Variational methods and applications to water waves

By G. B. Whitham, F.R.S.
California Institute of Technology

This paper reviews various uses of variational methods in the theory of nonlinear dispersive waves, with details presented for water waves. The appropriate variational principle for water waves is discussed first, and used to derive the long-wave approximations of Boussinesq and Korteweg & de Vries. The resonant near-linear interaction theory is presented briefly in terms of the Lagrangian function of the variational principle. Then the author's theory of slowly varying wavetrains and its application to Stokes's waves are reviewed. Luke's perturbation theory for slowly varying wavetrains is also given. Finally, it is shown how more general dispersive relations can be formulated by means of integro-differential equations; an important application of this, developed with some success, is towards resolving long-standing difficulties in understanding the breaking of water waves.

1. Variational principle for water waves

Certain investigations in nonlinear wave theory can be given a general form if the basic equations are governed by a variational principle

$$\delta \iint L \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = 0. \tag{1}$$

At the same time, the mathematical manipulations, which may be formidable otherwise, become simple in terms of the 'Lagrangian function' L. There seems to be no general method, other than experienced guesswork, for finding variational principles for given systems of equations. However, they are known for many important cases. Strangely enough, a suitable variational formulation for water waves does not seem to be given in the literature and certainly is not widely known. Water waves are the prime example considered in this paper, as being typical for dispersive waves, so the first two sections present the appropriate variational principle and the approximations for long waves.

In fluid dynamics, it is known that Hamilton's principle with L equal to kinetic energy minus potential energy must apply since, as a last resort, the fluid may be treated as a system of particles. However, the direct formulation of Hamilton's principle gives difficulty in the Eulerian description and various side conditions have to be introduced by means of Lagrange multipliers (see, for example, Serrin 1959).

For irrotational water waves, at least, a more convenient variational principle, free of side conditions, is (1) with

$$L = \int_0^{h(\mathbf{x},t)} \{\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gy\} \, \mathrm{d}y, \tag{2}$$

where y is the vertical coordinate, $\mathbf{x} = (x_1, x_2)$ is the horizontal coordinate, $\phi(\mathbf{x}, y, t)$ is the velocity potential, $y = h(\mathbf{x}, t)$ is the equation of the free surface and g is the acceleration of gravity. Variations of ϕ within the flow region lead to

$$\nabla^2\phi = 0, \tag{3}$$

the variation of h gives the pressure condition

$$\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gy = 0 \quad \text{on} \quad y = h(\mathbf{x}, t) \tag{4}$$

and, with some integration by parts, the variation of ϕ at the upper surface leads to the 'natural' boundary condition

$$h_t + \phi_{x_i} h_{x_i} - \phi_y = 0 \quad \text{on} \quad y = h(\mathbf{x}, t). \tag{5}$$

This formulation was pointed out explicitly by Luke (1967). Bateman (1944) writes down a form of which (2) is a special case, but he does not note that the free surface conditions (the main difficulty in water waves) also follow from (2).

It should be noted that Hamilton's principle would have

$$L_1 = \int_0^{h(\mathbf{x},t)} \{\tfrac{1}{2}(\nabla\phi)^2 - gy\}\,\mathrm{d}y; \tag{6}$$

variations of this function would give Laplace's equation for the flow, but incorrect boundary conditions at the surface. It is easily shown that

$$L_1 = -L - [\phi\phi_y]_{y=0} - [\phi(h_t + \phi_{x_i} h_{x_i} - \phi_y)]_{y=h} - \int_0^h \phi\nabla^2\phi\,\mathrm{d}y + \frac{\partial}{\partial t}\int_0^h \phi\,\mathrm{d}y + \frac{\partial}{\partial x_i}\int_0^h \phi\phi_{x_i}\mathrm{d}y.$$

Apart from the divergence expression, the extra terms all concern conservation of mass. When ϕ is a *solution* of the water wave equations, these extra terms vanish and L_1 differs from $-L$ by the divergence expression. For the theory described in §4, the average value of L is used; since the divergence would average to zero, the average values of L and $-L_1$ are the same.

2. Long waves

Boussinesq equations

Approximations for long waves may be derived by expanding ϕ in a power series in y. The solution of Laplace's equation subject to $\partial\phi/\partial y = 0$ on the bottom $y = 0$ is

$$\phi = f(\mathbf{x}, t) - \tfrac{1}{2}y^2\nabla^2 f(\mathbf{x}, t) + O(h^4/\lambda^4), \tag{7}$$

where λ is a typical wavelength. Then the Lagrangian in (2) becomes

$$L = h(f_t + \tfrac{1}{2}f_{x_i}^2) + \tfrac{1}{2}gh^2 - \tfrac{1}{6}h^3\{\nabla^2 f_t + f_{x_i}\nabla^2 f_{x_i} - (\nabla^2 f)^2\} + O(h^5/\lambda^5). \tag{8}$$

The term in h^3 is the dispersive correction to the usual shallow water theory. The variational equations for (8) give two differential equations for the functions $f(\mathbf{x}, t)$ and $h(\mathbf{x}, t)$. In this form the equations are complicated and it is simpler to work with the value of the potential at the surface, i.e.

$$F(\mathbf{x}, t) = f - \tfrac{1}{2}h^2\nabla^2 f + O(h^4/\lambda^4),$$

instead of f, the value on the bottom.

For, apart from a term

$$\frac{\partial}{\partial t}(\tfrac{1}{3}h^3\nabla^2 f) + \frac{\partial}{\partial x}(\tfrac{1}{3}h^3 f_{x_i}\nabla^2 f),$$

which does not contribute in the variational principle since it can be integrated out,

$$L = h(F_t + \tfrac{1}{2}F_{x_i}^2) + \tfrac{1}{2}gh^2 - \tfrac{1}{6}h^3(\nabla^2 F)^2 + O(h^5/\lambda^5). \tag{9}$$

The variational equations from this Lagrangian are

$$\delta F: \quad h_t + (hF_{x_i})_{x_i} + \nabla^2(\tfrac{1}{3}h^3\nabla^2 F) = 0, \tag{10}$$

$$\delta h: \quad F_t + \tfrac{1}{2}F_{x_i}^2 + gh - \tfrac{1}{2}h^2(\nabla^2 F)^2 = 0. \tag{11}$$

The highest order derivatives give the dispersive correction to shallow water theory. It is usually considered sufficient to have the linearized form for these correction terms; that is

$$\left.\begin{aligned} h_t + (hF_{x_i})_{x_i} + \tfrac{1}{3}h_0^3\nabla^4 F &= 0, \\ F_t + \tfrac{1}{2}F_{x_i}^2 + gh &= 0. \end{aligned}\right\} \tag{12}$$

This additional approximation assumes that the amplitude parameter a/h_0 is small in addition to h_0^2/λ^2. The fully linearized shallow water equations correspond to $a/h_0 \to 0$, $h_0^2/\lambda^2 \to 0$ and may be written

$$h_t + h_0 F_{x_i x_i} = 0, \quad F_t + gh = 0.$$

Equations (12) include the next order corrections in a/h_0 and h_0^2/λ^2.

An alternative form is obtained when the mean value of ϕ over the depth is introduced in place of F. The mean value $\mathscr{F}$ is given by

$$\begin{aligned} \mathscr{F} &= f - \tfrac{1}{6}h^2\nabla^2 f + O(h^4/\lambda^4) \\ &= F + \tfrac{1}{3}h^2\nabla^2 F + O(h^4/\lambda^4), \end{aligned}$$

and, in place of (12), we have

$$\left.\begin{aligned} h_t + (h\mathscr{F}_{x_i})_{x_i} &= 0, \\ \mathscr{F}_t + \tfrac{1}{2}\mathscr{F}_{x_i}^2 + gh - \tfrac{1}{3}h_0^2\nabla^2\mathscr{F}_t &= 0. \end{aligned}\right\} \tag{13}$$

From the first of these, $h_t = -h_0\nabla^2\mathscr{F}$ plus smaller terms, so (13) can be written in an equivalent form

$$\left.\begin{aligned} h_t + (h\mathscr{F}_{x_i})_{x_i} &= 0, \\ \mathscr{F}_t + \tfrac{1}{2}\mathscr{F}_{x_i}^2 + gh + \tfrac{1}{3}h_0 h_{tt} &= 0. \end{aligned}\right\} \tag{14}$$

Finally, if the mean horizontal velocity $U_i = \mathscr{F}_{x_i}$ is introduced, we may write

$$\left.\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial (hU_j)}{\partial x_j} &= 0, \\ \frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} + g\frac{\partial h}{\partial x_i} + \tfrac{1}{3}h_0\frac{\partial^3 h}{\partial x_i \partial t^2} &= 0, \end{aligned}\right\} \tag{15}$$

and this seems to be the form usually quoted for Boussinesq's equations. The Lagrangian corresponding to (14) is

$$L = h(\mathscr{F}_t + \tfrac{1}{2}\mathscr{F}_{x_i}^2) + \tfrac{1}{2}gh^2 - \tfrac{1}{6}h_0 h_t^2.$$

Korteweg & de Vries equation

Korteweg & de Vries (1895) obtained an equation for waves propagating in one direction only. It may be obtained as the 'simple wave' solution of the shallow water equations corrected for the third-order dispersion term in (15). It may be verified that

$$\left.\begin{aligned} U &= c_0\left(\frac{\eta}{h_0}-\frac{1}{4}\frac{\eta^2}{h_0^2}\right)+\tfrac{1}{6}c_0h_0\eta_{xx}, \\ \eta_t+c_0\left(1+\frac{3}{2}\frac{\eta}{h_0}\right)\eta_x &+ \tfrac{1}{6}c_0h_0^2\eta_{xxx} = 0, \\ \eta &= h-h_0, \end{aligned}\right\} \tag{16}$$

is a solution of (15) with errors of second order in a/h_0 and h_0^2/λ^2.

It is not clear how (16) could be obtained from the variational principle (9). However, a variational principle can be found directly when (16) is written in the form

$$\left.\begin{aligned} \psi_{xt}+c_0(1+\psi_x)\,\psi_{xx}+\tfrac{1}{6}c_0h_0^2\,\chi_{xx} &= 0, \\ \psi_{xx}-\chi &= 0, \end{aligned}\right\} \tag{17}$$

where $3\eta/2h_0 = \psi_x$. Equations (17) follow from

$$\delta\iint\{\tfrac{1}{2}\psi_x\psi_t+\tfrac{1}{2}c_0\psi_x^2+\tfrac{1}{6}c_0\psi_x^3+\tfrac{1}{12}c_0h_0^2(\chi^2+2\chi_x\psi_x)\}\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{18}$$

3. Resonant interactions

One way to tackle nonlinear waves is to use a perturbation theory for small amplitude, based on the linearized theory as the lowest order approximation. The naïve expansion gives rise to secular terms growing linearly in t, owing to resonance of higher order products of linear terms with the original terms in the linear theory. This topic has been studied extensively by the contributors to this discussion and most of the following papers will centre around this approach.

It seems worthwhile to note briefly how variational methods are used for resonant interactions in order to contrast this approach with the one of slowly varying waves described in the next section. A simple example will suffice for this purpose, but it should be stressed from the start that the algebraic calculations increase considerably in other examples. This one is unusually easy.

The simple example† is

$$u_t+3u^2u_x+u_{xxx} = 0.$$

To obtain a variational principle, we introduce $u = \phi_x$, $\chi = \phi_{xx}$, and write the equivalent pair

$$\left.\begin{aligned} \phi_{xt}+3\phi_x^2\phi_{xx}+\chi_{xx} &= 0, \\ \phi_{xx}-\chi &= 0. \end{aligned}\right\} \tag{19}$$

† This was proposed by Professor D. J. Benney as a simple model for discussions of resonant interactions.

The Lagrangian is
$$L = \tfrac{1}{2}\phi_x\phi_t + \phi_x\chi_x + \tfrac{1}{2}\chi^2 + \tfrac{1}{4}\phi_x^4. \tag{20}$$
Consider now a superposition of waves expressed as
$$\phi = \Sigma \frac{1}{\mathrm{i}k_\alpha} A_\alpha(t)\,\mathrm{e}^{\mathrm{i}k_\alpha x} \quad (\alpha = \pm 1, \ldots, \pm N),$$
where $k_{-\alpha} = -k_\alpha$ and $A_{-\alpha} = A_\alpha^*$ in order that ϕ be real. (The asterisk denotes complex conjugates.) The Lagrangian L becomes, apart from a divergence term,
$$L = \Sigma\Sigma\left\{\frac{1}{2\mathrm{i}k_\alpha} A_\beta \frac{\mathrm{d}A_\alpha}{\mathrm{d}t} + \tfrac{1}{2}k_\alpha k_\beta A_\alpha A_\beta\right\}\exp\{\mathrm{i}(k_\alpha+k_\beta)x\}$$
$$+\tfrac{1}{4}\Sigma\Sigma\Sigma\Sigma A_\alpha A_\beta A_\gamma A_\delta \exp\{\mathrm{i}(k_\alpha+k_\beta+k_\gamma+k_\delta)x\}. \tag{21}$$
In the variational principle L is integrated over an arbitrary rectangle in the (x,t) plane. Take a rectangle with $-l \leqslant x \leqslant l$ and consider
$$\hat{L} = \lim_{l\to\infty}\frac{1}{2l}\int_{-l}^{l} L\,\mathrm{d}x. \tag{22}$$
Only the 'resonant terms' with
$$k_\alpha + k_\beta = 0, \quad k_\alpha+k_\beta+k_\gamma+k_\delta = 0,$$
contribute in the limit. The resonant duets are $(k_\alpha, k_\beta) = (k_n, -k_n)$ in either order, where $n = 1, 2, \ldots, N$. In the quartets there are various possibilities. There will always be
$$(k_n, k_n, -k_n, -k_n) \text{ in some order,}$$
and
$$(k_m, k_n, -k_m, -k_n),\ m \neq n, \text{ in some order.}$$
But special cases may be posed in the given initial modes; for example, wavenumbers $k_0, k_+ = k_0+\mu, k_- = k_0-\mu$ are supposed to be among the initial set in the resonance problem considered by Dr Benjamin. Then the quartets
$$(k_0, k_0, -k_+, -k_-) \quad \text{and} \quad (-k_0, -k_0, k_+, k_-) \text{ resonate.}$$
The Lagrangian $\hat{L}$ becomes
$$\hat{L} = \Sigma\frac{\mathrm{i}}{2k_n}\left(A_n\frac{\mathrm{d}A_n^*}{\mathrm{d}t} - A_n^*\frac{\mathrm{d}A_n}{\mathrm{d}t}\right) - \Sigma k_n^2 A_n A_n^*$$
$$+\tfrac{3}{2}\Sigma(A_nA_n^*)^2 + 6\sum_{m\neq n}\sum A_mA_m^*A_nA_n^* + 3\{A_0^2A_+^*A_-^* + A_0^{*2}A_+A_-\}. \tag{23}$$
The variational principle is reduced to
$$\delta\int_{t_1}^{t_2}\hat{L}\,\mathrm{d}t = 0.$$
In the linear theory, only the quadratic terms are retained in $\hat{L}$ and variation with respect to A_n^* gives
$$\frac{\mathrm{i}}{k_n}\frac{\mathrm{d}A_n}{\mathrm{d}t} = -k_n^2 A_n. \tag{24}$$
The solution is $A_n = a_n\,\mathrm{e}^{-\mathrm{i}\omega_n t}$ with ω_n satisfying the linear dispersion relation
$$\omega_n = -k_n^3. \tag{25}$$

With the full expression (23), variation with respect to A_+^*, for example, gives

$$\frac{\mathrm{i}}{k_+}\frac{\mathrm{d}A_+}{\mathrm{d}t} = -\{k_+^2 - 3A_+A_+^* - 6\Sigma' A_n A_n^*\} A_+ + 3A_0^2 A_-^*, \tag{26}$$

where Σ' denotes summation over all modes except the one with $k = k_+$. The second term in the coefficient of A_+ is the change in frequency due to the nonlinear effects of a single mode; the third term is the change in frequency due to nonlinear coupling with the other modes. The term $3A_0^2A_-^*$ gives the change in A_+ due to the special resonance between k_0, k_+ and k_-. If A_+ and A_- are small compared with A_0, A_0 can be assumed to be unaffected by A_+ and A_- to first order, so that

$$A_0 = a_0 \mathrm{e}^{-\mathrm{i}\omega_0 t}, \quad \omega_0 = -k_0^3 + 3k_0 a_0^2$$

and a_0 is taken to be real without loss of generality.

Then (26), and the same equation with k_+, A_+ replaced by k_-, A_-, have approximate solutions

$$\left.\begin{aligned} A_+ &= a_+ \mathrm{e}^{-\mathrm{i}\omega_+ t}, \quad A_- = a_- \mathrm{e}^{-\mathrm{i}\omega_- t}, \\ \omega_\pm &= -k_\pm^3 + 6k_\pm a_0^2 + 3k_0(a_0^2 - \mu^2) + \mathrm{i}\sqrt{\{9k_0^2\mu^2(2a_0^2 - \mu^2)\}}. \end{aligned}\right\} \tag{27}$$

This is the kind of 'instability' discovered by Dr Benjamin for deep water waves. The 'resonance' of the frequencies, $\omega_+ + \omega_-^* = 2\omega_0$, which can be seen directly from the exponents of the exponentials in (26), should be specially noted.

In more general examples, the complications mentioned earlier arise because the resonance may not appear until two orders beyond linear theory. This whole question, and the use of diagram techniques to manage these complications, is discussed in the paper by Dr Hasselmann.

4. Averaging for slowly varying nonlinear wavetrains

The interaction approach is limited to nearly linear problems. With the idea that true nonlinear concepts may be missed in this approach, an attempt was made to find some fully nonlinear solutions in addition to the periodic uniform wavetrains which were known to exist in typical problems. Some simplifying feature other than linearization was sought and the obvious possibility seemed to be slowly varying wavetrains, with solutions close to the exact solutions for the uniform wavetrains. A general theory was developed (see Whitham 1965*a*, *b*) and the main steps will be reviewed in this section. A simple way of comparing the 'interaction approach' and the 'slowly varying approach' is to think of the elementary discussion of beats. For a linear dispersive problem, a solution with two neighbouring modes may be written either as

$$a_1 \cos(k_1 x - \omega_1 t) + a_2 \cos(k_2 x - \omega_2 t)$$

or as

$$a \cos(kx - \omega t - \epsilon),$$

where

$$\left.\begin{aligned} a^2 &= a_1^2 + a_2^2 + 2a_1 a_2 \cos\{\tfrac{1}{2}(k_2 - k_1)x - \tfrac{1}{2}(\omega_2 - \omega_1)t\}, \\ \tan\epsilon &= \frac{a_1 - a_2}{a_1 + a_2}\tan\{\tfrac{1}{2}(k_2 - k_1)x - \tfrac{1}{2}(\omega_2 - \omega_1)t\}, \\ k &= \tfrac{1}{2}(k_1 + k_2), \quad \omega = \tfrac{1}{2}(\omega_1 + \omega_2). \end{aligned}\right\} \tag{28}$$

In a nonlinear analogue we may discuss either changes in a_1, k_1, a_2, k_2 corresponding to the interaction of modes or the changes in the slowly varying functions a, k, ω.

In developing the theory of a slowly varying nonlinear wavetrain, the mathematical manipulations became impossible except for the easiest cases, until it was realized that all relevant expressions could be determined in terms of a Lagrangian function. But then the whole derivation could be given from the variational principle.

Take the case of water waves with

$$L = \int_0^{h(\mathbf{x},t)} \{\phi_t + \tfrac{1}{2}(\nabla^2\phi) + gy\}\,\mathrm{d}y.$$

There exist uniform wavetrains in which

$$\phi = \beta_i x_i - \gamma t + \Phi(\theta, y), \quad \theta = \kappa_i x_i - \omega t, \quad h = H(\theta), \tag{29}$$

where κ_i, ω, β_i, γ are constant parameters. The terms proportional to x_i and t must be included in ϕ for complete generality. The solution is normalized so that the change of the phase θ in one period is 2π and so that Φ is the periodic part of ϕ; these conditions may be written

$$[\theta] = 2\pi, \quad [\Phi] = 0, \tag{30}$$

where [ ] denotes the change in one period. Then, κ_i is the wavenumber, ω is the frequency, β_i is the mean horizontal velocity. There are two further parameters, which may be taken to be the mean height b and the amplitude of the waves a. Thus the solution depends on two triads $(\boldsymbol{\kappa}, \omega, a)$ and $(\boldsymbol{\beta}, \gamma, b)$. Two relations between these parameters are provided by the normalization conditions (30), but it is crucial to leave them independent at this stage.

For the full theory of water waves, this uniform solution is not known explicitly. For the long wave approximations in § 2, the uniform wavetrain solution is known in terms of elliptic functions. It is also known more generally in the Stokes approximation for small amplitude waves; this would be going back to the near-linear case, but the general point of view is valuable.

A slowly varying wavetrain is close to this uniform solution and may be approximated by the same expressions with $(\boldsymbol{\kappa}, \omega, a)$, $(\boldsymbol{\beta}, \gamma, b)$ slowly varying functions of space and time in the sense that the relative change of each of them in one wavelength or one period is small. Then, an *average* Lagrangian $\mathscr{L}$ is derived as

$$\begin{aligned}\mathscr{L}(\boldsymbol{\kappa}, \omega, a; \boldsymbol{\beta}, \gamma, b) &= \frac{1}{2\pi}\int_0^{2\pi} L\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{H} \{-(\gamma + \omega\Phi_\theta) + \tfrac{1}{2}(\beta_i + \kappa_i\Phi_\theta)^2 + \tfrac{1}{2}\Phi_y^2 + gy\}\,\mathrm{d}y\,\mathrm{d}\theta. \end{aligned} \tag{31}$$

It is then argued that the 'averaged equations' for the slowly varying functions $(\boldsymbol{\kappa}, \omega, a)$, $(\boldsymbol{\beta}, \gamma, b)$ can be obtained from the 'averaged variational principle',

$$\delta \iint \mathscr{L}(\boldsymbol{\kappa}, \omega, a; \boldsymbol{\beta}, \gamma, b)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0, \tag{32}$$

with

$$\left.\begin{aligned} \kappa_i &= \frac{\partial\theta}{\partial x_i}, \quad \omega = -\frac{\partial\theta}{\partial t}, \\ \beta_i &= \frac{\partial\psi}{\partial x_i}, \quad \gamma = -\frac{\partial\psi}{\partial t}, \end{aligned}\right\} \tag{33}$$

as the appropriate generalization of $\theta = \kappa_i x_i - \omega t$, $\psi = \beta_i x_i - \gamma t$ in the uniform solution. This variational principle is intuitively correct but is not expressed as a formal perturbation procedure. It was not clear how to apply a formal procedure to higher orders in a variational approach. But Luke (1966) has established how a detailed formal procedure on the differential equation leads to the same results in a special case; the main steps are reviewed in § 5.

The variational principle (32), with restraints (33), yields

$$\mathscr{L}_a = 0, \quad \mathscr{L}_b = 0, \tag{34}$$

$$\frac{\partial}{\partial t}\mathscr{L}_\omega - \frac{\partial}{\partial x_i}\mathscr{L}_{\kappa i} = 0, \quad \frac{\partial}{\partial t}\mathscr{L}_\gamma - \frac{\partial}{\partial x_i}\mathscr{L}_{\beta i} = 0. \tag{35}$$

Since only the derivatives of θ and ψ are involved, it is more convenient to add the consistency relations

$$\left.\begin{aligned} \frac{\partial \kappa_i}{\partial t} - \frac{\partial \omega}{\partial x_i} = 0, \quad \operatorname{curl} \boldsymbol{\kappa} = 0, \\ \frac{\partial \beta_i}{\partial t} - \frac{\partial \gamma}{\partial x_i} = 0, \quad \operatorname{curl} \boldsymbol{\beta} = 0, \end{aligned}\right\} \tag{36}$$

rather than substitute (33). The functional relations (34) give exactly the normalization conditions (30). The dispersion relation $[\theta] = 2\pi$ is $\mathscr{L}_a = 0$.

4·1. *Adiabatic invariants*

The problem of slowly varying wavetrains is analogous to that of slowly varying oscillations in classical mechanics. The elementary problem usually quoted is to find the variations in the amplitude of a simple pendulum when the string is pulled slowly over the support. In general it concerns the behaviour of a Hamiltonian system when an external parameter varies slowly with time. The theory is usually developed from Hamilton's equations with much use of canonical transformations. The corresponding transformations do not exist in the case of more independent variables (Rüssman 1961), so similar methods cannot be found in the waves problem. On the other hand, the averaged Lagrangian method can be applied to the mechanics problem, at least in simple cases.

Consider a mechanical system with Lagrangian $L(q, \dot{q}, \lambda)$ where $\lambda(t)$ denotes an external parameter. Suppose there are periodic solutions $q = q(\theta)$, $\dot{\theta} = \omega$, for λ = constant, with an energy integral

$$\dot{q}(\partial L/\partial \dot{q}) - L = E.$$

These correspond to the uniform wavetrains; here (E, ω) are parameters corresponding to $(a, \boldsymbol{\kappa}, \omega)$, etc. Now calculate the average Lagrangian

$$\begin{aligned} \mathscr{L}(\omega, E, \lambda) &= \frac{1}{2\pi}\int_0^{2\pi} L \,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial L}{\partial \dot{q}}\dot{q}\,\mathrm{d}\theta - E \\ &= \frac{\omega}{2\pi}\oint p\,\mathrm{d}q - E, \end{aligned} \tag{37}$$

where $p = \partial L/\partial \dot{q}$ is calculated as a function $p(q, \lambda, E)$ from the energy equation.

The variational equations are

$$\mathscr{L}_E = 0, \quad \mathrm{d}\mathscr{L}_\omega/\mathrm{d}t = 0.$$

If the action integral,

$$\frac{1}{2\pi}\oint p\,\mathrm{d}q,$$

is denoted by $I(\lambda, E)$, they reduce to

$$\frac{1}{\omega} = \frac{\partial I}{\partial E}, \quad I = \text{const.}$$

These are the classical results and

$$I = \frac{1}{2\pi}\oint p\,\mathrm{d}q = \mathscr{L}_\omega$$

is the 'adiabatic invariant'.

In the waves problem, then, the first equation in (35) may be interpreted as the balance between the changes in a timelike adiabatic variable $\mathscr{L}_\omega$ and the changes in spacelike adiabatic variables $\mathscr{L}_{\kappa_i}$, as the energy is transferred slowly to different parts of the wavetrain. In simple cases, the averaged Lagrangian follows closely the form in (37) (see (57) below). Also the function W in an early version (Whitham 1965*a*) is an analogue of I.

4.2. *Linear theory*

While developed specifically for nonlinear waves, this theory provides a new general treatment of linear dispersive waves. First, the water waves example is considered, then the general situation is discussed.

In the linearized theory of one-dimensional water waves, (29) becomes

$$H = h_0 + b + a\cos\theta,$$

$$\Phi = \frac{a\omega}{\kappa}\frac{\cosh\kappa y}{\sinh\kappa h_0}\sin\theta,$$

where h_0 is the undisturbed depth. It is expected that (β, γ, b) will not be required in a linear theory, but they will be carried along in calculating $\mathscr{L}$ to keep the comparison with the nonlinear case. The averaged Lagrangian $\mathscr{L}$ is calculated by substitution in (31) and we have

$$\mathscr{L} = (\tfrac{1}{2}\beta^2 - \gamma)(h_0 + b) + \tfrac{1}{2}gb^2 + \tfrac{1}{4}ga^2\left(1 - \frac{\omega^2}{g\kappa\tanh\kappa h_0}\right).$$

As expected, changes in mean velocity β and mean height b uncouple from the wave motion and we may take $b = \beta = \gamma = 0$ to obtain the usual linear theory. Then

$$\mathscr{L} = \tfrac{1}{4}ga^2\left(1 - \frac{\omega^2}{g\kappa\tanh\kappa h_0}\right).$$

Quite generally, for a linear system the Lagrangian L is quadratic in the perturbations so that it will always turn out that

$$\mathscr{L} = G(\omega, \boldsymbol{\kappa})\,a^2. \tag{38}$$

Since $\mathscr{L}_a = 0$, we have
$$G(\omega, \boldsymbol{\kappa}) = 0, \tag{39}$$
and it is clear that the function G will always give the dispersion relation.

The other variational equation is
$$\frac{\partial \mathscr{L}_\omega}{\partial t} - \frac{\partial \mathscr{L}_{\kappa i}}{\partial x_i} = 0, \tag{40}$$
which becomes
$$\frac{\partial (G_\omega a^2)}{\partial t} - \frac{\partial (G_{\kappa i} a^2)}{\partial x_i} = 0. \tag{41}$$

While this is like an energy equation, it corresponds rather to an equation for the adiabatic invariants discussed above. The companion equations to be solved with (41) are
$$\frac{\partial \kappa_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0, \quad \operatorname{curl} \boldsymbol{\kappa} = 0, \quad G(\omega, \boldsymbol{\kappa}) = 0; \tag{42}$$
ω can be treated as a function of $\boldsymbol{\kappa}$ and we have
$$\frac{\partial \kappa_i}{\partial t} + C_j(\boldsymbol{\kappa}) \frac{\partial \kappa_i}{\partial x_j} = 0, \tag{43}$$
where
$$C_j(\boldsymbol{\kappa}) = -G_{\kappa_j}/G_\omega \tag{44}$$
is the linear group velocity. From (43) and (44), it is clear that (41) can be transformed into
$$\frac{\partial}{\partial t}(F(\boldsymbol{\kappa})\, a^2) + \frac{\partial}{\partial x_i}(C_i F(\boldsymbol{\kappa})\, a^2) = 0 \tag{45}$$
for any function $F(\boldsymbol{\kappa})$. In particular $F(\boldsymbol{\kappa}) = 1$ transforms (41) to
$$\frac{\partial a^2}{\partial t} + \frac{\partial (C_i a^2)}{\partial x_i} = 0. \tag{46}$$

This may be treated as the 'energy equation', but the physical energy is, in general, another choice of $F(\kappa)$, see §4·5.

Equations (43) and (46) can be solved by integration along characteristics
$$\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{C}; \tag{47}$$
$\boldsymbol{\kappa}$ remains constant along characteristics and, once $\boldsymbol{\kappa}$ is found, the changes of a are determined from
$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{1}{2}\frac{\partial C_i}{\partial x_i} a. \tag{48}$$

Equation (48) shows the decrease in amplitude due to divergence of the group. Thus the group velocity is a double characteristic velocity for the system and determines the propagation of changes in $\boldsymbol{\kappa}$ and a. It should also be noted that an 'energy velocity' may be defined as energy flux divided by energy density. According to (45) this is also $\mathbf{C}$ for a linear system. Conceptually, however, the characteristic velocity is different from the energy velocity, and it turns out that the two are not the same for a nonlinear system.

4·3. *Type of the equations for a nonlinear system*

In the nonlinear theory $\mathscr{L}_a = 0$ does not give a relation independent of a, and

$$\omega = \omega(\boldsymbol{\kappa}, a),$$

even in a simple case in which the other variables $\boldsymbol{\beta}$, γ, b do not arise. Thus, the equations for $\boldsymbol{\kappa}$, a are no longer uncoupled and constitute a system of differential equations to be studied. The first important question is whether the equations are elliptic or hyperbolic. This can be decided by standard methods. As a simple first step beyond linear theory one can suppose, for a one-dimensional case, that

$$\omega = \omega_0(\kappa) + \omega_1(\kappa)\, a^2$$

to bring in the nonlinear effects, but assume that the 'energy equation'

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x}(C_0(\kappa)\, a^2) = 0$$

still holds with $$C_0(\kappa) = \omega_0'(\kappa).$$

When this is coupled with $$\frac{\partial \kappa}{\partial t} + \frac{\partial}{\partial x}(\omega_0 + \omega_1 a^2) = 0,$$

the characteristic velocities are easily found to be

$$C = C_0(\kappa) \pm a\sqrt{(\omega_1 C_0')} + O(a^2). \tag{49}$$

The equations are hyperbolic when $\omega_1 C_0' > 0$ and elliptic when $\omega_1 C_0' < 0$.

In the hyperbolic case, the double characteristic velocity of linear theory splits into two separate velocities and provides a generalization of the group velocity to nonlinear problems.

When the elliptic case is found the indication is that the original uniform wave train is unstable in a certain sense. For, small sinusoidal disturbances in κ and a will be given by solutions of the form

$$e^{i\mu(x-Ct)}, \tag{50}$$

where C is the value calculated from (49) for the unperturbed values of κ and a. When C is complex, corresponding to the elliptic case, the modulations given by (50) grow exponentially; in this sense the wave train is unstable. If this simple argument is applied to Stokes waves in deep water, the elliptic case is found since

$$\omega = \sqrt{(g\kappa)}\,(1 + \tfrac{1}{2}\kappa^2 a^2) + O(a^4). \tag{51}$$

Hence, $\omega_1 C_0' < 0$, the velocities in (49) are imaginary, and Stokes waves in deep water are unstable. At first this result seemed surprising and probably wrong! And it was put aside until a complete discussion of water waves could be given. However, when Lighthill (1965) looked at this whole theory, he came across this result for the Stokes waves and immediately saw it must be correct! He then studied elliptic cases in detail since they had largely been ignored up to that point.

The complete investigation for the Stokes waves in water of arbitrary depth has now been carried through (Whitham 1966). In addition to the nonlinearity introduced in the dispersion relation, there is a coupling of the wave motion with changes in the mean height b and velocity β; for deep water, this can be ignored and the above result is correct. For finite depth this coupling produces effects to counteract the growth of modulations, and for shallow water the equations change type and the wavetrains are stable.

For arbitrary depth, the average Lagrangian is found to be

$$\mathscr{L} = (\tfrac{1}{2}\beta^2-\gamma)(h_0+b)+\tfrac{1}{2}gb^2+\frac{1}{2}\left\{1-\frac{(\omega-\beta\kappa)^2}{g\kappa\tanh\kappa(h_0+b)}\right\}E+\frac{1}{2}\frac{\kappa^2 D_0}{g\tanh\kappa h_0}E^2+\ldots,$$

where $E = \frac{1}{2}ga^2$,

$$D_0 = \frac{9\tanh^4\kappa h_0-10\tanh^2\kappa h_0+9}{8\tanh^3\kappa h_0}.$$

The averaged equations are

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega_0}\right)+\frac{\partial}{\partial x}\left(\frac{C_0 E}{\omega_0}\right) = 0,$$

$$\frac{\partial b}{\partial t}+\frac{\partial}{\partial x}\left(\beta h_0+\frac{E}{c_0}\right) = 0,$$

$$\frac{\partial \kappa}{\partial t}+\frac{\partial}{\partial x}\left(\omega_0+\frac{\kappa^2 D_0}{c_0}E+\frac{\kappa B_0}{h_0}b+\kappa\beta\right) = 0,$$

$$\frac{\partial \beta}{\partial t}+\frac{\partial}{\partial x}\left(gb+\frac{B_0}{c_0 h_0}E\right) = 0,$$

where
$$\omega_0^2 = g\kappa\tanh\kappa h_0, \quad C_0 = \omega_0'(\kappa), \quad c_0 = \omega_0/\kappa_0,$$

$$B_0 = C_0-\tfrac{1}{2}c_0.$$

It is found that the equations are elliptic if $\kappa h_0 > 1{\cdot}36$ and hyperbolic if $\kappa h_0 < 1{\cdot}36$. Further details may be seen in the paper cited.

The theory has been applied to the Korteweg–de Vries and Boussinesq equations (without any approximation to small amplitude) and to a similar problem in plasma waves. For these and for further general discussion, reference may be made to the earlier papers (Whitham 1965 *a*, *b*).

4·4. *Relation with Benjamin's theory of instability*

A slowly varying wavetrain which is nearly linear can be written in the form

$$\phi = \tfrac{1}{2}a\,\mathrm{e}^{\mathrm{i}\theta}+\tfrac{1}{2}a^*\,\mathrm{e}^{-\mathrm{i}\theta},$$

where a, θ_x, θ_t are slowly varying functions. It is assumed that the amplitude is small enough for the wavetrain to keep the sinusoidal form, but the phase function θ will still have the nonlinear dependence on the amplitude given by the nonlinear dispersion relation. If we consider the special case in which these slowly varying functions are close to constant values, we may write

$$a = a_0+a_1, \quad \theta = \theta_0+\theta_1, \quad \theta_0 = \kappa_0 x-\omega(\kappa_0, a_0)t,$$

where a_0, κ_0 are constants and a_1, θ_1 represent small perturbations in the amplitude and the phase. Assuming that θ_1 is bounded (unlike θ_0), we may expand the solution to first order in a_1 and θ_1 as

$$\phi = \tfrac{1}{2}a_0 e^{i\theta_0} + \tfrac{1}{2}a_1 e^{i\theta_0} + \tfrac{1}{2}i\theta_1 a_0 e^{i\theta_0} + \text{conjugate}.$$

The small perturbations a_1 and θ_1 will satisfy the linearized approximation of the averaged equations. These have constant coefficients depending upon (κ_0, a_0) and admit solutions in the form

$$a_1 = B_+(t)\, e^{i\mu x} + B_-(t)\, e^{-i\mu x},$$
$$\theta_1 = \Theta_+(t)\, e^{i\mu x} + \Theta_-(t)\, e^{-i\mu x};$$

the functions B_+, etc., are exponentially growing or oscillatory depending upon whether the equations are elliptic or hyperbolic. Finally, then, ϕ may be expressed as

$$\phi = A_0 e^{i\kappa_0 x} + A_+ e^{i(\kappa_0+\mu)x} + A_- e^{i(\kappa_0-\mu)x} + \text{conjugate}.$$

This approach can now be identified with the resonant interaction approach of Benjamin (see §3). Notice that the *two* side band modulations introduced in Benjamin's analysis are equivalent to the coupled modulations of amplitude and phase in the averaging approach. The perturbations a_1 and θ_1 are slowly varying functions provided $\mu \ll \kappa_0$. Benjamin's approach is free of this restriction, but, on the other hand, applies only to near-linear problems.

4·5. *Conservation equations and shocks*

When the averaged equations are hyperbolic, certain solutions will 'break' in the sense that an initially continuous solution becomes multivalued. This is analogous to the appearance of shock waves in gas dynamics. However, in this treatment of water waves, it could correspond simply to the superposition of two parts of the wavetrain and not require discontinuities. The prediction of this occurrence from an initial form close to a single periodic wavetrain is itself interesting. Of course, after this occurrence the averaged equations developed here no longer apply. An extended theory with the possibility of more than one principal mode would be necessary. It is possible that such a theory could be developed and the interaction treatment for nearly linear modes would help in this connexion.

Another intriguing possibility is that a discontinuity in the averaged equations—a 'shock'—is the required solution in some cases. The argument would be that the averaged equations break down, because the assumption of a slowly varying wavetrain is no longer valid. However, just as in gas dynamics, the solution may be saved without appeal to the original detailed equations by fitting in discontinuities which satisfy the appropriate conservation equations. Mathematically this is the appeal to 'weak solutions'.

The jump conditions are taken from conservation equations. But in these non-linear problems, it seems to be an essential feature that there are always more conservation equations than the number of required shock conditions. For example, from the usual differential equations of inviscid gas dynamics one can obtain the equation for conservation of entropy

$$(\rho S)_t + \nabla . (\rho \mathbf{u} S) = 0,$$

in addition to conservation of mass, momentum and energy. But the corresponding jump condition should not be applied across a discontinuity, since it is known by physical arguments that the entropy is not conserved across a shock. A similar situation arises here.

The equations in (35) and (36) are already in conservation form. Other important ones may be obtained from the variational principle (32) by use of Noether's theorem. Since $\mathscr{L}$ is invariant with respect to an arbitrary translation in time, it follows that

$$\frac{\partial}{\partial t}(\omega\mathscr{L}_\omega+\gamma\mathscr{L}_\gamma-\mathscr{L})-\frac{\partial}{\partial x_i}(\omega\mathscr{L}_{\kappa_i}+\gamma\mathscr{L}_{\beta_i}) = 0 \tag{52}$$

this is the energy equation. Similarly, since $\mathscr{L}$ is invariant with respect to a translation in space, it follows that

$$-\frac{\partial}{\partial t}(\kappa_j\mathscr{L}_\omega+\beta_j\mathscr{L}_\gamma)+\frac{\partial}{\partial x_i}(\kappa_j\mathscr{L}_{\kappa_i}+\beta_j\mathscr{L}_{\beta_i}-\mathscr{L}\delta_{ij}) = 0; \tag{53}$$

this is the momentum equation. The invariance of $\mathscr{L}$ with respect to arbitrary constant changes in θ and ψ reproduces the variational equations (35). For water waves it turns out that the second equation in (35) corresponds to conservation of mass.

For any conservation equation in the form

$$\frac{\partial P}{\partial t}+\frac{\partial Q_i}{\partial x_i} = 0$$

the corresponding jump condition across a moving discontinuity surface with unit normal n_i and normal speed V is

$$n_i[Q_i] = V[P],$$

where [ ] denotes the magnitude of the discontinuity. However, a choice of the conservation equations has to be made and only the corresponding jump conditions are applicable. This choice must be made from additional information which is not contained in the averaged equations. Since the original differential equations still apply, we choose those conservation equations which are the averaged form of corresponding conservation equations in the original equations. For water waves the choice is then: (i) conservation of energy (52), (ii) conservation of momentum (53), (iii) conservation of mass which is the second one in (35), and (iv) the second set in (36), which result from elimination of ψ and seem to have no general physical significance. (The details of this choice can be found in Whitham (1965*b*).)

The conservation equations which must be omitted, because they can only be found in the averaged form for slowly varying wavetrains, are

$$\frac{\partial \kappa_i}{\partial t}+\frac{\partial \omega}{\partial x_i} = 0, \quad \operatorname{curl}\boldsymbol{\kappa} = 0, \tag{54}$$

and

$$\frac{\partial \mathscr{L}_\omega}{\partial t}-\frac{\partial \mathscr{L}_{\kappa_i}}{\partial x_i} = 0. \tag{55}$$

The set (54) comes from the existence of a phase function θ, but it may be interpreted more forcefully as the conservation of waves both in space and time. When curl $\boldsymbol{\kappa} = 0$, the line integral around a closed circuit is zero, i.e.

$$\oint \boldsymbol{\kappa} . \mathrm{d}\mathbf{s} = 0;$$

in the wave pattern at any instant there are the same number of waves (e.g. wave crests) entering the contour as leaving. Similarly, from the first equation in (54),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1^{(1)}}^{x_1^{(2)}} \kappa_1 \mathrm{d}x_1 = \omega^{(2)} - \omega^{(1)},$$

so that the number of waves in the interval changes at a rate given by the net flux of waves into the interval. But these integrated forms are precisely the ones that cannot be used across a discontinuity; they cannot be established directly but only from (54) which is valid for continuous parts of the solution.

The quantities $\mathscr{L}_\omega$, $\mathscr{L}_{\kappa_i}$ in (55) are similar to adiabatic invariants in classical mechanics (see §4·1). Equation (55) represents a balance between the changes of spacelike adiabatic invariants $\mathscr{L}_{\kappa_i}$ and a timelike adiabatic invariant $\mathscr{L}_\omega$. But this refers to slow changes and is not valid across a discontinuity in the system (a quantum jump!).

Equations (54) and (55) are on a similar footing from this point of view to the entropy in gas shocks. One naturally asks about the sign of the jump at a discontinuity. In a simple case (discussed in Whitham 1965*a*), it could be shown that as the waves cross a discontinuity the frequency, relative to the moving surface, is always increased. This seems to be the right way round for 'irreversibility' and might be expected to hold generally; a general proof has not yet been found. The result acquires special interest when it is remembered that the original equations are reversible! It may have relevance to the theory of smooth bores with waves behind them, and to the possibility of collisionless shocks in plasmas.

This idea was raised tentatively in earlier papers. Since then, Zabusky & Kruskal (1965) have performed numerical computations on the Korteweg–de Vries equation in which, effectively, two groups of waves follow one another. In that case the waves are found to pass through each other when the second group overtakes the first. It is clear from the outset, of course, that if it can exist at all, this kind of shock requires special conditions. In linear theory two wavetrains can always be superposed to give a new solution; so highly nonlinear waves are needed. Again it seems clear that two groups with very different velocities will go straight through each other with complicated interactions; so very small relative velocity is needed. Some relevant evidence bearing on this last point is the work of Benney & Luke (1964). They study the collision of two cnoidal waves ('cnoidal waves' are the uniform wavetrain solutions of the Korteweg–de Vries equation). Their theory determines the nonlinear interaction between the two, and the interaction terms tend to infinity as the strengths and directions of the two waves approach each other.

5. Full perturbation expansion

Luke (1966) considers a nonlinear Klein–Gordon equation

$$u_{tt}-u_{xx}+V'(u) = 0, \tag{56}$$

where $V(u)$ is any nonlinear potential energy which yields oscillatory solutions. The Lagrangian is

$$L = \tfrac{1}{2}u_t^2-\tfrac{1}{2}u_x^2-V(u),$$

and the average Lagrangian is easily found to be

$$\mathscr{L}(\omega,\kappa,E) = \{2(\omega^2-\kappa^2)\}^{\frac{1}{2}}\frac{1}{2\pi}\oint\sqrt{\{E-V(u)\}}\,\mathrm{d}u-E \tag{57}$$

(which is an interesting comparison with (37)).

Luke introduces a perturbation expansion

$$u(x,t) = U(\theta,X,T)+\epsilon U_1(\theta,X,T)+\ldots,$$
$$X = \epsilon x,\quad T = \epsilon t,\quad \theta = \epsilon^{-1}\Theta(X,T).$$

This is a generalization of the geometrical optics expansion for linear problems which would have all the $U_n(\theta,X,T)\propto \mathrm{e}^{\mathrm{i}\theta}$. It is also a generalization of various methods of Krylov–Bogoliubov, and particularly the work of Kuzmak (1959), from ordinary differential equations to partial differential equations. For $\epsilon = 0$, $u(x,t) = U(\theta)$ is the uniform wavetrain. This expansion is substituted in (56) and the coefficients of powers of ϵ are equated to zero. The first two orders give

$$(\omega^2-\kappa^2)\,U_{\theta\theta}+V'(U) = 0, \tag{58}$$

$$(\omega^2-\kappa^2)\,U_{1\theta\theta}+V''(U)\,U_1 = 2\omega U_{\theta T}+2\kappa U_{\theta X}+\omega_T U_\theta+\kappa_X U_\theta, \tag{59}$$

where $\omega = -\theta_t = -\Theta_T$, $\kappa = \theta_x = \Theta_X$. Equation (58) is the equation for a uniform wavetrain, and may be considered as an ordinary differential equation in the variable θ even though U,ω,κ are also functions of (X,T); the dependence on (X,T) gives the slow variation. A first integral of (58) is

$$\tfrac{1}{2}(\omega^2-\kappa^2)\,U_\theta^2+V(U) = E(X,T). \tag{60}$$

At this stage ω (X,T), $\kappa(X,T)$, $E(X,T)$ are undetermined.

Turn attention now to (59). As an ordinary differential equation in θ, the right-hand side is known and the left-hand side is linear in U_1. Moreover, a solution of the homogeneous equation is $U_1 = U_\theta$, since the left-hand side is then the derivative of (58). In principle, (59) can be solved by the substitution $U_1 = fU_\theta$. Here it is sufficient to note that (59) can be recast as

$$(\omega^2-\kappa^2)\frac{\partial}{\partial\theta}(U_{1\theta}\,U_\theta-U_1\,U_{\theta\theta}) = (\omega U_\theta^2)_T+(\kappa U_\theta^2)_X.$$

The right-hand side is periodic in θ. Hence U_1 is bounded in θ only if the integral of the right-hand side over one period vanishes. The condition is

$$\frac{\partial}{\partial T}\int_0^{2\pi}\omega\,U_\theta^2\,\mathrm{d}\theta+\frac{\partial}{\partial X}\int_0^{2\pi}\kappa U_\theta^2\,\mathrm{d}\theta = 0.$$

It may be verified that this may be rewritten from (57) and (60) as

$$\frac{\partial \mathscr{L}_\omega}{\partial T} - \frac{\partial \mathscr{L}_\kappa}{\partial X} = 0. \tag{61}$$

Thus, the 'orthogonality condition', to avoid secular terms in solving (59), gives the result of the averaged variational principle.

The discussion of higher order terms is an intricate one with introduction of a further orthogonality condition. The details are given by Luke (1966).

6. Integral equations for more general dispersion

Linear partial differential equations in $(\mathbf{x}, t)$, with constant coefficients, can only give polynomial dispersion functions; the correspondence is

$$\frac{\partial}{\partial t} \leftrightarrow -\mathrm{i}\omega, \quad \frac{\partial}{\partial x} \leftrightarrow \mathrm{i}\kappa.$$

Water waves have

$$\omega^2 = g\kappa \tanh \kappa h_0, \tag{62}$$

but this is through the dependence on an extra coordinate y, which in this sense is not part of the (x, t) space in which the wave propagation occurs. One method of obtaining more general dispersion in an (x, t) problem is to consider, for example,

$$\frac{\partial \eta}{\partial t} + \int_{-\infty}^{\infty} K(x-\xi)\, \eta_\xi(\xi, t)\, \mathrm{d}\xi = 0, \tag{63}$$

where $K(x)$ is a suitably chosen kernel. Solutions

$$\eta = \mathrm{e}^{\mathrm{i}(\kappa x - \omega t)}$$

satisfy (63) provided that

$$c = \frac{\omega}{\kappa} = \int_{-\infty}^{\infty} K(x-\xi)\, \mathrm{e}^{-\mathrm{i}\kappa(x-\xi)}\, \mathrm{d}\xi. \tag{64}$$

This means that any phase velocity $c(\kappa)$ can be obtained by choosing $K(x)$ to be the Fourier transform of $c(\kappa)$;

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\kappa)\, \mathrm{e}^{\mathrm{i}\kappa x}\, \mathrm{d}\kappa. \tag{65}$$

For the polynomial cases, $K(x)$ is a sum of δ functions. In the example

$$c(\kappa) = c_0 + c_2\kappa^2, \quad K(x) = c_0\delta(x) - c_2\delta''(x), \tag{66}$$

(63) becomes the linearized Korteweg–de Vries equation

$$\eta_t + c_0\eta_x - c_2\eta_{xxx} = 0.$$

An equation combining the general dispersion of the integral with typical non-linearity would be

$$\eta_t + \alpha\eta\eta_x + \int_{-\infty}^{\infty} K(x-\xi)\, \eta_\xi(\xi, t)\, \mathrm{d}\xi = 0. \tag{67}$$

The Korteweg–de Vries equation follows when (66) is used. An interesting extension is to consider other kernels and, in particular, for water waves to take

$$c(\kappa) = \left(\frac{g}{\kappa}\tanh \kappa h_0\right)^{\frac{1}{2}}, \quad K_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\kappa)\, e^{i\kappa x}\, d\kappa. \tag{68}$$

The Korteweg–de Vries case takes the first two terms in the long wave expansion $\kappa h_0 \ll 1$.

While the Korteweg–de Vries equation gives solitary waves and cnoidal wavetrains, it is inadequate to give the waves of greatest height with the Stokes 120° angle at the crest. Moreover, the alternative breaking into bores, described by the simpler shallow water equations, is lost because it seems certain that the η_{xxx} term will always prevent breaking (although a proof does not seem to have been given). Both are high frequency effects which are lost by the long wave expansion $\kappa h_0 \ll 1$. Equation (67) is not limited in this way.

Uniform wavetrains are obtained with $\eta = \eta(X)$, $X = x - Ut$, and (67) is

$$(U - \alpha\eta)\,\eta' = \int_{-\infty}^{\infty} K(X-\zeta)\,\eta'(\zeta)\, d\zeta. \tag{69}$$

This can be integrated once to the form

$$A + U\eta - \tfrac{1}{2}\alpha\eta^2 = \int_{-\infty}^{\infty} K(X-\zeta)\,\eta(\zeta)\, d\zeta, \tag{70}$$

where A is an integration constant. For the choice K_g in (68), the explicit solution of (70) cannot be given, but it can be shown that a limiting form occurs when $U = \alpha\eta$ and the crest becomes cusped with a vertical tangent. The cusp instead of the Stokes 120° angle is attributed to the remaining inadequacies of the nonlinear terms. However, a wave of greatest height is predicted so the desired qualitative effect is in.

The kernel in (68) normalized to $g = 1$, $h_0 = 1$, has the properties

$$K_g(x) = K_g(-x),$$
$$K_g(x) \sim (2\pi x)^{-\frac{1}{2}} \quad \text{as} \quad x \to 0,$$
$$K_g(x) \sim (\tfrac{1}{2}\pi^2 x)^{-\frac{1}{2}}\, e^{-\frac{1}{2}\pi x} \quad \text{as} \quad x \to \infty,$$
$$\int_{-\infty}^{\infty} K_g(x)\, dx = 1.$$

If a model kernel

$$K_0(x) = \tfrac{1}{4}\pi\, e^{-\frac{1}{2}\pi|x|}, \quad c(\kappa) = \frac{1}{1+(2\kappa/\pi)^2} \tag{71}$$

is tried, the integrals in (67) and (70) can be eliminated by applying the operator

$$\left(\frac{\partial^2}{\partial x^2} - \tfrac{1}{4}\pi^2\right),$$

since (71) is the Green function for this operator. Further details can then be worked out. With $\alpha = 3c_0/2h_0$, corresponding to the Korteweg–de Vries equation (see (16)), the solitary wave of maximum height has

$$\frac{\eta_{\max.}}{h_0} = \frac{8}{9},$$

and, amazingly (since it is fortuitous) this case has a finite angle of 110° at the crest! The finite angle instead of a cusp is because $K_0(x)$ is regular at $x = 0$, whereas $K_g(x)$ has a singularity there. Thus the angle result should not be taken seriously. However, the result for the maximum height may be taken more seriously since it depends upon the whole profile. The result compares reasonably well with McCowan's value of 0·78 obtained by a sort of Pohlhausen method.

As regards breaking into a bore, (67) also looks hopeful. There is no longer a higher derivative to prevent breaking. The integral could be more analogous to the example

$$\eta_t + (c_0 + \alpha\eta)\,\eta_x + \beta\eta = 0$$

which breaks if the initial slope η_x is ever negative and

$$|\eta_x| > \beta/\alpha.$$

Some first results in this direction have been found by Mr R. L. Seliger and will be published later.

If this type of integro-differential equation proves worthy of much further study, it can be derived from a variational principle and all the theory of slowly varying wavetrains, etc., would follow. The equation (67) is first written with $\alpha\eta = \psi_x$ as

$$\psi_{xt} + \psi_x\psi_{xx} + \int_{-\infty}^{\infty} K(x-\xi)\,\psi_{\xi\xi}(\xi, t)\,\mathrm{d}\xi = 0.$$

This equation follows from the variational principle

$$\delta \iint L\,\mathrm{d}x\,\mathrm{d}t = 0$$

with

$$L = \tfrac{1}{2}\psi_x\psi_t + \tfrac{1}{6}\psi_x^3 + \tfrac{1}{2}\psi_x \int_{-\infty}^{\infty} K(x-\xi)\,\psi_\xi(\xi, t)\,\mathrm{d}\xi.$$

This research was supported by the National Science Foundation and the Office of Naval Research of the U.S. Navy.

Appendix: Variational principle for Rossby waves

The question of a suitable variational principle for Rossby waves was raised in the discussion. This question was investigated after the meeting by Mr R. L. Seliger and the author with results as follows. The simple formulation for Rossby waves in the 'β-plane' approximation is

$$\left.\begin{aligned} \mathrm{D}u/\mathrm{D}t - f(y)\,v &= -p_x,\\ \mathrm{D}v/\mathrm{D}t + f(y)\,u &= -p_y,\\ u_x + v_y &= 0. \end{aligned}\right\} \tag{A 1}$$

Variational principles are most conveniently obtained in terms of potentials. In this case we introduce generalized potentials ϕ, α, β with the representation

$$u = \phi_x + \alpha\beta_x - \alpha, \tag{A 2}$$

$$v = \phi_y + \alpha\beta_y - f\beta, \tag{A 3}$$

$$-p = \phi_t + \alpha\beta_t + \tfrac{1}{2}\{(\phi_x + \alpha\beta_x - \alpha)^2 + (\phi_y + \alpha\beta_y - f\beta)^2\}. \tag{A 4}$$

This is an extension of the Clebsch transformation (Lamb's *Hydrodynamics*, p. 248) to include the Coriolis terms. From (A 1) it is easily shown that the equations to be satisfied by ϕ, α, β are

$$\left.\begin{aligned} \mathrm{D}\alpha/\mathrm{D}t + fv &= 0, \\ \mathrm{D}\beta/\mathrm{D}t - u &= 0, \\ u_x + v_y &= 0, \end{aligned}\right\} \tag{A 5}$$

where u, v stand for the expressions given in (A 2), (A 3). The variational principle is again in terms of the expression for the pressure. The set (A 5) follows from the variational principle

$$\delta \iiint \{\phi_t + \alpha\beta_t + \tfrac{1}{2}(u^2+v^2)\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t = 0, \tag{A 6}$$

where u and v are again expressed by (A 2) and (A 3). The general theory can now be applied to Rossby waves using the Lagrangian in (A 6). Applications and a more detailed discussion of the use of the 'Clebsch potentials' for rotational flows will appear later.

References (Whitham)

Bateman, H. 1944 *Partial differential equations*. Cambridge University Press.

Benney, D. J. & Luke, J. C. 1964 *J. Math. Phys.* **43**, 309.

Korteweg, D. J. & de Vries, G. 1895 *Phil. Mag.* (5), **39**, 422.

Kuzmak, G. E. 1959 *Prikl. Mat. Mekh. Akad. Nauk SSSR* **23**, 515. (Translated in *Appl. Math. Mech.* **23**, 730.)

Lighthill, M. J. 1965 *J. Inst. Math. Appl.* **1**, 269.

Luke, J. C. 1966 *Proc. Roy. Soc.* A **292**, 403.

Luke, J. C. 1967 *J. Fluid Mech.* **27**, 395.

Rüssman, H. 1961 *Arch. Rat. Mech. Anal.* **8**, 353.

Serrin, J. 1959 *Handbook of Physics*, vol. VIII, part I, 144. (Edited by S. Flugge.) Springer.

Whitham, G. B. 1965*a* *Proc. Roy. Soc.* A **283**, 238.

Whitham, G. B. 1965*b* *J. Fluid Mech.* **22**, 273.

Whitham, G. B. 1967 *J. Fluid Mech.* **27**, 399.

Zabusky N. & Kruskal, M. 1965 *Bell. Telephone Labs. Tech. Rep.*