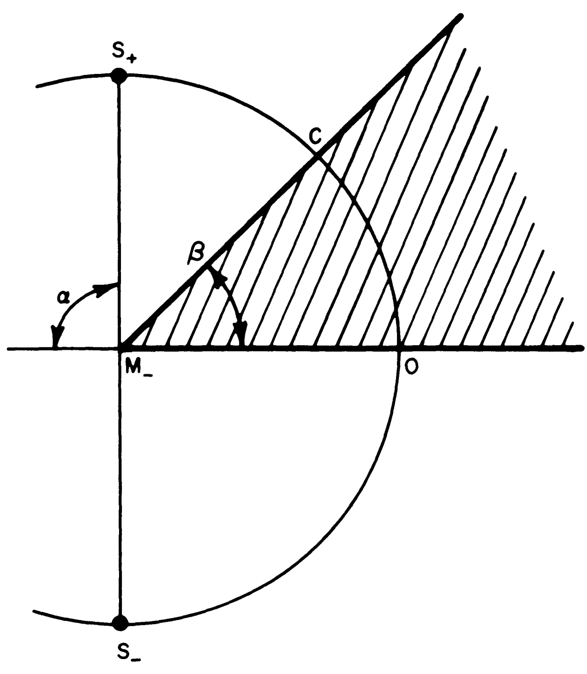

FIG. 3b

FIG. 3. *The mapping of the w-plane* (3a) *onto the q-plane* (3b). *The function* $q = e^{\beta w}$

For four values of β—10^{-6}, $\pi/8$, $\pi/4$ and $\pi/3$—a_n was computed to $n \geqq 200$. In three of these four cases a startling regularity emerged: the signs of a_n are periodic in n. This fact enables us to locate the directions of the singularities, i.e., the angles $\pm\alpha$ in Fig. 3. In order to arrive at this conclusion, we need a theorem from the theory of complex variables (see Hille (1959, pp. 133, 136)). To make use of the theorem, to be defined shortly, we rotate the solution for dz/dw by an angle θ:

$$\frac{dz}{dw} = 1 + \sum_{n=1}^{\infty} na_n\beta q^n = 1 + \sum_{n=1}^{\infty} na_n\beta p^n e^{in\theta}. \tag{13}$$

For the three cases, values of θ can be chosen so that the signs of each of the real part of the jth coefficient of p^j is the same. Specifically, for

$$\frac{dz}{dw} = 1 + \sum_{n=1}^{\infty} n\beta a_n[\cos(n\theta) + i\sin(n\theta)]p^n, \tag{14}$$

all $a_n \cos\theta$ have the same sign. The theorem found in Hille (1959) states that if $a_n \cos n\theta$ keeps the same sign for arbitrary n greater than some N, a singularity lies on the real axis of p. We use this theorem to identify the position of the singularity for all those values of β for which the signs of the original series are periodic. Table 3 lists a summary of the results for those cases run. For $\beta = 10^{-6}$, which governs a wave of minute amplitude, the period is 2 and the direction of the

TABLE 3
Position and nature of singularities

β	No of terms	Period	Position singularities	Nature singularities
10^{-6}	200	2	near π	$z \propto \frac{1}{1+q}$
$\frac{\pi}{8}$	200	8 to 40	near $\pm 3/4\pi$	unknown
$\frac{\pi}{4}$	200	4; reg.	$\pm\pi/2$	unknown
$\frac{\pi}{4} + 0.002$	200	4; irreg.	—	—
$\frac{\pi}{4} + 0.003$	200	4 to 150	—	—
$\frac{\pi}{3}$	551	6; reg.	$\pm\pi/3$	$(z - z_0) \propto (w - w_0)^{2/3}$
$\frac{\pi}{3} + 0.001$	200	6; irreg.	—	—
$\frac{\pi}{3} + 0.0040$	200	6 to 130	—	—

singularity in q-space is very close to π. The singularity is so nearly a simple pole that I cannot imagine that the singularity in the limit $\beta \to 0$ is anything other than a simple pole. For the case $\beta = \pi/4$, I think that $\alpha = \pm 2\beta = \pm\pi/2$ exactly. This opinion is based upon more than the fact that a change in the value of β lying between 0.002 and 0.003 upsets the periodicity in the signs of a_n. More to the point, a small change in the value of β, for example, 0.001, produces a series which tends toward an upset periodicity at some higher value of n. We shall come back to this later. In the case of $\pi/3$, calculations have been carried all the way to $n = 551$. The position of the singularities lies at or very close to $\pm\pi/3$. The nature of this singularity is that demanded by a wave of maximum amplitude, because when the singularities are at $\pm\pi/3$, one of the singularities lies on the free surface of the fluid.

Table 4 lists what is meant by regular and irregular in Table 3, using as an example the case $\beta = \pi/4$. If the periodicity in the signs is going to continue to

TABLE 4
Definition of regularity in high-n sequences

n	$10^3 a_n$		
184	0.297024		
185	0.228445		
186	−0.179990	For $\beta = \pi/4$:	$\dfrac{a_{190}}{a_{194}} \div \dfrac{a_{191}}{a_{195}} = 1 - 0.000008$
187	−0.110510		
188	0.288582		
189	0.221958	For $\beta = \pi/4 + 0.001$:	$\dfrac{a_{190}}{a_{194}} \div \dfrac{a_{191}}{a_{195}} = 1 + 0.02$
190	−0.174882		
191	−0.107373		
192	0.280592		
193	0.215818		
194	−0.170046		
195	−0.104403		

extremely high values of n, the ratio a_{190}/a_{194} should be approaching the same limit as a_{191}/a_{195}. The ratio of ratios should be of the order of n^2. As can be seen from Table 4, we are well within the expected bounds, and $\pi/4$ cannot be very different from the value of β which would place singularities at $\pm\pi/2$. On the other hand, a change in β of 0.001 produces a series for which the ratio of the ratios is not of the order of n^2. We expect that this series will ultimately have upset periodicity.

By applying a similar analysis to the case $\beta = \pi/3$, we can show that only if $\beta = \pi/3 \pm 0.00001$ does the ratio $(a_{3n-2}/a_{3n+1}) \div (a_{3n}/a_{3n+3})$ stay within $1 \pm O(n^2)$ for large n. I have faith that $\beta = \pi/3$ exactly, and has no other value close by, for the wave of maximum amplitude. This faith is nourished by the prominence of the integer 3 in the nature of the singularity, but, of course, I can offer no rigorous proof that $\pi/3$ marks the wave of maximum amplitude exactly.

One troublesome flaw in this set of arguments is that for the case of $\pi/3$ we are not exactly certain that we are computing the structure of the solitary wave. In the case of $\beta = \pi/4$, we know that we are not. In Fig. 4 we show a sketch of the profiles of the streamlines for various values of the stream function. The original series derived from (10)–(12) is employed for $-\infty < \phi < 0.3$. For each value of ψ a polynomial in ϕ is constructed using data centered at $\phi \cong 0.5$. The polynomial is then used to compute x and y in the domain $-0.3 < \phi \gtrsim +0.2$; this procedure is, then, an analytic continuation beyond the original radius of convergence (it is

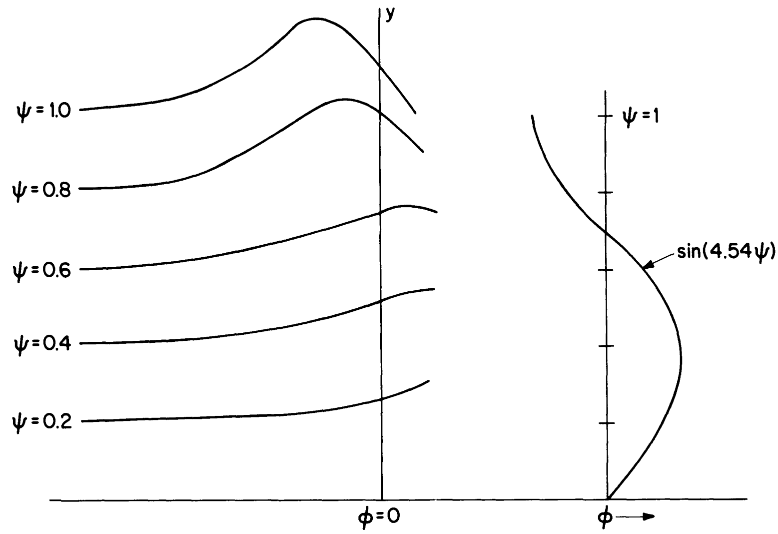

FIG. 4. *Sketches of the streamlines for $\beta = \pi/4$ after analytical continuation (left). The sketch at the right shows the first ignored eigenfunction. Its inclusion improves crest alignment*

not, however, a very efficient one but serves the purpose here). Notice that the crests are not aligned in the figure. If the crests are not aligned, there is no way of forming analytic continuation from the other side and completing the calculation of a solitary wave. On the other hand, there is nothing arbitrary about the calculation shown here using only the lowest value of β permitted by the dispersion relation. Evidently, something is missing from the theory, and we think that we have identified the missing component. In Table 5 we show, for five values of the stream function, the streamline slope, $\partial y/\partial\phi$, and the value of x for $\beta = \pi/4$. We identify the crest position as that value of ϕ which marks the radius of convergence of the expansion and call this $\phi = 0$. By the simple expedient of adding a contribution of the lowest "eigenfunction" permitted by the dispersion relation, which occurs for $\beta = 4.54$, we are able to greatly improve the solution. We choose the amplitude to be 0.0295 for the contribution from this mode, in order to make $\partial y/\partial\phi$ vanish at the crest of the wave. The inclusion of this term greatly improves the structure of the solutions. The variance in $\partial y/\partial\phi$ at $\phi = 0$ is reduced by a

TABLE 5
Effect of adding the most important contribution of the higher modes

ψ	$(\partial y/\partial\phi)_{\text{uncorr}}$	$(\partial y/\partial\phi)_{\text{corr}}$	x_{uncorr}	x_{corr}
1.0	−0.1317	—	0.57645	0.58157
0.8	−0.0715	−0.0088	0.55589	0.58189
0.6	+0.0058	−0.0486	0.54898	0.58588
0.4	+0.1113	−0.0184	0.56004	0.56716
0.2	+0.2845	+0.1791	0.59784	0.57971
$\sum_n (\partial y/\partial\varphi)^2$	0.1158	0.0349	0.00153	0.00015 $\sum_n (x - \bar{x})^2$

$(y_{\text{crest}})_{\text{uncorr}} = 0.2637$
$(y_{\text{crest}})_{\text{corr}} = 0.2728$
$(y_{\text{crest}})_{\text{accur}} = 0.2783$

factor of three, and the variance of $x(0, \psi)$ is reduced by a factor of ten. Also, the solitary wave amplitude is improved by a factor of approximately three. Thus, we conclude that our incomplete solution can be improved by the addition of the most important term ignored. The inclusion of all of the missing terms will probably lead to a symmetrical solution. For the case $\beta = \pi/3$, the amplitude of a potential missing term is less than 0.005. Figure 5 compares the profile of the improved solution with that given by the second order theory of Korteweg and

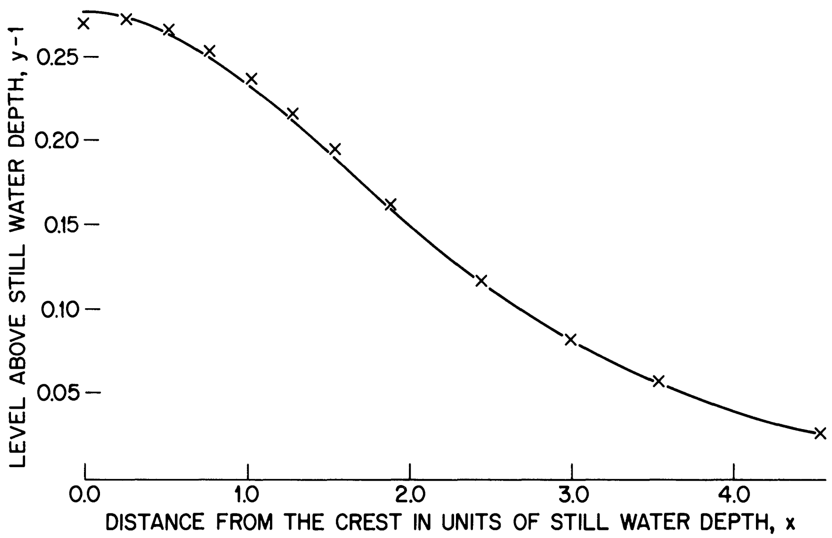

FIG. 5. *Comparison of the profiles of a solitary wave having* $\beta = \pi/4$; $\times$: *this theory*; —: *second order theory from Wehausen and Laitone* (1960)

deVries (1895) as found in Wehausen and Laitone (1960). The two theories produce very similar profiles. For values of x greater than 1.5 this theory is the more accurate; for values of x less than 1.5 the other theory is the more accurate.

The basic mathematical conclusions of this research are given as follows.

(a) The singularity closest to the fluid lies above the crest at a value of ψ given approximately by $\psi = (\pi - 2\beta)/\beta$; this relation is probably exact for $\beta = \pi/3$, $\beta = \pi/4$ and $\beta \to 0$.

(b) A solution including only integral powers of the lowest β satisfying the dispersion relation is incomplete, except in the limit $\beta \to 0$.

From these mathematical conclusions we infer the following.

(a) The wave of maximum amplitude has the following properties:

$$F^2 = 3\sqrt{3}/\pi, \qquad a = 3\sqrt{3}/2\pi = 0.826993\cdots, \tag{15}$$

subject to the reservation that our solution is incomplete and that the functions needed to complete the solution may shift singularities somewhat. The estimate that the amplitude of the missing function is $O(0.005)$ suggests that a may be incorrectly given in (15) by an amount $O(0.005)$ and F^2 may be incorrect by $O(0.01)$.

(b) Fenton's series solution, and by implication, all series expansions involving only integral powers of β, is asymptotic. By asymptotic we mean that in the limit $a \to 0$ it yields the correct result, and that for $a > 0$ it diverges. In the limit $a \to 0$, $\beta_0 \to 0$, so that the lowest value of $\beta_1 \not\to 0$ satisfying the dispersion relation is larger than $M\beta_0$ for arbitrarily large M. Thus, an expansion in powers of β_0 should approach the correct result at $\beta_0 \to 0$. Because the series leaves out part of the solution for finite a, however, it cannot converge. The number of terms which should be retained for maximum accuracy probably exceeds nine for all values of amplitude, since Fenton overestimates $F^2 = F^2(a)$ with nine terms and the second through the ninth are negative.

(c) Finally, we can understand why the expansion given by McCowan (1891), which satisfies the free surface boundary condition to high accuracy, is not particularly accurate. He writes the first term in a power series expansion in powers of $q' = \tanh(\beta w/2)$. The mapping of the w-plane shown in Fig. 3a to the q'-plane is shown in Fig. 6. McCowan's expansion parameter is q'. From this work, we note that the singularities are as close as the edge of the solitary wave

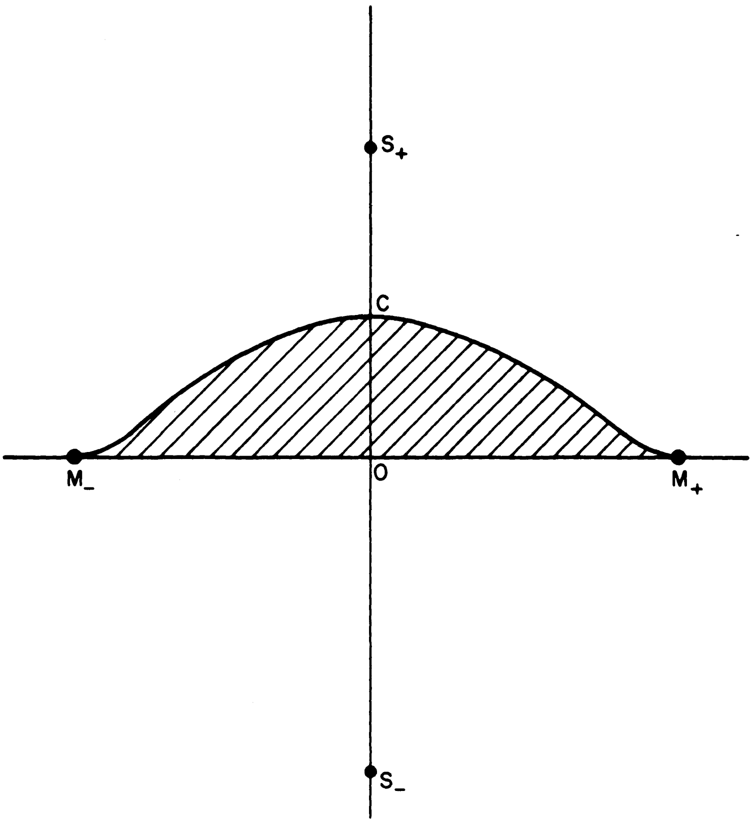

FIG. 6. *The mapping of Fig.* 3a *onto the q'-plane. The function* $q' = \tanh(\beta w/2)$

for all values of $\beta > \pi/4$. Thus, we cannot expect McCowan's apparently accurate calculation to be very good except when $\beta \ll \pi/4$, which limits its validity to small-amplitude solitary waves.

In the time that remains, I shall discuss some preliminary calculations of the evolution of the undular bore. Unlike the analytical/numerical approach to the structure of the solitary wave, we use here a raw numerical approach. The marker and cell method for computing the structure of time-dependent flows with a free surface is employed. This work is a joint effort with B. D. Nichols of the Los Alamos Scientific Laboratory performing the calculations on the Los Alamos computer facility. The motivation is the fact that the lowest order description appropriate for the evolution of fairly long waves, the Korteweg–deVries equation, has recently been shown to possess remarkable properties (see, e.g., Zabusky and Kruskal (1965), Zabusky (1967), (1968), Gardner et al. (1967), Lax (1968), Miura et al. (1968), Kruskal et al. (1970), Witting (1972)). Solitary waves according to the Korteweg–deVries equation run right through another, interacting nonlinearly, but emerging as the same solitary waves as before. Peregrine (1966) studied the evolution of the undular bore by numerical integration of a variant of the Korteweg–deVries equation. By means of direct numerical calculations we sought to determine whether the *qualitative* statements about undular bores which arise from the approximate Korteweg–deVries equation are true of "actual" (i.e., inviscid, incompressible but fully nonlinear) undular bores. Figure 7 illustrates the problem at hand. For most of the examples run, we have still water to the

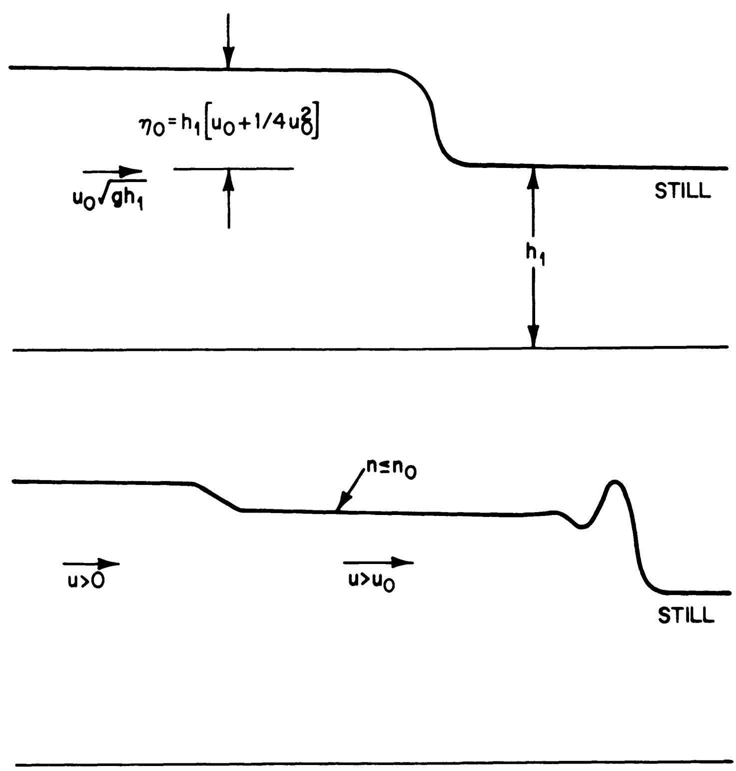

FIG. 7. *Sketch of the profile and fluid speeds for undular bores*; (a) *initially*; (b) *later. The horizontal scale is greatly compressed*

right and a semi-infinite block of higher water to the left which moves with a constant speed. The fluid everywhere has an initial velocity prescribed by that given by the (exact) long wave theory:

(16) $$v = 0,$$

(17) $$\eta/h_1 = F_1 + \tfrac{1}{4}F_1^2,$$

where

(18) $$F_1 \equiv u/\sqrt{gh_1}.$$

If the slope is small initially, we expect that disturbances initially propagate only toward the right, steepen to form a continuously evolving undular bore, perhaps with reflections toward the left. When the initial slope is steeper, so that at least at early times vertical accelerations are not negligible when compared to the acceleration of gravity, we expect two sets of disturbances, one propagating toward the right, steepening to form an undular bore, and the other propagating toward the left as a wave of depression which unsteepens. The Korteweg–deVries equation is capable of describing waves traveling in only one direction. Initially we wished merely to compare our results with Peregrine's, and so we started with a profile which had small initial slope; indeed, we encountered only waves propagating toward the right. We then relaxed the assumption of small slopes and generated the bores with a piston moving into still water. For most runs, the piston accelerates uniformly until a speed u is attained, and moves at u thereafter. By fixing u but varying the acceleration, we are able to vary the fluid's vertical acceleration near the piston, in particular, to values larger than g. For a while, the elevation above still water at the piston is *not* that given by the long wave relationship. Nonetheless, after transients die out (this takes several $\sqrt{h_i/g}$), the fluid near the piston is horizontal and has a height given by (17) *exactly* (to the limits of the numerical experiment—approximately four significant figures). I have found no way of either deriving or understanding this results on analytical grounds.

Returning to the initial aims of the numerical experiment, we find that indeed there is qualitative agreement between actual water waves and the results of the Korteweg–deVries equation. Figures 8a–8g show the evolution of the undular bore in time. We nondimensionalize x based upon the still water depth and time based upon the still water depth and g. Note that the amplitude of the first wave grows in time and the distance between the first two crests also increases in time. An important qualitative result of the Korteweg–deVries equation is that the first wave approaches a solitary wave, i.e., it grows but continues to grow no further. A second result of the Korteweg–deVries equation is that the first two crests separate logarithmically in time. Conclusions are based upon runs having $u_0 = 0.1$ and $u_0 = 0.25$. For both bores the first crest grows at a rate which is very close to being of the form

(19) $$a_1 = A(u_0)[1 - e^{-\alpha(u_0)t}],$$

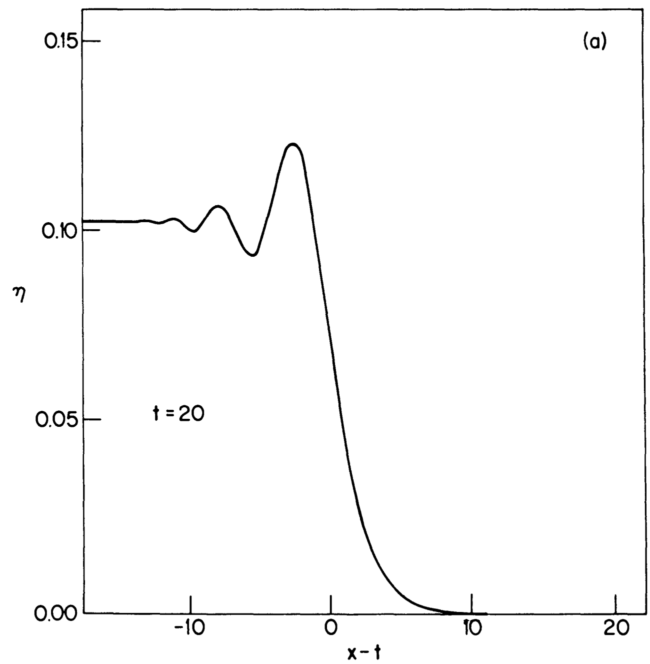
0.15
0.10
0.05
0.00
(a)
η
t = 20
-10
0
10
20
x - t

FIG. 8a

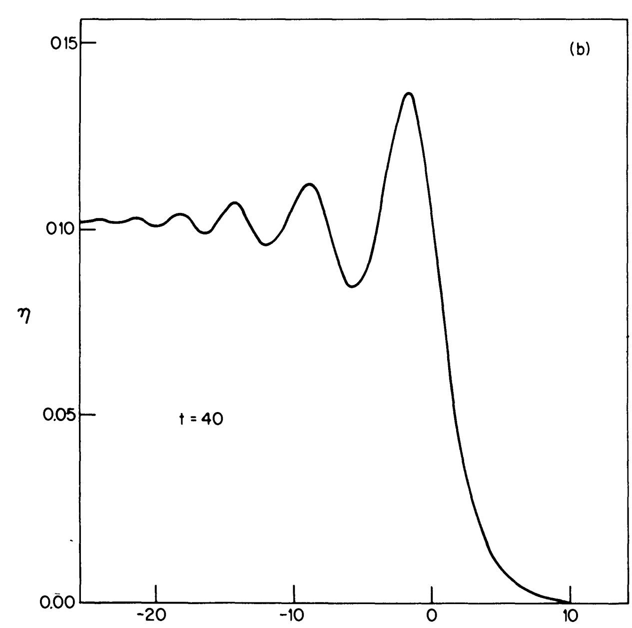
0.15
0.10
0.05
0.00
(b)
η
t = 40
-20
-10
0
10
x - t

FIG. 8b

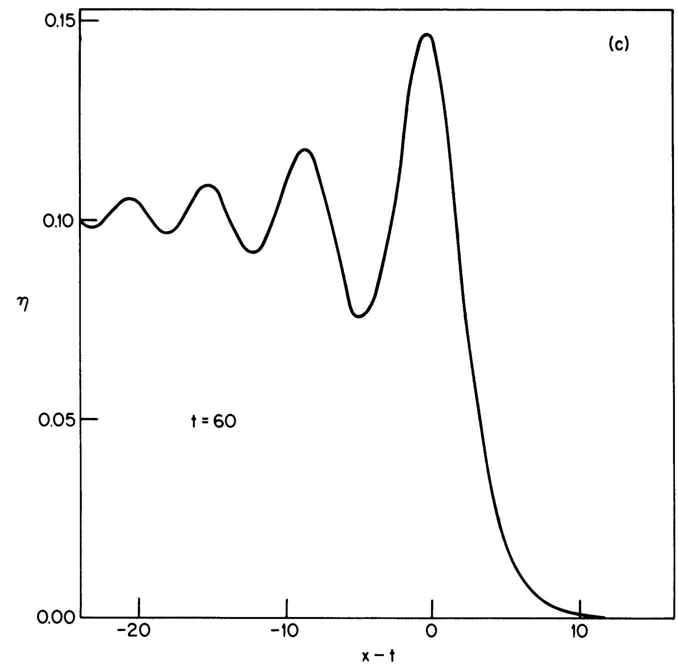
0.15
0.10
0.05
0.00
(c)
η
t = 60
-20
-10
0
10
x − t

FIG. 8c

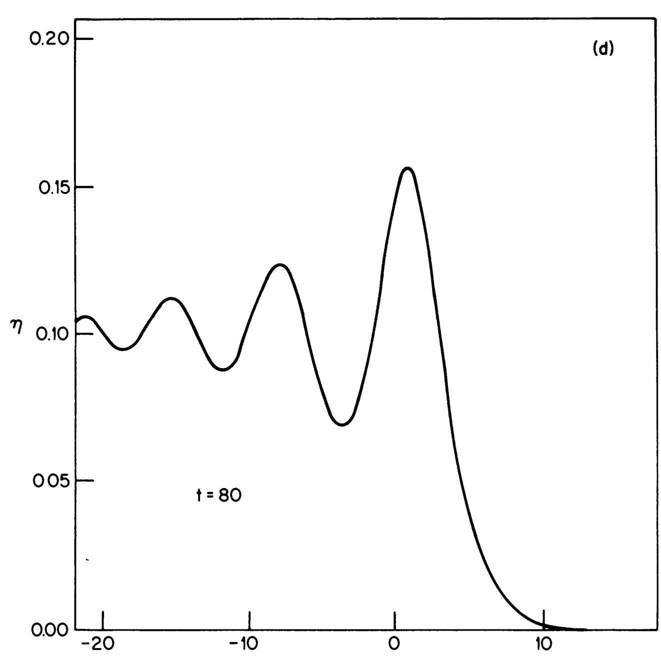
0.20
0.15
0.10
0.05
0.00
(d)
η
t = 80
-20
-10
0
10
x − t

FIG. 8d

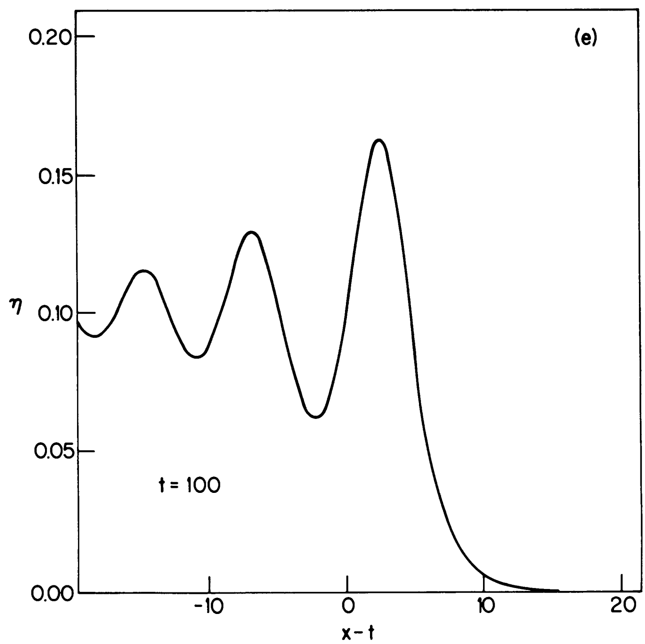
0.20
0.15
0.10
0.05
0.00
η
(e)
t = 100
-10
0
10
20
x - t

FIG. 8e

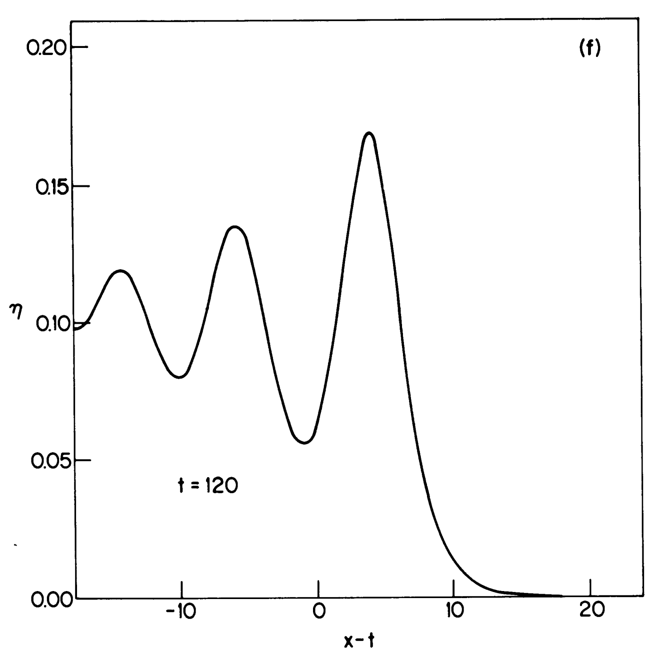
0.20
0.15
0.10
0.05
0.00
η
(f)
t = 120
-10
0
10
20
x - t

FIG. 8f

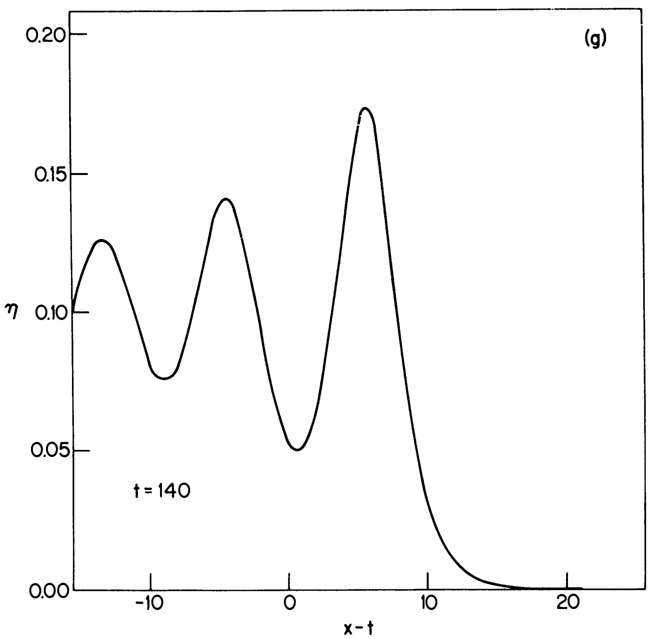

FIG. 8g

FIG. 8. *Profiles of undular bores as a function of time. The initial value* $u_0 = 0.1$

where $\alpha(u_0)$ increases with increasing u_0. The distance between the first two crests grows like

$$\lambda_{1,2} = B(u_0) + C \log \beta(u_0)t \tag{20}$$

for times greater than approximately twenty.

A detailed comparison between our results anu those of Peregrine (1966) is possible for the case $u_0 = 0.1$. The rates of evolution and the wave amplitudes differ by an amount which is of the order of the square of the amplitude. This is the order of the first term neglected in the Koretweg–deVries equation. These results are only preliminary; we expect to report the results of the full computations at a later time.

As is so often the case, in both the solitary wave study and the undular bore study, unexpected and rather surprising results emerged. In the case of undular bores, the relationship between the elevation above still water depth and the speed of the fluid given by the long wave theory is preserved even though initial or boundary conditions flagrantly violate the conditions under which the theory is valid. This is certainly grist for a theorist to chew. In the case of the solitary waves, I started computing, confident that the solution of the solitary wave would be carried to an order higher than imagined using other methods. Instead, we find that expansion methods, while here perhaps producing the conditions for the wave of maximum amplitude, are inadequate unless all possible modes are included—a formidable task indeed.

Note added in proof. In a recent article, Longuet-Higgins and Fenton (*On the mass, momentum, energy and circulation of a solitary wave. II*, Proc. Roy. Soc. London Ser. A, 340 (1974), pp. 471–493) compute solitary wave properties to high accuracy, by using what essentially corresponds to $1 - u_c^2$ as an expansion parameter (u_c is the speed at the crest) and Padé approximants to speed convergence. They obtain high precision all the way up to the highest wave. Their work independently shows the asymptotic nature of expansions in powers of a. Furthermore, they compute $a = 0.827$ for the highest wave, which agrees with the value cited here. Their theory still involves expansions, however, and an independent assessment of their accuracy is in order. Therefore, J. Bergin and I have used the method of Yamada (1957) and a modern computer to make what we regard as highly accurate calculations of the highest wave properties, as well as those of weaker solitary waves.

Preliminary results show virtually perfect agreement with the results of Longuet-Higgins and Fenton only up to amplitudes $\cong 0.75$. Thereafter, small differences show up. Our highest wave has an amplitude of 0.8332, which is 0.0062 higher than the value cited in (15). If these preliminary new results are valid, then (i) forming a complete solution does shift the singularity for $\beta = \pi/3$ by $O(0.005)$, and (ii) Padé approximants do not fully cure the problems associated with incomplete expansions. At this writing, the exact (to 3 decimal places) amplitude of the highest wave cannot be considered known.

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