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THE STOCHASTIC CAUSTIC*

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Abstract. The propagation of a high frequency initially plane wave through a homogeneous and isotropic random medium with small, order $O(\sigma)$, fluctuations in index of refraction is investigated using geometrical optics. It is shown that caustics occur along every ray in a distance scale of order $O(\sigma^{-2/3}a)$ where a is the correlation length of the medium. On this scale the ray angle deviations are small, of order $O(\sigma^{2/3})$, while the ray position deviates $O(1)$ from its deterministic value. Furthermore, it is shown that if $t=\gamma^{1/3}s$ where s is arclength along a ray,

$$\gamma=2\int_0^\infty dr\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 R(r),$$

and $R(r)$ is the correlation function of the medium, then as a function of t the probability density of distance to first caustic is as $\sigma\to 0$ a universal curve, with no free parameters and thus does not depend on the detailed statistics of the random medium. For small values of t this density is given by the asymptotic formula $f(t)\approx(a_1/t^4)\exp\{-a_2/t^3\}$, with $a_1=1.7399^+$, $a_2=.6565^+$. These results parallel results of V. Kulkarny and B. S. White for two-dimensional random media, where the analogous small t formula has been shown to be valid into the initial region of caustic formation, and may thus be used to determine, in an experimental situation, whether or not caustic formation is likely.

1. Introduction. In [1] the propagation of high frequency waves or weak shocks in a two-dimensional medium with small statistically homogeneous and isotropic random inhomogeneities was investigated using geometrical optics. It was shown that despite the assumed smallness, of order $O(\sigma)$, of the index of refraction fluctuations, a plane wave propagating long distances, of order $O(\sigma^{-2/3}a)$, where a is the medium correlation length, will develop singular amplitudes, or caustics. Furthermore the probability distribution of the distance along a ray to first caustic formation is given, for small σ, by a universal curve; that is, under general hypotheses, the detailed statistics of the random medium are irrelevant to the form of the caustic probability curve, contributing only a single scale factor computable from the medium correlation function.

Recently, the two-dimensional theory has been verified by the Monte-Carlo simulations of Hesselink [2], who has experimentally reproduced the universal caustic formation curve with the predicted scale factor.

In this paper, the main results of [1] will be extended to treat the fully three-dimensional problem. It is assumed that the normalized propagation speed (reciprocal of the index of refraction) is of the form

$$c(x)=1+\sigma\hat{c}(x), \tag{1.1}$$

where $0<\sigma\ll 1$, and $\hat{c}$ is a mean zero homogeneous and isotropic random field, with correlation function

$$R(|x|)=E[\hat{c}(x')\hat{c}(x'+x)]. \tag{1.2}$$

Then, if

$$\gamma_2=\left(2\int_0^\infty\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 R(r)\,dr\right)^{1/2} \tag{1.3}$$

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and

$$t = (\sigma\gamma_2)^{2/3} s, \tag{1.4}$$

where s is arclength along a ray, then the probability density of distance to first caustic, written as a function of t, is asymptotically as $\sigma \to 0$ a universal curve. For small values of t this curve takes the simple form

$$f(t) \sim \frac{a_1}{t^4} \exp\left\{-\frac{a_2}{t^3}\right\}, \tag{1.5}$$

with approximate values of a_1, a_2 given by

$$a_1 = 1.7399^+, \qquad a_2 = .6565^+. \tag{1.6}$$

The analogous small t expression for the two-dimensional problem has been shown to be valid into the initial region of caustic formation [1]. It may thus be used to determine, in an experimental situation, whether or not large amplitudes are likely.

In [1] the two-dimensional formula analogous to (1.5) was also used as part of an approximation giving $f(t)$ for all values of t. To accomplish this for the three-dimensional problem it is necessary to solve numerically a linear parabolic partial differential equation with singular coefficients, as will be shown here. The numerical work will be deferred to a future publication.

2. Rays and wavefront curvatures. The phase, Ψ, of a wave propagating according to the laws of geometrical optics satisfies the eiconal equation [3]

$$|\nabla\Psi|^2 = \frac{1}{c^2}, \tag{2.1}$$

where $c = c(x)$ is the (normalized) propagation speed. The surfaces of constant Ψ are wavefronts and the curves everywhere orthogonal to the wavefronts are rays. Equations for the rays can be obtained from (2.1) and this orthogonality condition. We denote the ray emanating from x_0 by $x(s)$ where s is arclength along the ray. Then if I is the identity matrix, the ray equations take the form

$$\frac{dx}{ds} = V_1 \qquad \frac{dV_1}{ds} = -\frac{1}{c}(I - V_1 V_1^T)\nabla c, \tag{2.2}$$

with initial conditions

$$x(0) = x_0, \qquad V_1(0) = U_0. \tag{2.3}$$

Here V_1 is the unit tangent to $x(s)$. Let V_2, V_3 be the unit normal and binormal respectively. Then the Frenet formulae [4] give equations for V_1, V_2, V_3:

$$\frac{dV_1}{ds} = \kappa V_2, \quad \frac{dV_2}{ds} = -\kappa V_2 + \tau V_3, \quad \frac{dV_3}{ds} = -\tau V_2, \tag{2.4}$$

where κ is the curvature and τ the torsion of the space curve $x(s)$. Equations (2.5)–(2.10) are discussed in more detail in the Appendix, § 8. Let

$$c_i = \nabla c \cdot V_i, \qquad c_{ij} = V_i^T \nabla\nabla c V_j. \tag{2.5}$$

Then

$$\kappa = -\frac{c_2}{c}, \qquad \tau = \frac{c_{13}}{c_2}. \tag{2.6}$$

Now a caustic occurs when one of the two principal normal curvatures of the wavefront becomes infinite. From the Appendix the two principal normal curvatures of the wavefront at the point $x(s)$ are represented as the eigenvalues of a 2×2 symmetric matrix $\tilde{\mathbf{R}}$, with elements

$$\tilde{R}_{ij}(s)=c(x(s))V_{i+1}^T(s)\nabla\nabla\Psi(x(s))V_{j+1}(s), \qquad i,j=1,2. \tag{2.7}$$

$\tilde{\mathbf{R}}$ satisfies the propagation equation

$$\frac{d}{ds}\tilde{\mathbf{R}}=-\tilde{\mathbf{R}}^2+\tau(\tilde{\mathbf{M}}\tilde{\mathbf{R}}-\tilde{\mathbf{R}}\tilde{\mathbf{M}})+\frac{c_1}{c}\tilde{\mathbf{R}}+\tilde{\mathbf{Q}}. \tag{2.8}$$

Here

$$\tilde{\mathbf{M}}=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}, \qquad \mathbf{Q}=-\frac{1}{c}\begin{bmatrix} c_{22} & c_{23}\\ c_{23} & c_{33}\end{bmatrix}. \tag{2.9}$$

In (2.8), (2.9) τ, c_i, c_{ij} are evaluated along the ray, that is, at $x(s)$.

For a wavefront which is initially plane, we have the initial condition

$$\tilde{\mathbf{R}}(0)=0. \tag{2.10}$$

It is now assumed that c is of the form (1.1) with σ small, and $\hat{c}$ a mean zero homogeneous and isotropic random field. We introduce the scaled variables t, U, y defined by

$$t=\sigma^{2/3}s, \quad V_1=U_0+\sigma^{2/3}U, \quad x=x_0+\frac{t}{\sigma^{2/3}}U_0+y. \tag{2.11}$$

Thus, for t, of order $O(1)$, s will be large. Furthermore, we have anticipated that the fluctuations in V_1 are $O(\sigma^{2/3})$, those in x, $O(1)$. Then from (2.2), (1.1) we have

$$\begin{aligned}\frac{dy}{dt}&=U,\\ \frac{dU}{dt}&=-\frac{1}{\sigma^{1/3}}(I-U_0U_0^T)\nabla\hat{c}+O(\sigma^{2/3}+\sigma^{1/3}|U|).\end{aligned} \tag{2.12}$$

Here $\hat{c}$ is evaluated along the ray, i.e.

$$\nabla\hat{c}=\nabla\hat{c}\left(x_0+\frac{t}{\sigma^{2/3}}U_0+y\right). \tag{2.13}$$

We have also the initial conditions

$$y(0)=U(0)=0. \tag{2.14}$$

Now by definition (2.11), $V_1=U_0+O(\sigma^{2/3}|U|)$. We will derive similar approximations for V_2, V_3 valid for long distances $s=t/\sigma^{2/3}$ with t of $O(1)$.

For $j=2,3$

$$0=V_j\cdot V_1=V_j\cdot U_0+O(\sigma^{2/3}|U|),$$

so that

$$V_j\cdot U_0=O(\sigma^{2/3}|U|) \quad \text{for } j=2,3. \tag{2.15}$$

We now work in a fixed Cartesian coordinate system with $V_1(0)=U_0=(1,0,0)^T$, $V_2(0)=(0,1,0)^T$, $V_3(0)=(0,0,1)^T$. Then

$$V_1=(1,0,0)^T+O(\sigma^{2/3}|U|),$$
$$V_j=(0,V_j^{(2)},V_j^{(3)})+O(\sigma^{2/3}|U|),\qquad j=2,3. \tag{2.16}$$

Note that from (2.5), (2.6), (1.1) the curvature, κ, of the ray is small, of order $O(\sigma)$, while the torsion, τ, is order $O(1)$. We next approximate $V_j^{(2)}$, $V_j^{(3)}$ for $j=2,3$. From (2.4), (2.12) we obtain

$$\frac{dV_2}{dt}=\frac{\tau}{\sigma^{2/3}}V_3-\sigma^{1/3}\frac{\hat{c}_2}{c}V_1,\qquad \frac{dV_3}{dt}=-\frac{\tau}{\sigma^{2/3}}V_2. \tag{2.17}$$

Thus on the distance scale on which t is $O(1)$, V_2, V_3 undergo rapid random rotations with angular speed $O(\sigma^{-2/3})$. Let the 2×2 matrix $\mathbf{P}$ be defined as

$$\mathbf{P}=\begin{bmatrix} V_2^{(2)} & V_3^{(2)} \\ V_2^{(3)} & V_3^{(3)} \end{bmatrix}. \tag{2.18}$$

From (2.16), (2.17), (2.18) we have that

$$\frac{d\mathbf{P}}{dt}=-\frac{\tau}{\sigma^{2/3}}\mathbf{P}\tilde{\mathbf{M}}+O(\sigma|U|),\qquad \mathbf{P}(0)=I. \tag{2.19}$$

Now let $\mathbf{E}$ satisfy

$$\frac{d\mathbf{E}}{dt}=-\frac{\tau}{\sigma^{2/3}}\mathbf{E}\tilde{\mathbf{M}},\qquad \mathbf{E}(0)=I. \tag{2.20}$$

Clearly

$$\mathbf{E}=\begin{bmatrix} \cos\bar{\theta} & -\sin\bar{\theta} \\ \sin\bar{\theta} & \cos\bar{\theta} \end{bmatrix}, \tag{2.21}$$

where

$$\bar{\theta}(t)=\int_0^{t/\sigma^{2/3}}\tau(h)\,dh. \tag{2.22}$$

Since $\mathbf{E}$ is a rotation matrix, $\mathbf{E}^T=\mathbf{E}^{-1}$, and $\|\mathbf{E}\|$ is bounded. Therefore, from (2.19), (2.20)

$$\frac{d}{dt}\mathbf{P}\mathbf{E}^T=O(\sigma|U|),$$

and hence

$$\mathbf{P}(t)=-\mathbf{E}(t)+O(\sigma\|U\|), \tag{2.23}$$

where $\|U\|=|U(t)|+\int_0^t|U(h)|\,dh$. Thus V_1, V_2, V_3 are determined approximately from (2.16), (2.18), (2.23).

We next change basis of the representation of $\tilde{\mathbf{R}}$ in such a way as to remove, approximately, the troublesome torsion terms in (2.8). Thus, let

$$\hat{\mathbf{R}}=\mathbf{E}\tilde{\mathbf{R}}\mathbf{E}^T. \tag{2.24}$$

Clearly the eigenvalues of $\hat{\mathbf{R}}$ are the same as those of $\tilde{\mathbf{R}}$, and therefore are the principal normal curvatures of the wavefront at $x(s)$. Substitution of (2.24) into (2.8) yields,

$$\frac{d}{ds}\hat{\mathbf{R}}=\sigma^{2/3}\frac{d\hat{\mathbf{R}}}{dt}=-\hat{\mathbf{R}}^2+\frac{c_1}{c}\hat{\mathbf{R}}+\mathbf{E}\tilde{\mathbf{Q}}\mathbf{E}^T. \tag{2.25}$$

Now let $\partial/\partial x_i$ be a derivative in the fixed direction given by $V_i(0)$, and let $\mathbf{Q}(t)$ be the 2×2 matrix with elements

$$Q_{ij}(t) = -\frac{1}{c}\frac{\partial^2 c}{\partial x_{i+1}\,\partial x_{j+1}}\left(x_0 + \frac{t}{\sigma^{2/3}}U_0 + y\right). \tag{2.26}$$

For $i, j = 1, 2$ we have from (2.15), since $\tilde{\mathbf{Q}}$ is $O(\sigma)$, that

$$\tilde{\mathbf{Q}}_{ij} = -\frac{1}{c}\sum_{k,l=2}^{3} V_{i+1}^{(k)}\frac{\partial^2 c}{\partial x_k\,\partial x_l}V_{j+1}^{(l)} + O(\sigma^{5/3}|U|). \tag{2.27}$$

Therefore from (2.18), (2.26), (2.27)

$$\tilde{\mathbf{Q}} = \mathbf{P}\mathbf{Q}\mathbf{P}^T + O(\sigma^{5/3}\|U\|). \tag{2.28}$$

Substitution of (2.23) into (2.28) then yields

$$\tilde{\mathbf{Q}} = \mathbf{E}\mathbf{Q}\mathbf{E}^T + O(\sigma^{5/3}\|U\|). \tag{2.29}$$

Substitution of (2.29) into (2.25) then yields

$$\frac{d}{ds}\hat{\mathbf{R}} = \sigma^{2/3}\frac{d}{dt}\hat{\mathbf{R}} = -\hat{\mathbf{R}}^2 + \frac{c_1}{c}\hat{\mathbf{R}} + \mathbf{Q} + O(\sigma^{5/3}\|U\|). \tag{2.30}$$

The equation (2.30) now has the torsion terms removed, and the matrix $\mathbf{Q}$ of second derivatives of c now has those derivatives, via (2.26) in fixed directions that do not rotate with V_1, V_2, V_3.

Direct approximations to $\hat{\mathbf{R}}$ are not possible since, as will be shown, the eigenvalues of $\hat{\mathbf{R}}$ become infinite for finite (random) t. Small errors in the determination of this value of t will thus give infinite errors in any approximation to $\hat{\mathbf{R}}$. To circumvent this difficulty, the Ricatti equation (2.30) will be reduced to a linear equation by the inclusion of more variables. Thus, let $\bar{\mathbf{A}}$ be defined by

$$\frac{d}{ds}\bar{\mathbf{A}} = \hat{\mathbf{R}}\bar{\mathbf{A}}, \qquad \bar{\mathbf{A}}(0) = I. \tag{2.31}$$

Letting $\bar{\mathbf{B}} = \hat{\mathbf{R}}\bar{\mathbf{A}}$ we obtain the linear matrix equations

$$\frac{d}{ds}\bar{\mathbf{A}} = \bar{\mathbf{B}}, \qquad \frac{d}{ds}\bar{\mathbf{B}} = \mathbf{Q}\bar{\mathbf{A}} + O(\sigma^{5/3}\|U\|\,\|\bar{\mathbf{A}}\| + \sigma\|\bar{\mathbf{B}}\|). \tag{2.32}$$

Now letting $t = \sigma^{2/3}s$ and

$$\hat{\mathbf{A}} = \bar{\mathbf{A}}, \qquad \hat{\mathbf{B}} = \sigma^{-2/3}\bar{\mathbf{B}}, \tag{2.33}$$

we have that on the t-scale

$$\begin{aligned}&\frac{d}{dt}\hat{\mathbf{A}} = \hat{\mathbf{B}},\\ &\frac{d}{dt}\hat{\mathbf{B}} = -\frac{1}{\sigma^{1/3}}\begin{bmatrix}\hat{c}_{x_2x_2} & \hat{c}_{x_2x_3}\\ \hat{c}_{x_2x_3} & \hat{c}_{x_3x_3}\end{bmatrix}\hat{\mathbf{A}} + O(\sigma^{2/3} + \sigma^{1/3}\|U\|\,\|\hat{\mathbf{A}}\| + \sigma^{1/3}\|\hat{\mathbf{B}}\|).\end{aligned} \tag{2.34}$$

Here

$$\hat{c}_{x_ix_j} = \frac{\partial^2}{\partial x_i\,\partial_{x_j}}\hat{c}\left(x_0 + \frac{t}{\sigma^{2/3}}U_0 + y\right). \tag{2.35}$$

Equations (2.34) are now properly scaled for the limit of the next section. We note here what this scaling implies qualitatively for the actual curvatives, the eigenvalues of $\hat{\mathbf{R}}$. Note that

$$\hat{\mathbf{R}} = \sigma^{2/3}\hat{\mathbf{B}}\hat{\mathbf{A}}^{-1}. \tag{2.36}$$

Now $\hat{\kappa} = \det \hat{\mathbf{R}}$, the product of the principal normal curvatures, is the Gaussian curvature of the wavefront at $x(s)$. With this interpretation, the scaling (2.36) shows the behavior of $\hat{\kappa}$ if well behaved limits for $\hat{\mathbf{B}}, \hat{\mathbf{A}}$ are obtained. As long as $\det \hat{\mathbf{A}}$ is bounded away from zero, $\hat{\kappa}$ is small, of order $O(\sigma^{4/3})$. However, in the relatively small distance (when viewed on the $t = \sigma^{2/3}s$ scale) that it takes for $\det \hat{\mathbf{A}}$ to pass through zero $\hat{\kappa}$ must become infinite. The geometrical picture is then this: the patch of wavefront traveling with the ray $x(s)$ first propagates substantially as a plane wave, i.e., its curvature is small; the formation of a caustic appears almost instantaneous on the t distance scale. If this phenomenon is observed experimentally, it may well be misinterpreted as implying a strong inhomogeneity in the velocity field in a neighborhood of the caustic. However, this impulsive behavior is predicted by the scaling law (2.36) when fluctuations in index of refraction are small.

3. The limit equations. We now apply the Papanicolaou–Kohler theorem [5] to the results of the last section, after dropping the error terms. From (2.12) we have the two vector equations

$$\frac{dy}{dt} = U, \qquad \frac{dU}{dt} = -\frac{1}{\sigma^{1/3}}\begin{bmatrix} 0 \\ \hat{c}_{x_2} \\ \hat{c}_{x_3} \end{bmatrix}, \tag{3.1}$$

with initial conditions

$$y(0) = U(0) = 0, \tag{3.2}$$

and the two 2×2 matrix equations

$$\frac{d\hat{\mathbf{A}}}{dt} = \hat{\mathbf{B}}, \qquad \frac{d\hat{\mathbf{B}}}{dt} = -\frac{1}{\sigma^{1/3}}\begin{bmatrix} \hat{c}_{x_2x_2} & \hat{c}_{x_2x_3} \\ \hat{c}_{x_2x_3} & \hat{c}_{x_3x_3} \end{bmatrix}\hat{\mathbf{A}}, \tag{3.3}$$

with initial conditions

$$\hat{\mathbf{A}}(0) = I, \qquad \hat{\mathbf{B}}(0) = 0. \tag{3.4}$$

Here, of course,

$$\hat{c}_{x_i} = \frac{\partial}{\partial_{x_i}}\hat{c}\left(x_0 + \frac{t}{\sigma^{2/3}}U_0 + y\right),$$

with a similar expression for $\hat{c}_{x_ix_j}$.

Note that, from (3.1), $u^{(1)} = y^{(1)} = 0$, so that taking x_0 as the origin and writing in Cartesian coordinates we have that

$$\frac{1}{\sigma^{1/3}}\hat{c}_{x_i} = \frac{1}{\sigma^{1/3}}\hat{c}_{x_i}\left(\frac{t}{\sigma^{2/3}}, y^{(2)}, y^{(3)}\right),$$

with a similar expression for $\hat{c}_{x_ix_j}$. To apply the limit theorem, we need to assume a "mixing hypothesis". This means roughly, that the values of c in sets $\mathcal{A}, \mathcal{B}$ of physical space become asymptotically statistically independent as the distance between $\mathcal{A}$ and $\mathcal{B}$ becomes large.

We now apply the Papanicolaou–Kohler theorem, using the easily proven identities that if $R(r)$ is the correlation function of $\hat{c}$ [1.2] and if

$$
\begin{aligned}
q_{ij}(r) &= E\left[\frac{\partial \hat{c}(x_1+r, x_2, x_3)}{\partial x_i}\,\frac{\partial \hat{c}(x_1, x_2, x_3)}{\partial x_j}\right],\\
q_{ijk}(r) &= E\left[\frac{\partial^2 \hat{c}(x_1+r, x_2, x_3)}{\partial x_i\,\partial x_j}\,\frac{\partial \hat{c}(x_1, x_2, x_3)}{\partial x_k}\right],\\
q_{ijkl}(r) &= E\left[\frac{\partial^2 \hat{c}(x_1+r, x_2, x_3)}{\partial x_i\,\partial x_j}\,\frac{\partial^2 \hat{c}(x_1, x_2, x_3)}{\partial x_k\,\partial x_l}\right],
\end{aligned}
\tag{3.5}
$$

then

$$
\begin{aligned}
q_{ij}(r) &= -\frac{1}{r}\frac{\partial}{\partial r}R(r)\,\delta_{ij} \quad \text{for } i \neq 1, j \neq 1,\\
q_{ijk}(r) &= 0 \quad \text{for } i \neq 1, j \neq 1, k \neq 1,\\
q_{ijkl}(r) &= \left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 R(r)\right][\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]\\
&\qquad \text{for } i \neq 1, j \neq 1, k \neq 1, l \neq 1.
\end{aligned}
\tag{3.6}
$$

Here δ_{ij} is the Kronecker delta.

The result of the limit theorem is then that $(y, U, \hat{\mathbf{A}}, \hat{\mathbf{B}})$ converges weakly as a stochastic process (on $\mathscr{C}[0, t_0]$ for any t_0) to $(\bar{y}, \bar{U}, \mathbf{A}, \mathbf{B})$, where $(\bar{y}, \bar{U})$ is statistically independent of $\mathbf{A}, \mathbf{B}$ (this independence is basically a consequence of $q_{ijk} = 0$ in (3.6)). Then $(\bar{y}, \bar{U})$ satisfy the Ito stochastic differential equations

$$
\begin{aligned}
&\bar{y}^{(1)} = \bar{u}^{(1)} = 0,\\
&d\bar{y}^{(2)} = \bar{u}^{(2)}\,dt, \qquad d\bar{u}^{(2)} = \gamma_1\, d\tilde{\beta}_1,\\
&d\bar{y}^{(3)} = \bar{u}^{(3)}\,dt, \qquad d\bar{u}^{(2)} = \gamma_1\, d\tilde{\beta}_2.
\end{aligned}
\tag{3.7}
$$

Here $\tilde{\beta}_2, \tilde{\beta}_3$ are independent Brownian motions and

$$
\gamma_1 = \left(-2\int_0^\infty \frac{1}{r}\frac{\partial R}{\partial r}\,dr\right)^{1/2}.
\tag{3.8}
$$

Thus, on this scale the ray deviations, $\bar{u}^{(2)}, \bar{u}^{(3)}$ are independent Brownian motions, and the positions $\bar{y}^{(2)}, \bar{y}^{(3)}$ are their integrals. This corresponds to a result of Chernov's [6], but is here derived by an honest method in the sense of J. B. Keller [7]. We also see the scale on which this result holds, i.e., propagation distance is of order $O(\sigma^{-2/3})$, the ray angles are of order $O(\sigma^{2/3})$, but the ray positions have deviated $O(1)$ in the plane orthogonal to the initial direction of propagation.

We now turn to the limit equations for $\mathbf{A}, \mathbf{B}$. Letting, for $i, j = 1, 2$, A_{ij}, B_{ij} be the elements of $\mathbf{A}, \mathbf{B}$ we obtain the infinitesimal generator (Kolmogorov backward operator) in the form

$$
\mathscr{L} = \frac{1}{2}a^{ijkl}\frac{\partial^2}{\partial B_{ij}\,\partial B_{kl}} + B_{ij}\frac{\partial}{\partial A_{ij}},
\tag{3.9}
$$

where the summation convention is implied. The coefficients a^{ijkl} are given by

$$
a^{ijkl} = \gamma_2^2[\delta_{im}\delta_{kp} + \delta_{ik}\delta_{mp} + \delta_{ip}\delta_{mk}]A_{mj}A_{pl},
\tag{3.10}
$$

where

$$(3.11)\qquad \gamma_2=\left(2\int_0^\infty\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 R(r)\,dr\right)^{1/2}.$$

The generator $\mathscr{L}$ corresponds to the Ito stochastic differential equation

$$(3.12)\qquad dA_{ij}=B_{ij}\,dt,\qquad dB_{ij}=h_{imnr}A_{mj}\,d\beta_{nr},$$

where β_{nr} are independent Brownian motions. Since

$$a^{ijkl}\,dt=dB_{ij}\,dB_{kl}=h_{imnr}h_{kpnr}A_{mj}A_{pl}\,dt,$$

h_{imnr} may be determined, via (3.10), by the (nonunique) factorization

$$(3.13)\qquad h_{imnr}h_{kpnr}=\gamma_2^2[\delta_{im}\delta_{kp}+\delta_{ik}\delta_{mp}+\delta_{ip}\delta_{mk}].$$

Equation (3.13) may be solved by writing the 16 components of h_{imnr} as a 4×4 matrix Σ with components

$$(3.14)\qquad h_{imnr}=\Sigma_{k(i,m),k(n,r)},$$

with $k(1,1)=1$, $k(2,2)=2$, $k(1,2)=3$, $k(2,1)=4$. The equation for h then becomes

$$(3.15)\qquad \Sigma\Sigma^T=\gamma_2^2\begin{bmatrix}3&1&0&0\\1&3&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix},$$

with a conveniently chosen factorization

$$(3.16)\qquad \Sigma=\gamma_2\begin{bmatrix}\sqrt{2}&1&0&0\\\sqrt{2}&-1&0&0\\0&0&1&0\\0&0&1&0\end{bmatrix}.$$

Then $h_{imnr}\,d\beta_{nr}$ may be determined from

$$(3.17)\qquad \Sigma\begin{bmatrix}d\beta_{11}\\d\beta_{22}\\d\beta_{12}\\d\beta_{21}\end{bmatrix}=\gamma_2\begin{bmatrix}\sqrt{2}\,d\beta_{11}+d\beta_{22}\\\sqrt{2}\,d\beta_{11}-d\beta_{22}\\d\beta_{12}\\d\beta_{12}\end{bmatrix},$$

so that $h_{imnr}\,d\beta_{nr}$ may be written as the matrix of stochastic differentials

$$(3.18)\qquad H=\gamma_2\begin{bmatrix}\sqrt{2}\,d\beta_{11}+d\beta_{22}&d\beta_{12}\\d\beta_{12}&\sqrt{2}\,d\beta_{11}-d\beta_{22}\end{bmatrix}.$$

Therefore we obtain from the expression (3.12)

$$(3.19)\qquad dA_{ij}=B_{ij}\,dt,\qquad dB_{ij}=(HA)_{ij}.$$

Now the constant γ_2 may be set equal to unity by the rescaling

$$(3.20)\qquad t'=\gamma_2^{2/3}t,\qquad B'_{ij}=\gamma_2^{-2/3}B_{ij}.$$

We then obtain universal equations for **A**, **B** with no free parameters. We will drop the primes in (3.20) and write the equations fully. They are

$$(3.21)\qquad \begin{aligned} &d\mathbf{A}=\mathbf{B}\,dt,\\ &d\mathbf{B}=\left(\sqrt{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}d\beta_{11}+\begin{bmatrix}1&0\\0&-1\end{bmatrix}d\beta_{22}+\begin{bmatrix}0&1\\1&0\end{bmatrix}d\beta_{12}\right)\mathbf{A},\\ &\mathbf{A}(0)=I,\qquad \mathbf{B}(0)=0.\end{aligned}$$

Thus the random coefficient in (3.21) consists of the sum of three independent white noises multiplied by three Pauli spin matrices.

We may now transform to find the equation satisfied by $\mathbf{R}\equiv\mathbf{B}\mathbf{A}^{-1}$. **R** satisfies the stochastic Ito–Riccati equation

$$(3.22)\qquad \begin{aligned} &d\mathbf{R}=-\mathbf{R}^2\,dt+\sqrt{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}d\beta_{11}+\begin{bmatrix}1&0\\0&-1\end{bmatrix}d\beta_{22}+\begin{bmatrix}0&1\\1&0\end{bmatrix}d\beta_{12},\\ &\mathbf{R}(0)=0.\end{aligned}$$

4. Evolution of the principal normal curvatures. Now the two principal normal curvatures of the wavefront are related to the eigenvalues, λ_1, λ_2, of **R** by scaling. To determine the first distance at which a caustic occurs, we look for values of t at which one of λ_1, λ_2 becomes infinite. To obtain evolution equations for λ_1, λ_2, we diagonalize **R**. Let

$$(4.1)\qquad \mathbf{D}=\begin{bmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{bmatrix}$$

be the rotation matrix that puts **R** in diagonal form, i.e.,

$$(4.2)\qquad \mathbf{D}^T\mathbf{R}\mathbf{D}=\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}.$$

Now $\theta, \lambda_1, \lambda_2$ are functions of **R**. $d\theta$, $d\lambda_1$, $d\lambda_2$ are therefore stochastic differentials, e.g.,

$$d\theta=\phi_0\,dt+\phi_1\,d\beta_{11}+\phi_2\,d\beta_{12}+\phi_3\,d\beta_{22},$$

where ϕ_i are nonanticipating functionals of $\beta_{11}, \beta_{22}, \beta_{12}$. To obtain equations for $d\theta$, $d\lambda_1$, $d\lambda_2$, Ito calculus [8] must be used. Thus, many terms appear in the following calculations which would not appear in ordinary calculus, e.g.,

$$(4.3)\qquad dD=\begin{bmatrix}-\sin\theta&-\cos\theta\\ \cos\theta&-\sin\theta\end{bmatrix}d\theta+\frac{1}{2}\begin{bmatrix}-\cos\theta&\sin\theta\\ -\sin\theta&-\cos\theta\end{bmatrix}(d\theta)^2.$$

By proceeding in this way and using (3.22) the following three equations are obtained from the expression for

$$d\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix};$$

$$(4.4)\qquad \begin{aligned} d\lambda_1=&-\lambda_1^2\,dt+\sqrt{2}\,d\beta_{11}+\cos 2\theta\,d\beta_{22}+\sin 2\theta\,d\beta_{12}\\ &+(\lambda_2-\lambda_1)(d\theta)^2-2\sin 2\theta\,d\beta_{22}\,d\theta+2\cos 2\theta\,d\beta_{12}\,d\theta,\end{aligned}$$

$$(4.5)\qquad \begin{aligned} d\lambda_2=&-\lambda_2^2\,dt+\sqrt{2}\,d\beta_{11}-\cos 2\theta\,d\beta_{22}-\sin 2\theta\,d\beta_{12}\\ &+(\lambda_1-\lambda_2)(d\theta)^2+2\sin 2\theta\,d\beta_{22}\,d\theta-2\cos 2\theta\,d\beta_{12}\,d\theta,\end{aligned}$$

$$(4.6)\qquad \begin{aligned} 0=&(\lambda_2-\lambda_1)\,d\theta-\sin 2\theta\,d\beta_{22}+\cos 2\theta\,d\beta_{12}\\ &-2\cos 2\theta\,d\beta_{22}\,d\theta-2\sin 2\theta\,d\beta_{12}\,d\theta.\end{aligned}$$

Multiplication of (4.6) successively by $d\beta_{11}, d\beta_{22}, d\beta_{12}$ yields the three equations

$$d\beta_{11}\,d\theta = 0, \tag{4.7}$$

$$(\lambda_2-\lambda_1)\,d\theta\,d\beta_{22} = \sin 2\theta\,dt, \tag{4.8}$$

$$(\lambda_2-\lambda_1)\,d\theta\,d\beta_{12} = -\cos 2\theta\,dt. \tag{4.9}$$

Putting (4.8), (4.9) into (4.6) yields

$$(\lambda_1-\lambda_2)\,d\theta = -\sin 2\theta\,d\beta_{22} + \cos 2\theta\,d\beta_{12}. \tag{4.10}$$

Squaring (4.10) we obtain

$$(\lambda_1-\lambda_2)^2(d\theta)^2 = dt. \tag{4.11}$$

Now substitution of (4.7)–(4.11) into (4.4), (4.5) yields

$$d\lambda_1 = -\left(\lambda_1^2 + \frac{1}{(\lambda_2-\lambda_1)}\right)dt + \sqrt{2}\,d\beta_{11} + \cos 2\theta\,d\beta_{22} + \sin 2\theta\,d\beta_{12}, \tag{4.12}$$

$$d\lambda_2 = -\left(\lambda_2^2 + \frac{1}{(\lambda_1-\lambda_2)}\right)dt + \sqrt{2}\,d\beta_{11} - \cos 2\theta\,d\beta_{22} - \sin 2\theta\,d\beta_{12}. \tag{4.13}$$

The relevant equations are now (4.10), (4.12), (4.13). These may be simplified further by defining $\beta_1, \beta_2, \beta_3$ by

$$\begin{aligned} d\beta_1 &= d\beta_{11}, \\ d\beta_2 &= \cos 2\theta\,d\beta_{22} + \sin 2\theta\,d\beta_{12}, \\ d\beta_3 &= -\sin 2\theta\,d\beta_{22} + \cos 2\theta\,d\beta_{12}. \end{aligned} \tag{4.14}$$

Note that $d\beta_i\,d\beta_j = \delta_{ij}\,dt$. $\beta_1, \beta_2, \beta_3$ are therefore independent standard Brownian motions. In terms of them (4.10), (4.12), (4.13) become

$$d\lambda_1 = -\left(\lambda_1^2 + \frac{1}{(\lambda_2-\lambda_1)}\right)dt + \sqrt{2}\,d\beta_1 + d\beta_2, \tag{4.15}$$

$$d\lambda_2 = -\left(\lambda_2^2 + \frac{1}{(\lambda_1-\lambda_2)}\right)dt + \sqrt{2}\,d\beta_1 - d\beta_2, \tag{4.16}$$

$$d\theta = \frac{d\beta_3}{(\lambda_1-\lambda_2)}. \tag{4.17}$$

These equations may now be contrasted with that of the two-dimensional theory, where the single wavefront curvature Z satisfies [1]

$$dZ = -Z^2\,dt + d\beta. \tag{4.18}$$

(The scaling convention is chosen somewhat differently for this case.) The three-dimensional theory is thus quite similar to that of (4.18), except that the two curvatures have correlated noise terms and repel each other inversely as their separation. Because of the repulsion term, λ_1 and λ_2 cannot coalesce for any $t>0$, although they may become arbitrarily close.

Now, for an initially plane wave, $\lambda_1(0)=\lambda_2(0)=0$. The eigenvalues separate instantaneously for $t>0$, say $\lambda_1<\lambda_2$. Then $\lambda_1<\lambda_2$ for all t. If λ satisfies

$$d\lambda = -\lambda^2\,dt + \sqrt{2}\,d\beta_1 + d\beta_2, \tag{4.19}$$

then

$$d\lambda_1+\lambda_1^2\,dt=d\lambda+\lambda^2\,dt+\frac{1}{[\lambda_1-\lambda_2]}\,dt<d\lambda+\lambda^2\,dt.$$

Therefore

$$d\left(\exp\int_0^t[\lambda_1(s)+\lambda_2(s)]\,ds\right)(\lambda_1(t)-\lambda_2(t))<0,$$

and hence

$$\lambda_1(t)<\lambda(t). \tag{4.20}$$

Since, by the two-dimensional theory, $\lambda\to-\infty$ for some finite t with probability one, (4.20) implies the same for λ_1. That is, *a caustic occurs in finite t-distance with probability one.*

This caustic will be formed by one of the principal normal curvatures becoming negatively infinite; that is, the ends of the patch of wavefront focusing will be further advanced along the direction of propagation than will be the middle of the patch. To rule out positively infinite curvatures, we consider the equations for

$$\xi=\lambda_1+\lambda_2,\qquad \rho=|\lambda_2-\lambda_1|. \tag{4.21}$$

For $\lambda_1<\lambda_2$ the Ito equations for ξ,ρ can be computed from (4.15), (4.16). They are

$$\begin{aligned} d\xi&=\tfrac{1}{2}(\xi^2+\rho^2)\,dt+2\sqrt{2}\,d\beta_1,\\ d\rho&=\left(\frac{2}{\rho}-\rho\xi\right)dt-2\,d\beta_2. \end{aligned} \tag{4.22}$$

We may compare ξ to $\bar\lambda$ defined by

$$d\bar\lambda=-\tfrac{1}{2}\bar\lambda^2+2\sqrt{2}\,d\beta_1,\qquad \bar\lambda(0)=0.$$

As in the comparison of λ_1 with λ, we obtain that $\xi<\bar\lambda$, and hence that $\xi\to-\infty$ for some finite t with probability one. Furthermore, the probability that $\xi\to+\infty$ is zero since $+\infty$ is an entrance boundary for $\bar\lambda$[1].

Finally, note that (λ_1,λ_2) form a diffusion Markov process without the necessity of including θ in the state space. The angle θ merely performs a one-dimensioal random walk on a random time scale governed by $(\lambda_1-\lambda_2)$; the rotation speeds up when the curvatures are close and slows down when they are far apart.

The forward Kolmogorov (Fokker–Planck) equation for the probability density $P(t,\lambda_1,\lambda_2)$ of (λ_1,λ_2) at distance t is given, from (4.15), (4.16) as

$$\begin{aligned} P_t&=\frac{3}{2}P_{\lambda_1\lambda_1}+P_{\lambda_1\lambda_2}+\frac{3}{2}P_{\lambda_2\lambda_2}+\frac{\partial}{\partial\lambda_1}\left(\left[\lambda_1^2-\frac{1}{(\lambda_1-\lambda_2)}\right]P\right)\\ &\quad+\frac{\partial}{\partial\lambda_2}\left(\left[\lambda_2^2-\frac{1}{(\lambda_2-\lambda_1)}\right]P\right). \end{aligned} \tag{4.23}$$

For an initially plane wave the initial condition for (4.23) is

$$P(0,\lambda_1,\lambda_2)=\delta(\lambda_1)\delta(\lambda_2). \tag{4.24}$$

Equivalently, we may consider ξ,ρ defined by (4.21). The probability density $\bar P(t,\xi,\rho)$ of ξ,ρ is given by the solution of

$$\bar P_t=4\bar P_{\xi\xi}+2\bar P_{\rho\rho}+\frac{\partial}{\partial\xi}\left(\frac{1}{2}[\xi^2+\rho^2]P\right)+\frac{\partial}{\partial\rho}\left(\left[\rho\xi-\frac{2}{\rho}\right]P\right), \tag{4.25}$$

with initial condition for a plane wave

$$P(0,\xi,\rho)=\delta(\xi)\delta(\rho). \tag{4.26}$$

5. Small t: probability rays in curvature space. We will first investigate the initial separation of λ_1, λ_2 for very short distances $t \ll 1$. For extremely small t ξ, ρ are also small so that we may neglect quadratic terms in (4.22) to get

$$d\xi \approx 2\sqrt{2}\, d\beta_1, \qquad d\rho = \frac{2}{\rho}\, dt - 2\, d\beta_2. \tag{5.1}$$

Thus, while ξ, ρ are both still small, they are approximately independent, with ξ performing a random walk, and ρ behaving like a Bessel process; that is, the equation for ρ is the same as that satisfied by the radial part of a two-dimensional Brownian motion viewed in polar coordinates. Thus the joint probability density $\bar{P}(t,\xi,\rho)$ of ξ, ρ can be written down easily as

$$\bar{P}(t,\xi,\rho) \approx \frac{1}{16\sqrt{\pi t^3}}\rho \exp\left\{-\frac{1}{16t}(\xi^2+2\rho^2)\right\}. \tag{5.2}$$

The corresponding density for λ_1, λ_2 is then

$$P(t,\lambda_1,\lambda_2) \approx \frac{|\lambda_1-\lambda_2|}{16\sqrt{\pi t^3}} \exp\left\{-\frac{1}{16t}[3\lambda_1^2-2\lambda_1\lambda_2+3\lambda_2^2]\right\}. \tag{5.3}$$

Evidently, the probability of caustic formation can be neglected on the scale for which (5.3) is valid. We shall next obtain an asymptotic expansion valid into the initial region of caustic formation. Equation (5.3) will be used as a matching condition for extremely small t.

Let $0<\varepsilon \ll 1$, and define

$$t' = \varepsilon^{-1/3} t, \qquad \lambda_i' = \varepsilon^{1/3} t. \tag{5.4}$$

Thus for ε fixed, t' order one, t will be small. We therefore obtain an approximation for small t if t' is held fixed and $\varepsilon \to 0$. It is emphasized that ε has no physical significance and is introduced here only for convenience in applying an asymptotic method.

Putting (5.4) into (4.23), and dropping the prime notation, we obtain

$$\begin{aligned} P_t &= \varepsilon\left(\frac{3}{2}P_{\lambda_1\lambda_1}+P_{\lambda_1\lambda_2}+\frac{3}{2}P_{\lambda_2\lambda_2}\right)+\lambda_1^2 P_{\lambda_1}+\lambda_2^2 P_{\lambda_2} \\ &\quad +2(\lambda_1+\lambda_2)P+\varepsilon\left(\frac{[P_{\lambda_1}-P_{\lambda_2}]}{[\lambda_1-\lambda_2]}+\frac{2P}{[\lambda_1-\lambda_2]^2}\right). \end{aligned} \tag{5.5}$$

We proceed to apply a W.K.B. method developed by Ludwig [9] for parabolic equations. Thus, let

$$P \sim e^{-\phi/\varepsilon} u \tag{5.6}$$

to obtain

$$(5.6)\quad \begin{aligned} -\frac{\phi_t}{\varepsilon}u+u_t=&\frac{u}{\varepsilon}\left(\frac{3}{2}\phi_{\lambda_1}^2+\phi_{\lambda_1}\phi_{\lambda_2}+\frac{3}{2}\phi_{\lambda_2}^2\right)-\frac{\lambda_1^2\phi_{\lambda_1}}{\varepsilon}u-\frac{\lambda_2^2\phi_{\lambda_2}}{\varepsilon}u\\ &-(3\phi_{\lambda_1}u_{\lambda_1}+u_{\lambda_1}\phi_{\lambda_2}+u_{\lambda_2}\phi_{\lambda_1}+3\phi_{\lambda_2}u_{\lambda_2})+\lambda_1^2u_{\lambda_1}+\lambda_2^2u_{\lambda_2}\\ &-u\left(\frac{3}{2}\phi_{\lambda_1\lambda_1}+\phi_{\lambda_1\lambda_2}+\phi_{\lambda_2\lambda_2}\right)+\frac{(\phi_{\lambda_1}-\phi_{\lambda_2})}{(\lambda_1-\lambda_2)}u+2(\lambda_1+\lambda_2)u\\ &+\varepsilon\left(\frac{3}{2}u_{\lambda_1\lambda_1}+u_{\lambda_1\lambda_2}+\frac{3}{2}u_{\lambda_2\lambda_2}\right)-\varepsilon\frac{(u_{\lambda_1}-u_{\lambda_2})}{(\lambda_1-\lambda_2)}+\varepsilon\frac{2u}{(\lambda_1-\lambda_2)^2}.\end{aligned}$$

By equating, in (5.6), the coefficients of ε^{-1} we obtain

$$(5.7)\quad \phi_t+\tfrac{3}{2}\phi_{\lambda_1}^2+\phi_{\lambda_1}\phi_{\lambda_2}+\tfrac{3}{2}\phi_{\lambda_2}^2-\lambda_1^2\phi_{\lambda_1}-\lambda_2^2\phi_{\lambda_2}=0.$$

By equating the coefficient of ε^0 we obtain

$$(5.8)\quad \begin{aligned} u_t=&-u\left(\frac{3}{2}\phi_{\lambda_1\lambda_1}+\phi_{\lambda_1\lambda_2}+\frac{3}{2}\phi_{\lambda_2\lambda_2}-\frac{[\phi_{\lambda_1}-\phi_{\lambda_2}]}{[\lambda_1-\lambda_2]}-2[\lambda_1+\lambda_2]\right)\\ &+u_{\lambda_1}(\lambda_1^2-3\phi_{\lambda_1}-\phi_{\lambda_2})+u_{\lambda_2}(\lambda_2^2-3\phi_{\lambda_2}-\phi_{\lambda_1}).\end{aligned}$$

The method and terminology now mimic that of geometrical optics. Thus (5.7) is an "eiconal equation" for the "phase", ϕ of the solution given by (5.6). Equation (5.8) is a "transport equation" for the "amplitude", u of (5.6). The "eiconal" equation may be solved by the method of "rays", which are its characteristics. A matching condition, for small t, is given by considering the "canonical problem" (5.1), which has solution (5.2). For the remainder of §§ 5 and 6, the geometrical optics terminology will be used for these "probability rays in curvature space", without, it is hoped, causing confusion with the physical rays of the true eiconal, equation (2.1).

The equations of the rays are

$$(5.9)\quad \begin{aligned} &\dot t=1,\qquad \dot\phi_t=0,\\ &\dot\lambda_1=-\lambda_1^2+3\phi_{\lambda_1}+\phi_{\lambda_2},\qquad \dot\lambda_2=-\lambda_2^2+3\phi_{\lambda_2}+\phi_{\lambda_1},\\ &\dot\phi_{\lambda_1}=2\lambda_1\phi_{\lambda_1},\qquad \dot\phi_{\lambda_2}=2\lambda_2\phi_{\lambda_2}.\end{aligned}$$

The dots in (5.9) represent derivatives along the ray path. From the first of equations (5.9), these may be identified with t-derivatives. Since the rays emanate from $(\lambda_1,\lambda_2)=(0,0)$ at $t=0$, we have the initial conditions that

$$(5.10)\quad \lambda_1=\lambda_2=0,\quad \phi_{\lambda_1}=\alpha_1,\quad \phi_{\lambda_2}=\alpha_2,\quad \text{for } t=0.$$

Here α_1,α_2 parametrize the ray, and the mapping $(\alpha_1,\alpha_2)\to(\lambda_1,\lambda_2)$ gives, for fixed t, a covering of (λ_1,λ_2)-space by rays. To determine ϕ at a point (λ_1,λ_2) and fixed t we must determine the ray that reaches (λ_1,λ_2) at t. ϕ can then be found by integration along this ray of the equation

$$(5.11)\quad \dot\phi=\tfrac{3}{2}\phi_{\lambda_1}^2+\phi_{\lambda_1}\phi_{\lambda_2}+\tfrac{3}{2}\phi_{\lambda_2}^2,$$

with initial conditions

$$(5.12)\quad \phi=0\quad \text{for } t=0.$$

Note that once ϕ and its first and second derivatives have been determined, the transport equation (5.8) can be solved as a linear ordinary differential equation for u along a ray

$$\dot{u} = \left(-\frac{3}{2}\phi_{\lambda_1\lambda_1} - \phi_{\lambda_1\lambda_2} - \frac{3}{2}\phi_{\lambda_2\lambda_2} + 2[\lambda_1+\lambda_2] + \frac{[\phi_{\lambda_1} - \phi_{\lambda_2}]}{[\lambda_1 - \lambda_2]}\right) u. \tag{5.13}$$

Here we may think of λ_i, ϕ_{λ_i}, $\phi_{\lambda_i\lambda_j}$ as functions of t and the ray parameters, e.g., $\lambda_i = \lambda_i(t, \alpha_1, \alpha_2)$, etc.

Since second derivatives of ϕ are needed in (5.13), and these are not obtained directly from solution of (5.9), we will also need propagation equations for the second derivatives of ϕ along a ray. For α_1, α_2 fixed let $\phi_{\lambda\lambda}(t, \alpha_1, \alpha_2)$ be the 2×2 matrix with elements $\phi_{\lambda_i\lambda_j}(t, \alpha_1, \alpha_2)$. Then, by the chain rule,

$$\phi_{\lambda\lambda} = \phi_{\lambda\alpha}\alpha_\lambda = \phi_{\lambda\alpha}(\lambda_\alpha)^{-1} \tag{5.14}$$

where

$$(\phi_{\lambda\alpha})_{ij} = \frac{\partial}{\partial\alpha_i}\phi_{\lambda_i}(t, \alpha_1, \alpha_2), \qquad (\lambda_\alpha)_{ij} = \frac{\partial}{\partial\alpha_j}\lambda_i(t, \alpha_1, \alpha_2).$$

By differentiation of the ray equations (5.9) with respect to α_i, we obtain propagation equations for λ_α, $\phi_{\lambda\alpha}$:

$$\frac{d}{dt}\lambda_\alpha(t, \alpha_1, \alpha_2) = -2\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\lambda_\alpha + \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\phi_{\lambda\alpha}, \tag{5.15}$$

$$\begin{aligned}\frac{d}{dt}\phi_{\lambda\alpha}(t, \alpha_1, \alpha_2) &= 2\begin{bmatrix}\phi_{\lambda_1} & 0\\ 0 & \phi_{\lambda_2}\end{bmatrix} + 2\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\phi_{\lambda\lambda}\\ &\quad + 2\phi_{\lambda\lambda}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} - \phi_{\lambda\lambda}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\phi_{\lambda\lambda}.\end{aligned} \tag{5.16}$$

Thus, by differentiating (5.14) and using (5.15), (5.16) we obtain a matrix Ricatti equation for $\phi_{\lambda\lambda}$

$$\frac{d}{dt}\phi_{\lambda\lambda} = 2\begin{bmatrix}\phi_{\lambda_1} & 0\\ 0 & \phi_{\lambda_2}\end{bmatrix} + 2\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\phi_{\lambda\lambda} + 2\phi_{\lambda\lambda}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} - \phi_{\lambda\lambda}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\phi_{\lambda\lambda}. \tag{5.17}$$

Equivalently, we may obtain from (5.17) a matrix Ricatti equation for the propagation of $\phi_{\lambda\lambda}^{-1}$ along a ray

$$\frac{d}{dt}\phi_{\lambda\lambda}^{-1} = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix} - 2\phi_{\lambda\lambda}^{-1}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} - 2\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\phi_{\lambda\lambda}^{-1} - 2\phi_{\lambda\lambda}^{-1}\begin{bmatrix}\phi_{\lambda_1} & 0\\ 0 & \phi_{\lambda_2}\end{bmatrix}\phi_{\lambda\lambda}^{-1}. \tag{5.17$'$}$$

To complete the specifications, it remains to give initial conditions for u, $\phi_{\lambda\lambda}$ as $t\downarrow 0$. These quantities are necessarily singular as $t\downarrow 0$ since $\boldsymbol{P}$ must reduce to a delta function at $t=0$. The correct matching condition can be obtained by use of the canonical problem (5.1) with solution (5.3). By putting back the scaling (5.4), and matching amplitude and phase from (5.6), we obtain

$$u \sim \varepsilon^{-5/6}\frac{|\lambda_1 - \lambda_2|}{16\sqrt{\pi t^3}} \quad \text{as } t\downarrow 0, \tag{5.18}$$

$$\phi \sim \frac{1}{16t}(3\lambda_1^2 - 2\lambda_1\lambda_2 + 3\lambda_2^2) \quad \text{as } t\downarrow 0. \tag{5.19}$$

Now in ray coordinates we have from (5.9) that, for very small t,

$$\lambda_1-\lambda_2\sim 2(\phi_{\lambda_1}-\phi_{\lambda_2})t\sim 2(\alpha_1-\alpha_2)t,$$

so that (5.18) may be rewritten along a ray as

$$u\sim\varepsilon^{-5/6}\frac{|\alpha_1-\alpha_2|}{8\sqrt{\pi t}}\quad\text{as } t\downarrow 0. \tag{5.20}$$

Similarly, differentiation of (5.19) gives that

$$\phi_{\lambda\lambda}\sim\frac{1}{8t}\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}\quad\text{as } t\downarrow 0, \tag{5.21}$$

or equivalently,

$$\phi_{\lambda\lambda}^{-1}\sim t\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\to 0\quad\text{as } t\downarrow 0. \tag{5.22}$$

The equations of this section do not appear solvable in closed form. We shall, however, use them in § 6 to determine, in closed form, the caustic probability curve. For this purpose we need two implicit relations, equations (5.27) and (5.30) below, which will now be derived.

From (5.9), $\dot{\phi}_t=0$, and hence ϕ_t is constant along a ray. Therefore, from (5.7) we have that along a fixed ray

$$\tfrac{3}{2}\phi_{\lambda_1}^2+\phi_{\lambda_1}\phi_{\lambda_2}+\tfrac{3}{2}\phi_{\lambda_2}^2=\lambda_1^2\phi_{\lambda_1}+\lambda_2^2\phi_{\lambda_2}+C_1, \tag{5.23}$$

where

$$C_1=\tfrac{3}{2}\alpha_1^2+\alpha_1\alpha_2+\tfrac{3}{2}\alpha_2^2 \tag{5.24}$$

is constant along a ray. Putting (5.23) into (5.11) yields

$$\phi(t,\alpha_1,\alpha_2)=C_1t+\int_0^t(\lambda_1^2\phi_{\lambda_1}+\lambda_2^2\phi_{\lambda_2})(s,\alpha_1,\alpha_2)\,ds. \tag{5.25}$$

But, since from (5.9) $\dot{\phi}_{\lambda_i}=2\lambda_i\phi_{\lambda_1}$, we obtain from (5.25) upon integration by parts and further use of (5.19)

$$\begin{aligned}\phi&=C_1t+\frac{1}{2}\int_0^t(\lambda_1\dot{\phi}_{\lambda_1}+\lambda_2\dot{\phi}_{\lambda_2})\,ds\\ &=C_1t+\frac{1}{2}(\lambda_1\phi_{\lambda_1}+\lambda_2\phi_{\lambda_2})+\frac{1}{2}\int_0^t(\lambda_1^2\phi_{\lambda_1}+\lambda_2^2\phi_{\lambda_2})\,ds\\ &\quad-\int_0^t\left(\frac{3}{2}\phi_{\lambda_1}^2+\phi_{\lambda_1}\phi_{\lambda_2}+\frac{3}{2}\phi_{\lambda_2}^2\right)ds.\end{aligned} \tag{5.26}$$

Now use of (5.25), (5.26) and (5.11) yields, upon rearrangement, the final implicit formula for ϕ,

$$\phi(t,\alpha_1,\alpha_2)=\tfrac{1}{3}(\tfrac{3}{2}\alpha_1^2+\alpha_1\alpha_2+\tfrac{3}{2}\alpha_2^2)+\tfrac{1}{3}(\lambda_1\phi_{\lambda_1}+\lambda_2\phi_{\lambda_2}). \tag{5.27}$$

Next, an implicit relation will be derived for u. From (5.13),

$$\begin{aligned}u(t,\alpha_1,\alpha_2)=u(t_0,\alpha_1,\alpha_2)\exp\Bigg\{\int_{t_0}^t\Bigg(&-\frac{3}{2}\phi_{\lambda_1\lambda_1}-\phi_{\lambda_1\lambda_2}-\frac{3}{2}\phi_{\lambda_2\lambda_2}\\ &+2[\lambda_1+\lambda_2]+\frac{[\phi_{\lambda_1}-\phi_{\lambda_2}]}{[\lambda_1-\lambda_2]}\Bigg)ds\Bigg\}.\end{aligned} \tag{5.28}$$

Now from (5.9)

$$\frac{d}{dt}(\lambda_1-\lambda_2)=\lambda_2^2-\lambda_1^2+2(\phi_{\lambda_1}-\phi_{\lambda_2}),$$

so that

$$\int^t \frac{[\phi_{\lambda_1}-\phi_{\lambda_2}]}{[\lambda_1-\lambda_2]}\,ds=\frac{1}{2}\log|\lambda_1-\lambda_2|+\frac{1}{2}\int^t(\lambda_1+\lambda_2)\,ds.$$

Hence

$$u(t,\alpha_1,\alpha_2)=C_2(\alpha_1,\alpha_2)\frac{|\lambda_1-\lambda_2|^{1/2}}{t} \cdot \exp\left\{\frac{5}{2}\int_0^t(\lambda_1+\lambda_2)\,ds-\int_0^t\left(\frac{3}{2}\phi_{\lambda_1\lambda_1}+\phi_{\lambda_1\lambda_2}+\frac{3}{2}\phi_{\lambda_2\lambda_2}-\frac{1}{s}\right)ds\right\} \tag{5.29}$$

where the $1/s$ term has been inserted in the exponential to cancel the singularity at $s=0$ as determined by (5.22). The integration constant C_2 may be determined by matching to (5.20). Then the final implicit result for u is that

$$u(t,\alpha_1,\alpha_2)\sim\frac{\varepsilon^{-5/6}}{8t}\left|\frac{2(\alpha_1-\alpha_2)(\lambda_1-\lambda_2)}{\pi}\right|^{1/2} \cdot \exp\left\{\frac{5}{2}\int_0^t(\lambda_1+\lambda_2)\,ds-\int_0^t\left(\frac{3}{2}\phi_{\lambda_1\lambda_1}+\phi_{\lambda_1\lambda_2}+\frac{3}{2}\phi_{\lambda_2\lambda_2}-\frac{1}{s}\right)ds\right\}. \tag{5.30}$$

6. A closed form expression for the initial region of caustic formation. In this section, we will use the equations of § 5 to derive a simple expression for the caustic probability curve, valid into the initial region of caustic formation. We first write (5.5) in the form of a conservation law,

$$P_t=\nabla\cdot J, \tag{6.1}$$

where the probability flux vector is given by

$$J=\begin{bmatrix}\varepsilon\left(\frac{3}{2}P_{\lambda_1}+\frac{1}{2}P_{\lambda_2}\right)+\left(\lambda_1^2-\frac{\varepsilon}{[\lambda_1-\lambda_2]}\right)P\\ \varepsilon\left(\frac{3}{2}P_{\lambda_2}+\frac{1}{2}P_{\lambda_1}\right)+\left(\lambda_2^2-\frac{\varepsilon}{[\lambda_2-\lambda_1]}\right)P\end{bmatrix}. \tag{6.2}$$

Now if T is the (random) distance to the first caustic we have that

$$P\{T>t\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P(t,\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2, \tag{6.3}$$

since $P\{T>t\}$ is the probability that λ_1,λ_2 are still finite by distance t. If

$$f(t)=-\frac{\partial}{\partial t}P\{T>t\} \tag{6.4}$$

is the probability density of the random variable T, we have from (6.1), (6.3) that

$$f(t)=-\lim_{M\to\infty}\int_{-M}^{M}\int_{-M}^{M}(\nabla\cdot J)\,d\lambda_1\,d\lambda_2. \tag{6.5}$$

Let Ω_M be the square of side $2M$ centered at $(\lambda_1, \lambda_2) = (0, 0)$. From the divergence theorem,

(6.6) $$f(t) = -\lim_{M\to\infty} \int_{\partial\Omega_M} (J \cdot \nu)\, ds,$$

where ν is the outward-pointing normal to $\partial\Omega_M$. We label the sides of $\partial\Omega_M$ by Γ_i, $i = 1, 2, 3, 4$ as in Fig. 1. Let

(6.7) $$I_k(M) = -\int_{\Gamma_k(M)} (J \cdot \nu)\, ds, \qquad k = 1, 2, 3, 4.$$

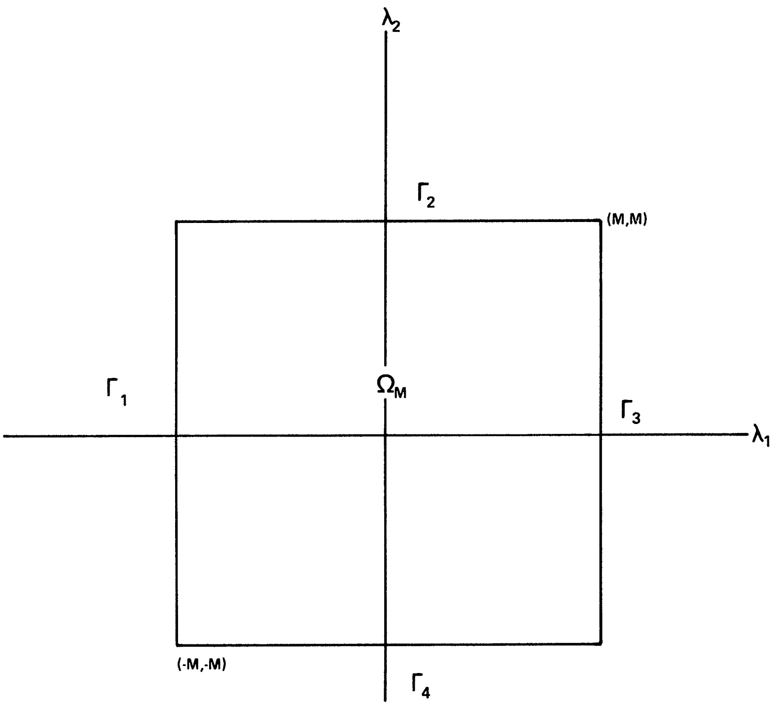

FIG. 1. *Region of integration in the* (λ_1, λ_2)*-plane.*

Then from (6.2) we have that

(6.8) $$I_2 = I_3 = \int_{-M}^{M} \left(\varepsilon\left[\frac{3}{2} P_{\lambda_2} + \frac{1}{2} P_{\lambda_1}\right] + \left[\lambda_2^2 - \frac{\varepsilon}{(\lambda_2 - \lambda_1)}\right] P \right) d\lambda_1,$$

(6.9) $$I_1 = I_4 = -\int_{-M}^{M} \left(\varepsilon\left[\frac{3}{2} P_{\lambda_1} + \frac{1}{2} P_{\lambda_2}\right] + \left[\lambda_1^2 - \frac{\varepsilon}{(\lambda_1 - \lambda_2)}\right] P \right) d\lambda_2.$$

Putting the W.K.B. Ansatz (5.6) into (6.8), (6.9) we obtain,

(6.10) $$I_1 \sim -\int_{-M}^{M} \left(\lambda_1^2 - \frac{3}{2} \phi_{\lambda_1} - \frac{1}{2} \phi_{\lambda_2} \right) u \exp\left\{-\frac{\phi}{\varepsilon}\right\} d\lambda_2 \bigg|_{\lambda_1 = -M},$$

(6.11) $$I_2 \sim \int_{-M}^{M} \left(\lambda_2^2 - \frac{3}{2} \phi_{\lambda_2} - \frac{1}{2} \phi_{\lambda_1} \right) u \exp\left\{-\frac{\phi}{\varepsilon}\right\} d\lambda_1 \bigg|_{\lambda_2 = M}.$$

The integrals in (6.10), (6.11) will be evaluated asymptotically as $\varepsilon \downarrow 0$ by Laplace's method; thus only points corresponding to a minimum of ϕ on $\partial\Omega_M$ will contribute to the final result. We will show that in fact $\phi \to +\infty$ on all of Γ_2 as $M \to \infty$, so that the contribution from I_2 can be neglected altogether. This is to be expected since, from the results of § 4, $\{\lambda_2 \to +\infty$ for finite $t\}$ is an event of zero probability. Furthermore, it will be shown that there is a single minimum of ϕ on Γ_1 which asymptotically gives the total contribution to I_1.

Equation (5.23) may be solved alternately for ϕ_1, ϕ_2 to obtain

$$(6.12)\quad 3\phi_{\lambda_1}+\phi_{\lambda_2}-\lambda_1^2=(\text{sgn}\,([3\alpha_1+\alpha_2])\sqrt{(\lambda_2^2-\phi_{\lambda_2})^2+6C_1+6\lambda_2^2\phi_{\lambda_2}-9\phi_{\lambda_2}^2},$$

$$(6.13)\quad 3\phi_{\lambda_2}+\phi_{\lambda_1}-\lambda_2^2=(\text{sgn}\,([3\alpha_2+\alpha_1])\sqrt{(\lambda_2^2-\phi_{\lambda_1})^2+6C_1+6\lambda_1^2\phi_{\lambda_1}-9\phi_{\lambda_1}^2}.$$

Comparison of (6.12), (6.13) with (5.9) now shows that

$$(6.14)\quad \begin{aligned}&\dot\lambda_1>0 \text{ for all } t \text{ if and only if } 3\alpha_1+\alpha_2>0,\\ &\dot\lambda_2>0 \text{ for all } t \text{ if and only if } 3\alpha_2+\alpha_1>0.\end{aligned}$$

We first consider the corner at (M, M). To reach this point along a ray, we must take $\alpha_1=\alpha_2$, whence $\lambda_1\equiv\lambda_2$, $\phi_{\lambda_1}\equiv\phi_{\lambda_2}$ for all t. Because of (6.14), we must take $\alpha_1>0$. Then, since from (5.9) we have that

$$(6.15)\quad \phi_{\lambda_i}(t,\alpha_1,\alpha_2)=\alpha_i \exp\left\{2\int_0^t \lambda_i(s,\alpha_1,\alpha_2)\,ds\right\},\qquad i=1,2$$

it follows that $\lambda_i\phi_{\lambda_i}\to+\infty$ as $M\to\infty$ if $\alpha_1=\alpha_2$ are chosen so that $\lambda_i(t,\alpha_1,\alpha_2)=M$. Thus, from (5.27) $\phi\to\infty$ also.

Similarly, we may consider the corner at $(-M, M)$. Because $\lambda_1<0, \lambda_2>0$ there we must have, from (6.14) that $3\alpha_1+\alpha_2<0$, $3\alpha_2+\alpha_1>0$. But these inequalities imply that $\alpha_1<0, \alpha_2>0$. Using (6.15) again, we have that $\lambda_1\phi_{\lambda_1}+\lambda_2\phi_{\lambda_2}\to+\infty$ as $M\to\infty, \lambda_1=-M, \lambda_2=+M$. So again $\phi\to+\infty$ by (5.27).

To complete the analysis of Γ_2, we look for a critical point of ϕ, $\phi_{\lambda_1}=0$ on Γ_2. By (6.15) this can be achieved only if $\alpha_1=0$, whence, by (6.14), we must take $\alpha_2>0$ to reach Γ_2. But then again, $\lambda_2\phi_{\lambda_2}\to+\infty$ at this point as $\lambda_2(t,0,\alpha_2)=M\to+\infty$. Hence by (5.27) $\phi\to+\infty$ also.

We next consider Γ_1, starting with the corner at $(-M, -M)$. The ray reaching this point at distance t must have $\alpha_1=\alpha_2<0$ for some value of α_1. Thus along this ray $\lambda_1\equiv\lambda_2$, $\phi_{\lambda_1}\equiv\phi_{\lambda_2}$. From (6.12) we have that on this ray

$$(6.16)\quad 4\phi_{\lambda_1}-\lambda_1^2=-\sqrt{\lambda_1^4+16\alpha_1^2},$$

and hence, from (5.9),

$$(6.17)\quad \dot\lambda_1=-\sqrt{\lambda_1^4+16\alpha_1^2}.$$

We define the function $g(x)$ by

$$(6.18)\quad g(x)=\int_0^x \frac{d\xi}{\sqrt{\xi^4+1}}.$$

Then the solution of (6.17) may be written implicitly as

$$(6.19)\quad g\left(\frac{|\lambda_1|}{2\sqrt{|\alpha_1|}}\right)=2\sqrt{|\alpha_1|}\,t.$$

Thus the ray parameters of the ray hitting $(-M, -M)$ are given by, as $M\to\infty$,

$$(6.20)\quad \alpha_1=\alpha_2=-\frac{k^2}{4t^2},$$

where

(6.21) $$k = g(\infty) = 1.854^{+}.$$

Then, from (6.16), we have that $\lambda_1 \phi_{\lambda_1} \to 0$ along this ray as $M \to \infty$. Hence, from (5.27), we get

(6.22) $$\phi\left(t, -\frac{k^2}{4t^2}, -\frac{k^2}{4t^2}\right) = \frac{k^4}{12t^3}.$$

It remains to look for a critical point on Γ_1. Since $\phi_{\lambda_2} = 0$ there, we must take $\alpha_2 = 0$ because of (6.15). Thus $\phi_{\lambda_2} \equiv 0$ all along this ray. Therefore, from (6.13) with $\alpha_1 < 0$

(6.23) $$3\phi_{\lambda_2} - \lambda_1^2 = -\sqrt{\lambda_1^4 + 9\alpha_1^2}.$$

Substitution of (6.23) into (5.9) gives

(6.24) $$\dot{\lambda}_1 = -\sqrt{\lambda_1^4 + 9\alpha_1^2},$$

with the implicit solution

(6.25) $$g\left(\frac{|\lambda_1|}{\sqrt{3|\alpha_1|}}\right) = \sqrt{3|\alpha_1|}\, t.$$

As $\lambda_1 = -M \to -\infty$, (6.24) yields

(6.26) $$\alpha_1 = -\frac{k^2}{3t^2}.$$

Now $\lambda_2 \phi_{\lambda_2} = 0$ and, as before, we show $\lambda_1 \phi_{\lambda_1} \to 0$ as $M \to \infty$. Thus, by (5.27),

(6.27) $$\phi\left(t, -\frac{k^2}{3t^2}, 0\right) = \frac{k^4}{18t^3}.$$

This point is clearly the minimum of ϕ. The situation is summarized in Fig. 2.

Thus, for fixed t the probability ray of maximum likelihood of caustic formation is given by $\alpha_1 = -k^2/3t^2, \alpha_2 = 0$. Evaluation of I_1 by Laplace's method gives that asymptotically

(6.28) $$f(t) \sim 2 \lim_{M \to \infty} I_1 \sim 2 \exp\left\{-\frac{k^4}{18\varepsilon t^3}\right\} \left(\frac{2\pi\varepsilon}{\phi^*_{\lambda_2 \lambda_2}}\right)^{1/2} \cdot \lim_{\substack{\lambda_1 \to -\infty \\ \lambda_2 = \lambda_2^*}} (\lambda_1^2 u),$$

where starred quantities refer to the maximum likelihood ray.

Now along this ray,

$$\phi_{\lambda_1}\left(s, -\frac{k^2}{3t^2}, 0\right) = \alpha_1 \exp\left\{2 \int_0^s \lambda_1 \, d\xi\right\} = -\frac{3\alpha_1^2}{[\lambda_1^2 + \sqrt{\lambda_1^4 + 9\alpha_1^2}]}.$$

Therefore as $s \uparrow t, \lambda_1 \to -\infty$

$$\exp\left\{\frac{5}{2} \int_0^s \lambda_1 \, d\xi\right\} \sim \left[\frac{|3\alpha_1|}{2\lambda_1^2}\right]^{5/4}.$$

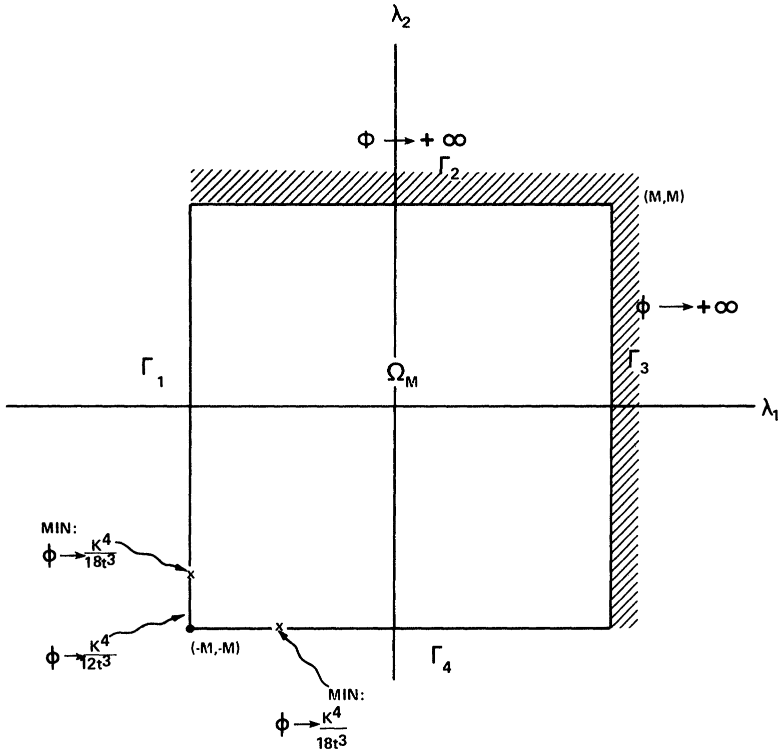

FIG. 2. *Behavior of ϕ as $M \to \infty$.*

Since also $|\lambda_1 - \lambda_2|^{1/2} \sim |\lambda_1|^{1/2}$ we have, on substituting into (5.29) and letting $s \to t$

$$f(t) \sim \frac{2^{3/4}\varepsilon^{-1/3}k^{7/2}}{16\sqrt{3\phi^*_{\lambda_2\lambda_2}}\,t^{9/2}} \exp\left\{-\frac{k^4}{18\varepsilon t^3}\right\} \tag{6.29}$$
$$\cdot \exp\left\{\frac{5}{2}\int_0^t \lambda_2\,ds - \int_0^t \left(\frac{3}{2}\phi_{\lambda_1\lambda_1} + \phi_{\lambda_1\lambda_2} + \frac{3}{2}\phi_{\lambda_2\lambda_2} - \frac{1}{s}\right) ds\right\},$$

where the integrals are along the maximum likelihood ray. These integrals are, in fact, constants! This fact follows from a scaling law of the ray equations (5.9), namely that for $\alpha_1 = -k^2/3t^2$, $s \leqq t$

$$\begin{aligned} \lambda_i(s, \alpha_1, 0) &= \sqrt{|\alpha_1|}\,\lambda_i(\sqrt{|\alpha_1|}\,s, -1, 0),\\ \phi_{\lambda_i}(s, \alpha_1, 0) &= |\alpha_1|\phi_{\lambda_i}(\sqrt{|\alpha_1|}\,s, -1, 0),\\ \phi^{-1}_{\lambda\lambda}(s, \alpha_1, 0) &= \frac{1}{\sqrt{|\alpha_1|}}\phi^{-1}_{\lambda\lambda}(\sqrt{|\alpha_1|}\,s, -1, 0). \end{aligned} \tag{6.30}$$

(6.30) may be verified by noting that equated quantities satisfy the same differential equations (5.9), (5.18) and initial conditions. We also obtain from the scaling laws

$$\phi^*_{\lambda_2\lambda_2} = \phi_{\lambda_2\lambda_2}\left(t, -\frac{k^2}{3t^2}, 0\right) = \frac{k}{\sqrt{3}t}\phi_{\lambda_2\lambda_2}\left(\frac{k}{\sqrt{3}}, -1, 0\right). \tag{6.31}$$

Thus, all quantities in (6.29) can be computed by numerical integration of the single ray, $\alpha_1=-1, \alpha_2=0$. This has been done, and the final answer, obtained after setting $\varepsilon=1$ is

$$f(t) \sim \frac{a_1}{t^4} \exp\left\{-\frac{a_2}{t^3}\right\}, \tag{6.32}$$

where numerical values for a_1, a_2 are

$$a_1 = 1.7399^+, \qquad a_2 = .6565^+. \tag{6.33}$$

7. Comparison with Monte Carlo experiments. The limiting equations (3.21) for the eight entries of the matrices **A**, **B** have been simulated digitally using pseudorandom numbers. Now the first value of t for which $\det \mathbf{A}=0$ corresponds to the first occurrence of a caustic. For $n=10{,}000$ simulations the histogram of this value has been plotted as the x in Fig. 3. Plotted also are the small t approximations of (6.32) and (6.33), and an exponential curve of the form

$$a_3 \exp\{-a_4 t\},$$

with the values $a_3=1.000$, $a_4=.893$.

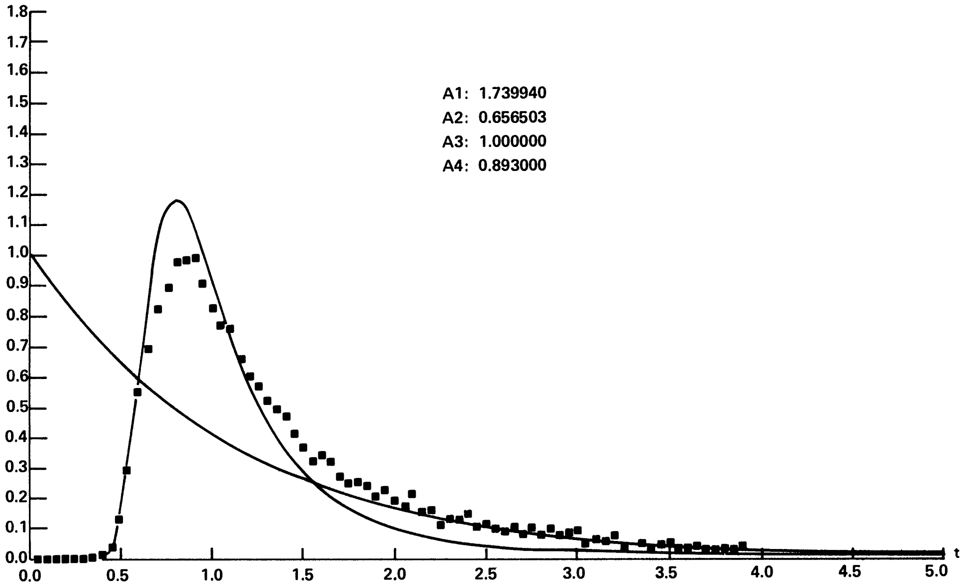

FIG. 3. *Monte-Carlo simulations, small and large t approximations.*

The qualitative form of the curve is quite similar to that of the two-dimensional theory [2] with the small t approximation valid into the region where the probability of caustic formation is appreciable.

For large values of t, the curve will be approximately a decaying exponential if it is assumed that the operator on the right-hand side of (4.23) has a discrete spectrum; the decay rate would then be the lowest eigenvalue of this operator. Although it has not been shown that the operator in (4.23) does have a discrete spectrum, the exponential plotted in Fig. 3 appears to fit the data quite well.

8. Appendix. In this section we derive propagation equations for the principal normal curvatures of the wavefront.

The eiconal equation is

$$|\nabla\psi|^2 = \frac{1}{C^2}, \tag{8.1}$$

where C is the space-varying propagation speed and ψ is the phase of the disturbance. We follow a single ray $X(s)$ parametrized by arclength s and emanating from X_0 at $s=0$. Let $V_1(s)$ be the unit tangent to $X(s)$. Since the rays are orthogonal to surfaces of constant ψ,

$$V_1(s) = C(X(s))\nabla\psi(X(s)). \tag{8.2}$$

By differentiating (8.1) we obtain

$$\sum_{j=1}^{3} \psi_{x_i x_j}\psi_{x_j} = -\frac{1}{C^3}C_{x_i}. \tag{8.3}$$

Then by differentiating (8.2) and using (8.3), we obtain the ray propagation equations in the form

$$\frac{d}{ds}X = V_1, \qquad \frac{d}{ds}V_1 = -\frac{1}{C}\nabla C + \left(\frac{1}{C}\nabla C \cdot V_1\right)V_1. \tag{8.4}$$

Let $\kappa(s)$ be the curvature of the ray, $\tau(s)$ the torsion, $V_2(s)$ the unit normal, and $V_3(s)$ the unit binormal. If $\mathbf{P}(s)$ is the 3×3 matrix with V_i as its ith column, then the Frenet–Serret formulae [4] give

$$\frac{d}{ds}\mathbf{P}(s) = \mathbf{P}(s)\mathbf{M}(s), \tag{8.5}$$

where

$$\mathbf{M}(s) = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}. \tag{8.6}$$

Let

$$\mathbf{D}(s) = \mathbf{P}^T(s)\nabla\nabla\psi(X(s))\mathbf{P}(s). \tag{8.7}$$

A propagation equation may be written for $\mathbf{D}$ by differentiating (8.7), using (8.5) and the identity

$$\sum_{i=1}^{3} (\psi_{x_i x_j x_k}\psi_{x_i} + \psi_{x_i x_j}\psi_{x_i x_k}) = \frac{\partial}{\partial x_k}\left(\frac{1}{C^3}C_{x_j}\right) \tag{8.8}$$

obtained by differentiation of (8.3). $\mathbf{D}(s)$ satisfies the matrix Ricatti equation

$$\frac{d}{ds}\mathbf{D} = \mathbf{DM} - \mathbf{MD} - \mathbf{C}\mathbf{D}^2 + \mathbf{Q}_1, \tag{8.9}$$

where

$$\mathbf{Q}_1 = \frac{3}{C^3}(P^T\nabla C)(P^T\nabla C)^T - \frac{1}{C^2}P^T\nabla\nabla CP. \tag{8.10}$$

Now from (8.4), (8.5), (8.6), we obtain

$$\kappa V_2 = (\nabla C \cdot V_1)\frac{V_1}{C} + C\nabla\nabla\psi V_1. \tag{8.11}$$

By taking the dot product of (8.11) with, respectively V_1, V_2, V_3, and recalling the definition (8.7) we obtain

$$D_{11} = -\frac{1}{C^2}\nabla C \cdot V_1, \quad D_{12} = D_{21} = \frac{\kappa}{C}, \quad D_{13} = D_{31} = 0. \tag{8.12}$$

Since **D** is symmetric, (8.9) contains six scalar equations. By writing these out, and using the three equations (8.12), we can derive the matrix Ricatti equation (2.8) for the 2×2 symmetric matrix $\tilde{\mathbf{R}}$, where

$$\tilde{\mathbf{R}} = C\begin{bmatrix} D_{22} & D_{23} \\ D_{32} & D_{33} \end{bmatrix}. \tag{8.13}$$

The eigenvalues of $\tilde{\mathbf{R}}$ are the two principal normal curvatures of the wavefront at $X(s)$. The expressions (2.6) are also obtained from this calculation.

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