J. Fluid Mech. (1999), *vol.* 386, *pp.* 329–344. Printed in the United Kingdom

Wave action on currents with vorticity

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(Received 2 March 1998 and in revised form 9 November 1998)

The interaction of waves on deep water with spatially varying currents may be described by a ray theory, with the wave amplitudes determined by the principle of conservation of wave action (CWA). However, all previous deep water derivations of CWA are restricted to the case of an irrotational current. In this paper, both the ray theory and CWA are derived by a WKB method without the assumption of irrotationality. Also derived is a new equation for a spatially varying phase shift which is not predicted by the usual ray theory, and which, in general, displaces the positions of the wave crests by a distance on the order of a wavelength. This phase shift, which is caused by variations of the current velocity with depth, vanishes in the irrotational case, and so is in accord with the irrotational theory.

1. Introduction

When waves on deep water encounter a spatially varying current, their speed, direction, wavelength, height and even their shape can be altered by the interaction. For linear waves many features of this phenomenon can be described by a ray theory (Whitham 1974; Peregrine 1976; Jonsson 1990). Furthermore, the equations of the relevant ray system are derived by a simple heuristic argument, as follows:

In the absence of current, deep-water surface gravity waves are characterized by the dispersion relation

$$\bar{\Omega}(\boldsymbol{k}) = \pm(g|\boldsymbol{k}|)^{1/2}, \tag{1}$$

where g is the acceleration due to gravity and $\boldsymbol{k} = (k_1, k_2)^T$ is the wavenumber. That is, the frequency ω is related to the wavenumber through the relationship $\omega = \bar{\Omega}(\boldsymbol{k})$.

Let $\boldsymbol{U}$ be the current velocity. We denote the two horizontal components of $\boldsymbol{U}$, restricted to the water's surface, by $\tilde{\boldsymbol{U}} = (U^1, U^2)^T$. We will use this tilde notation generally to denote two-dimensional horizontal projections of three-dimensional vectors, restricted to the water's surface.

If $\boldsymbol{U}$ is constant the dispersion relation is altered to $\omega = \Omega(\boldsymbol{k})$, where

$$\Omega = \bar{\Omega} + \boldsymbol{k} \cdot \tilde{\boldsymbol{U}}. \tag{2}$$

Equation (2), which is just (1) in a coordinate system moving with the current, prescribes the usual Doppler shift.

Now suppose that $\tilde{\boldsymbol{U}} = \tilde{\boldsymbol{U}}(\boldsymbol{x})$ varies with spatial coordinates $\boldsymbol{x} = (x, y)^T$ on the surface. We seek the local phase, $\theta(t, \boldsymbol{x})$, of the wave, which in the case of constant current is of the form $\theta = -\omega t + \boldsymbol{k} \cdot \boldsymbol{x}$. If frequency and wavenumber are generalized by the relations $\omega = -\partial\theta/\partial t$ and $\boldsymbol{k} = \tilde{\nabla}\theta$, where $\tilde{\nabla} = (\partial/\partial x, \partial/\partial y)^T$, then use of the spatially varying dispersion relation yields a first-order nonlinear partial differential equation for θ

$$\frac{\partial \theta}{\partial t} + \Omega(\boldsymbol{x}, \tilde{\nabla}\theta) = 0. \tag{3}$$

Equation (3) can be solved by the general theory of such equations (Courant & Hilbert 1962; Whitham 1974) using the system of characteristic curves, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{k}})$, which are defined by the ordinary differential equations (with suitable initial conditions)

$$\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} = \Omega_{,\boldsymbol{k}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{k}}), \quad \frac{\mathrm{d}\bar{\boldsymbol{k}}}{\mathrm{d}t} = -\tilde{\nabla}\Omega(\bar{\boldsymbol{x}}, \bar{\boldsymbol{k}}), \tag{4}$$

where

$$\Omega_{,\boldsymbol{k}}(\boldsymbol{x}, \boldsymbol{k}) = \frac{\partial}{\partial \boldsymbol{k}}\Omega(\boldsymbol{x}, \boldsymbol{k}) = \pm\tfrac{1}{2}g^{1/2}\frac{\boldsymbol{k}}{|\boldsymbol{k}|^{3/2}} + \tilde{\boldsymbol{U}}(\boldsymbol{x}), \tag{5}$$

and

$$\tilde{\nabla}\Omega(\boldsymbol{x}, \boldsymbol{k}) = \tilde{\nabla}(\boldsymbol{k} \cdot \tilde{\boldsymbol{U}}(x)). \tag{6}$$

Equations (4) are the ray equations. They determine phase information, including local wave direction, speed, wavenumber, etc., along with θ itself, which is determined from initial conditions and the additional equation for $\bar{\theta}$ to be solved along the rays

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}t} = \bar{\boldsymbol{k}} \cdot \Omega_{,\boldsymbol{k}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{k}}) - \Omega(\bar{\boldsymbol{x}}, \bar{\boldsymbol{k}}). \tag{7}$$

From the usual ray theory, $\theta = \bar{\theta}$ and $\tilde{\nabla}\theta = \bar{\boldsymbol{k}}$ for those values of $t, \boldsymbol{x}$ such that $\boldsymbol{x} = \bar{\boldsymbol{x}}$.

Equations for the amplitude, A, of the wave are not so easily inferred. There was some controversy about the correct principle, until the paper of Longuet-Higgins & Stewart (1961), in which conservation of wave action (CWA) was derived for some examples. CWA is expressed as

$$\frac{\partial}{\partial t}\left\{\frac{A^2}{\bar{\Omega}(\tilde{\nabla}\theta)}\right\} + \tilde{\nabla} \cdot \left\{\frac{A^2}{\bar{\Omega}(\tilde{\nabla}\theta)}\Omega_{,\boldsymbol{k}}(\boldsymbol{x}, \tilde{\nabla}\theta)\right\} = 0. \tag{8}$$

Equation (8) can be written as a transport equation for the wave amplitude, along rays.

CWA has been established more generally by Whitham's method of the averaged Lagrangian (Whitham 1974), a powerful technique that even extends to nonlinear problems. However, the method relies in an essential way on the existence of a velocity potential, and so is restricted to irrotational currents. This is a serious limitation which would preclude many if not most of the applications, were it strictly observed. In this paper, the ray theory and CWA will be derived without the assumption of irrotationality.

The lack of a mathematical derivation has not inhibited the use of CWA in the rotational case, e.g. Peregrine (1976, Sec. II E) and Gerber (1993); also, see Peregrine & Thomas (1978, Sec. 4) for nonlinear effects. But the greatest use of CWA, for currents in deep water, is in the analysis of ocean waves, although ocean currents are seldom irrotational. Since most sea states consist of a mixture of waves of different frequencies, CWA is often used in a form that describes a continuous wave action spectrum; CWA is readily extended to this case using the theory of Longuet-Higgins (1957), who showed in general how a conservation law along rays for a monochromatic wave implies a related law for the transformation of a spectrum. As explicitly rotational examples, Gutshabash & Lavrenov (1986) used a shear flow to model the Agulhas Current, and Mapp, Welch & Munday (1985) analysed wave refraction by warm core rings. Ray theory and CWA have been used to analyse wave spectrum data for major ocean currents, despite the eddies and meanders of these currents, which are rotational. For example, Hayes (1980) got good agreement of theory with synthetic aperture radar (SAR), laser profilometer spectra and pitch and roll buoy data of waves in the Gulf Stream, and Irvine & Tilley (1988) explained significant enhancement of an SAR spectral peak for waves in the Agulhas current, aptly explaining their use of CWA as 'the presently accepted core of belief'. The main result of this paper is the mathematical confirmation of that belief.

More complex ocean models for wave action spectra allow for wind–wave interaction, wave breaking, nonlinearity and other effects in addition to wave–current interaction, e.g. Thompson & Gasparovic (1986), Holthuijsen & Tolman (1991), Komen *et al.* (1994). However, in these models, CWA is taken as a starting point and then generalized to a wave action balance equation that includes these other effects. Thus the theory of this paper also supports the use of these models for currents with non-zero vorticity. The fact that these models have been used successfully in the analysis of real ocean waves gives further experimental support to the present theory.

In this paper the basis for wave–current ray theory, i.e. equations (3) and (8), will be derived without the assumption that the current is irrotational. For simplicity, we will consider steady currents. We will apply a formal perturbation scheme to the fundamental nonlinear equations for inviscid incompressible surface gravity waves with a free boundary, utilizing two small parameters: δ, the ratio of a typical wave height to a typical wavelength, and ϵ, the ratio of a typical wavelength to the spatial scale of the current. For a linear theory, it is necessary to assume that $\delta \ll \epsilon$.

We will also derive an equation for a new, spatially varying phase shift χ which is not predicted by the usual ray theory, and which, in general, displaces the positions of the wave crests by a distance on the order of a wavelength. This phase shift, which is caused by variations of the current velocity with depth, vanishes in the irrotational case, and so is in accord with the irrotational theory. It also vanishes if the current velocity changes slowly enough with depth, i.e. if the vertical scale of the current is much larger than the horizontal scale. Nevertheless, since mesoscale eddies, for example, have both significant vorticity and significant depth dependence (Gründlingh 1988), a non-zero χ may be observable in ocean measurements. However χ cannot be observed by the usual spectral methods, since these methods do not record the positions of individual wave crests, but only the slow modulations of wave amplitude over scales much larger than a wavelength. So the phase shift part of the present theory, which does not affect either the ray equations or CWA, has not been observed experimentally.

Although this is the first general derivation of CWA for deep water, other authors have addressed the theory of wave–current interactions for rotational currents. Jonsson, Brink-Kjaer & Thomas (1978) considered water of finite depth, with a two-dimensional shear current of constant vorticity propagating over a gently sloping bed, and obtained a one-dimensional form of CWA. Stiassnie & Peregrine (1979) obtained CWA for water of finite depth and modulations of the waves that are long compared to the depth. But not all approaches have yielded CWA. For water of finite depth over a bed with small slope, and for a current that is vertically uniform with vorticity about a vertical axis, Christoffersen & Jonsson (1981) derived an alternative to CWA – an energy reference line method for computing wave amplitudes. Shrira (1993) considered deep water waves that are close to potential, on horizontally uniform shear currents, and obtained a power series solution to the resulting boundary value problem. Milewski (1992) followed an approach most similar to that of this paper, using a WKB method on deep water; however he chose the Froude number as a small parameter, in contrast to our choice of ϵ as described above, and did not obtain CWA. For a comparison of laboratory results with theory, see Thomas (1981).

Without loss of generality, we will study the single-frequency, time-harmonic case. For this case, $\theta = -\omega t + \psi(x, y)$, where ω is a constant. Substituting this expression into (3) gives

$$\Omega(\boldsymbol{x}, \tilde{\nabla}\psi) = \omega. \tag{9}$$

The time-harmonic amplitude, A, is given by the time-independent version of equation (8)

$$\tilde{\nabla} \cdot \left\{ \frac{A^2}{\bar{\Omega}(\tilde{\nabla}\psi)} \Omega_{,\boldsymbol{k}}(\boldsymbol{x}, \tilde{\nabla}\psi) \right\} = 0. \tag{10}$$

In what follows, we will derive equations (9) and (10), as well as an equation for the phase shift χ. For steady currents there is no loss of generality in treating the time-harmonic case, as is shown in the Appendix, since equations (3), (8), and the phase shift equation, can be recovered from the time-harmonic case by Fourier superposition and a stationary phase approximation.

Although we shall not do so in this paper, the amplitude of a wave can be written, via equation (10), in terms of the Jacobian of the mapping from ray coordinates to Cartesian coordinates. Equivalently, the Jacobian can be expressed in terms of the "raytube area", which is the (inifinitesimal) distance between two neighbouring rays. The Jacobian is usually determined by solving four more ordinary differential equations along a ray, in addition to the usual equations of ray theory (Jonsson 1990). White & Fornberg (1998) showed how in the single-frequency case the raytube area can be determined with only two additional equations, and demonstrated the efficacy of this method by implementing it in a general-purpose ray-tracing code.

A curve on which the Jacobian (or equivalently the raytube area) vanishes, is called a caustic, and CWA cannot be used directly there, since it predicts that the wave amplitude is infinite. To obtain the correct amplitude near a caustic CWA must be modified to include linear or possibly nonlinear corrections. For a discussion of caustics, including the chance of their occurrence, see White & Fornberg (1998) and references therein.

The outline of this paper is as follows.

In § 2, we state the nonlinear free boundary value problem for inviscid, incompressible surface gravity waves. We then scale the equations appropriately and introduce the two small parameters, ϵ and δ, as defined above. We show how the condition $\delta \ll \epsilon$ leads to linear equations for the waves, although the current may satisfy nonlinear equations.

In § 3 we introduce the WKB ansatz, including the phase (i.e. the eikonal) ϕ, and the WKB surface phase ψ_0, which is the restriction of ϕ to the free surface. It turns out that the WKB surface phase ψ_0 is not exactly the same as the physical phase ψ introduced in equations (9) and (10) above. However, as it is shown in § 4, the two phases are closely related, since the WKB phase is the leading-order term in the expansion of the physical phase in powers of ϵ. Substitution of the WKB ansatz into the equations of § 2 gives the equations of the WKB expansion.

In § 4 we use the WKB expansion to derive eikonal equations for the phases ϕ and ψ_0, and in § 5 we derive a transport equation for the pressure. This pressure equation is in the volume of underwater fluid, and must be restricted to the free surface to complete our ray theory description of the waves. This is accomplished in § 6, where alternative expressions are found for the various normal derivatives to the free surface that occur in the underwater pressure equation. The result of § 6 is then a surface transport equation for the wave amplitude.

In § 7 we show that this surface transport equation may be interpreted as CWA with a phase shift χ, and that χ is zero in the irrotational case. The phase shift is interpreted physically in § 8 as follows: suppose that the waves respond to the current not at the surface, but at some non-zero underwater *depth of influence*; then χ is accounted for, up to the accuracy of the perturbation expansion, if the depth of influence is identified as the local wavelength divided by 4π.

In the Appendix we show how the general case can be recovered from the time-harmonic equations by Fourier superposition and a stationary phase approximation.

2. Scaling and linearization

Let ρ be the density, $\boldsymbol{u}$ the velocity and p the pressure. Then for a free surface $\zeta = \zeta(t, x, y)$ the deep water surface gravity wave equations are

$$\left.\begin{array}{lll}\rho\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \nabla p = \boldsymbol{0} & \text{for} & z < \zeta, \\ \nabla\cdot\boldsymbol{u} = \boldsymbol{0} & \text{for} & z < \zeta, \\ p = \rho g \zeta & \text{on} & z = \zeta, \\ u^3 = \dfrac{\partial \zeta}{\partial t} + \boldsymbol{u}\cdot\nabla\zeta & \text{on} & z = \zeta,\end{array}\right\} \tag{11}$$

where u^3 is the vertical (z) component of $\boldsymbol{u}$ and ∇ is the three-dimensional gradient. Let $\bar{X}$ be a typical wavelength of the waves and $\bar{X}_c$ the spatial scale of the current, and define the small parameter

$$\epsilon = \bar{X}/\bar{X}_c \ll 1 \quad . \tag{12}$$

Typical velocity ($\bar{u}$) and time ($\bar{t}$) scales are defined as those of the waves in the absence of current

$$\bar{u} = (g\bar{X})^{1/2}, \quad \bar{t} = \bar{X}/\bar{u}, \tag{13}$$

when $\bar{u}/(2\pi)^{1/2}$ is the phase velocity of the waves. The following scaling and depth transformation puts the spatial coordinates on the scale of the current, and refers depth to distance below the free surface, which is mapped into $\{z' = 0\}$. Time and the height of the free surface are scaled by quantities appropriate to the waves:

$$\left.\begin{array}{l} x' = x/\bar{X}_c, \quad y' = y/\bar{X}_c, \quad z' = (z-\zeta)/\bar{X}_c, \\ \boldsymbol{u}' = \boldsymbol{u}/\bar{u}, \quad t' = t/\bar{t}, \\ p' = p/(\rho g \bar{X}), \quad \zeta' = \zeta/\bar{X}. \end{array}\right\} \tag{14}$$

Substitution of (14) into (11) and use of (12) and (13) yields, after dropping primes for notational convenience

$$\left.\begin{array}{lll}\dfrac{\partial \boldsymbol{u}}{\partial t} + \epsilon\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \epsilon\nabla p = \epsilon\dfrac{\partial \zeta}{\partial t}\dfrac{\partial \boldsymbol{u}}{\partial z} + \epsilon^2(\boldsymbol{u}\cdot\nabla\zeta)\dfrac{\partial \boldsymbol{u}}{\partial z} + \epsilon^2\nabla\zeta\dfrac{\partial p}{\partial z} & \text{for} & z < 0, \\ \nabla\cdot\boldsymbol{u} = \epsilon\nabla\zeta\cdot\dfrac{\partial \boldsymbol{u}}{\partial z} & \text{for} & z < 0, \\ p = \zeta & \text{for} & z = 0, \\ u^3 = \dfrac{\partial \zeta}{\partial t} + \epsilon\boldsymbol{u}\cdot\nabla\zeta & \text{for} & z = 0.\end{array}\right\} \tag{15}$$

The current is a time-independent solution $\boldsymbol{u} = \boldsymbol{U}$, $p = P$, $\zeta = \eta$ of (15). It is assumed that fluctuations in the height of the free surface created by the current are not much larger than a wavelength. Since in equations (14) the free surface is scaled by a typical wavelength, this condition implies that η is of $O(1)$ as $\epsilon \downarrow 0$. We therefore expand

$$\left.\begin{aligned} \boldsymbol{U} &= \boldsymbol{U}_0 + \epsilon \boldsymbol{U}_1 + O(\epsilon^2), \\ P &= P_0 + \epsilon P_1 + O(\epsilon^2), \\ \eta &= \eta_0 + \epsilon \eta_1 + O(\epsilon^2). \end{aligned}\right\} \tag{16}$$

A regular expansion of (15) then yields, for the $O(1)$ terms

$$\left.\begin{aligned} &\boldsymbol{U}_0 \cdot \nabla \boldsymbol{U}_0 + \nabla P_0 = 0 && \text{for} \quad z < 0, \\ &\nabla \cdot \boldsymbol{U}_0 = 0 && \text{for} \quad z < 0, \\ &P_0 = \eta_0 && \text{for} \quad z = 0, \\ &U_0^3 = 0 && \text{for} \quad z = 0. \end{aligned}\right\} \tag{17}$$

Note that equation (17) implies that, on the free surface

$$\frac{\partial P_0}{\partial z} = 0 \quad \text{for} \quad z = 0. \tag{18}$$

The $O(\epsilon)$ terms in the expansion (16) satisfy

$$\left.\begin{aligned} &\boldsymbol{U}_1 \cdot \nabla \boldsymbol{U}_0 + \boldsymbol{U}_0 \cdot \nabla \boldsymbol{U}_1 + \nabla P_1 = \boldsymbol{U}_0 \cdot \nabla \eta_0 \frac{\partial \boldsymbol{U}_0}{\partial z} + \nabla \eta_0 \frac{\partial P_0}{\partial z} && \text{for} \quad z < 0, \\ &\nabla \cdot \boldsymbol{U}_1 = \nabla \eta_0 \cdot \frac{\partial \boldsymbol{U}_0}{\partial z} && \text{for} \quad z < 0, \\ &P_1 = \eta_1 && \text{for} \quad z = 0, \\ &U_1^3 = \boldsymbol{U}_0 \cdot \nabla \eta_0 && \text{for} \quad z = 0. \end{aligned}\right\} \tag{19}$$

Let δ be a small parameter giving the scale of the wave amplitude. For linear waves it is assumed that $\delta \ll \epsilon$. The waves, characterized by $\hat{\boldsymbol{u}}, \hat{p}, \hat{\zeta}$, are then defined as perturbations of the current, so that the full solution of (15) is

$$\boldsymbol{u} = \boldsymbol{U} + \delta \hat{\boldsymbol{u}}, \quad p = P + \delta \hat{p}, \quad \zeta = \eta + \delta \hat{\zeta}. \tag{20}$$

In (20) $\boldsymbol{U}, P, \eta$ may also be small or even zero, so that the case of vanishing current is not excluded. Putting (20) into the first of equations (15) and using that $\boldsymbol{U}, P, \eta$ are also solutions yields

$$\begin{aligned} &\left\{ \frac{\partial}{\partial t} \hat{\boldsymbol{u}} + \epsilon (\boldsymbol{U} \cdot \nabla) \hat{\boldsymbol{u}} + \epsilon \nabla \hat{p} \right\} \\ &\quad = \epsilon \left\{ -(\hat{\boldsymbol{u}} \cdot \nabla)\, \boldsymbol{U} + \epsilon\, (\boldsymbol{U} \cdot \nabla \eta)\, \frac{\partial \hat{\boldsymbol{u}}}{\partial z} + \epsilon \nabla \eta \frac{\partial \hat{p}}{\partial z} + \left(\frac{\partial \hat{\zeta}}{\partial t} + \epsilon \boldsymbol{U} \cdot \nabla \hat{\zeta} \right) \frac{\partial \boldsymbol{U}}{\partial z} + \epsilon \nabla \hat{\zeta} \frac{\partial P}{\partial z} \right\} \\ &\qquad + \delta \left\{ -\epsilon (\hat{\boldsymbol{u}} \cdot \nabla) \hat{\boldsymbol{u}} + \epsilon \frac{\partial \hat{\zeta}}{\partial t} \frac{\partial \hat{\boldsymbol{u}}}{\partial z} + \epsilon^2 (\boldsymbol{U} \cdot \nabla \hat{\zeta}) \frac{\partial \hat{\boldsymbol{u}}}{\partial z} + \epsilon^2 \nabla \hat{\zeta} \frac{\partial \hat{p}}{\partial z} \right\} + \epsilon^2 \left\{ (\hat{\boldsymbol{u}} \cdot \nabla \eta) \frac{\partial \boldsymbol{U}}{\partial z} \right\} \\ &\qquad + \epsilon \delta \left\{ \epsilon (\hat{\boldsymbol{u}} \cdot \nabla \eta) \frac{\partial \hat{\boldsymbol{u}}}{\partial z} + \epsilon (\hat{\boldsymbol{u}} \cdot \nabla \hat{\zeta}) \frac{\partial \boldsymbol{U}}{\partial z} \right\} + \delta^2 \left\{ \epsilon^2 (\hat{\boldsymbol{u}} \cdot \nabla \hat{\zeta}) \frac{\partial \hat{\boldsymbol{u}}}{\partial z} \right\}, \qquad z < 0. \end{aligned} \tag{21}$$

In assessing the order of each term in equation (21), note that gradients of $\hat{\boldsymbol{u}}, \hat{p}, \hat{\zeta}$

are of $O(1/\epsilon)$. Therefore each expression enclosed in braces in this equation is of $O(1)$, and the order of each term is given by the factors of ϵ and δ multiplying the expressions in braces.

Let $\delta \ll \epsilon$. We will retain the leading-order term and one order higher in equation (21) since, when using the WKB method, two orders of accuracy in the equations are necessary to correctly calculate the leading-order term of the solution. (From equation (21) it is apparent that the assumption $\delta \ll \epsilon$ is necessary for a linear theory, since nonlinear terms appear at $O(\delta)$.) Thus we will retain terms of $O(1)$ and $O(\epsilon)$, while dropping the terms of $O(\delta), O(\epsilon^2), O(\epsilon\delta), O(\delta^2)$. We also substitute $\boldsymbol{U}, P, \eta$ from equations (16). Since the resulting equation is linear, we then substitute $\partial/\partial t \to -\mathrm{i}\omega$ for time-harmonic waves of frequency ω. The resulting equation is

$$-\mathrm{i}\omega\hat{\boldsymbol{u}} + \epsilon\boldsymbol{U}_0\cdot\nabla\hat{\boldsymbol{u}} + \epsilon\nabla\hat{p} = \epsilon\bigg\{ -(\hat{\boldsymbol{u}}\cdot\nabla)\,\boldsymbol{U}_0 - \epsilon\,(\boldsymbol{U}_1\cdot\nabla)\,\hat{\boldsymbol{u}} + \epsilon\,(\boldsymbol{U}_0\cdot\nabla\eta_0)\,\frac{\partial\hat{\boldsymbol{u}}}{\partial z}$$
$$+\epsilon\nabla\eta_0\frac{\partial\hat{p}}{\partial z} + (-\mathrm{i}\omega\hat{\zeta} + \epsilon\boldsymbol{U}_0\cdot\nabla\hat{\zeta})\frac{\partial\boldsymbol{U}_0}{\partial z} + \epsilon\nabla\hat{\zeta}\frac{\partial P_0}{\partial z}\bigg\}, \quad z<0. \quad (22)$$

Similar approximations in the other three of equations (15) yield

$$\nabla\cdot\hat{\boldsymbol{u}} = \epsilon\nabla\eta_0\cdot\frac{\partial\hat{\boldsymbol{u}}}{\partial z} + \epsilon\nabla\hat{\zeta}\cdot\frac{\partial\boldsymbol{U}_0}{\partial z}, \qquad z<0 \quad (23)$$

$$\hat{p} = \hat{\zeta}, \qquad z=0 \quad (24)$$

$$\hat{u}^3 = -\mathrm{i}\omega\hat{\zeta} + \epsilon\boldsymbol{U}_0\cdot\nabla\hat{\zeta} + \epsilon^2\boldsymbol{U}_1\cdot\nabla\hat{\zeta} + \epsilon\hat{\boldsymbol{u}}\cdot\nabla\eta_0, \qquad z=0. \quad (25)$$

3. WKB expansion

Let $\psi_0(x,y)$ be the the surface eikonal, that is, the WKB phase of the wave at the surface point $(x,y,z=0)$. It will be shown below that ψ_0 is closely related, but not identical, to the physical phase ψ, as explained in the Introduction. The phase ψ_0 is the restriction to $z=0$ of the underwater WKB phase, the eikonal $\phi(x,y,z)$, i.e.

$$\psi_0(x,y) = \phi(x,y,0). \quad (26)$$

Then the WKB ansatz, including both surface and underwater terms becomes

$$\hat{\boldsymbol{u}} \sim \bar{\boldsymbol{u}}\mathrm{e}^{\mathrm{i}\phi/\epsilon} + \bar{\boldsymbol{v}}\mathrm{e}^{\mathrm{i}\psi_0/\epsilon} \quad \hat{p} \sim \bar{p}\mathrm{e}^{\mathrm{i}\phi/\epsilon} + \bar{q}\mathrm{e}^{\mathrm{i}\psi_0/\epsilon} \quad \hat{\zeta} \sim \bar{\zeta}\mathrm{e}^{\mathrm{i}\psi_0/\epsilon}, \quad (27)$$

where the various complex amplitudes may be expanded as

$$\left.\begin{array}{l} \bar{\boldsymbol{u}} \sim \bar{\boldsymbol{u}}_0 + \epsilon\bar{\boldsymbol{u}}_1 + O(\epsilon^2), \quad \bar{p} \sim \bar{p}_0 + \epsilon\bar{p}_1 + O(\epsilon^2), \quad \bar{\zeta} \sim \bar{\zeta}_0 + \epsilon\bar{\zeta}_1 + O(\epsilon^2), \\ \bar{\boldsymbol{v}} \sim \epsilon\bar{\boldsymbol{v}}_1 + O(\epsilon^2), \quad \bar{q} \sim \epsilon\bar{q}_1 + O(\epsilon^2). \end{array}\right\} \quad (28)$$

It is anticipated in (28) that the surface phase terms have amplitudes $\bar{\boldsymbol{v}}$, $\bar{q}$ that are of $O(\epsilon)$, a fact that can be verified more laboriously by including more $O(1)$ terms.

The WKB method has had great success in the theory of bottom effects on water waves, since the pioneering work of Keller (1958). For a review, see Meyer (1979). For use of WKB for waves on a current, see Peregrine & Smith (1975) , and Milewski (1992). Our equations (27) differ from the usual WKB ansatz in that they contain the surface phase terms, proportional to $\mathrm{e}^{\mathrm{i}\psi_0/\epsilon}$ in the underwater equations for $\hat{\boldsymbol{u}}$ and $\hat{p}$. It is readily apparent that these terms are necessary here to balance the terms involving the surface amplitude $\hat{\zeta}$, which appear in the underwater equations (22) and (23).

These terms involving $\hat{\zeta}$ in equations (22) and (23) can be traced back to the transformation of z to z' in equations (14). That is, the surface phase terms in the bulk equations arise from our coordinate system transformation, where z' represents depth below the free surface. From this observation, we can anticipate their values with the following heuristic argument. Suppose that there are no surface phase terms in the original coordinate system. Then to transform pressure and velocity to the new coordinate system, we must add, to $O(\epsilon)$, their vertical gradients times the height of the free surface. Since the free surface has waves, a surface phase term will appear, as in equations (27). Through our systematic perturbation expansion, we will indeed confirm that these extra terms are simply the anticipated vertical gradients times the height of the surface, c.f. equations (47) and (48).

Insertion of the ansatz (27), (28) into (22) yields, from the $O(1)$ term

$$\bar{\boldsymbol{u}}_0 = \frac{\nabla\phi}{(\omega - \boldsymbol{U}_0 \cdot \nabla\phi)}\bar{p}_0, \tag{29}$$

and from the $O(\epsilon)e^{i\phi/\epsilon}$ terms

$$(\omega - \boldsymbol{U}_0 \cdot \nabla\phi)\bar{\boldsymbol{u}}_1 - \nabla\phi\bar{p}_1 + i\left[(\boldsymbol{U}_0 \cdot \nabla)\bar{\boldsymbol{u}}_0 + (\bar{\boldsymbol{u}}_0 \cdot \nabla)\boldsymbol{U}_0 + \nabla\bar{p}_0\right] + \left(\boldsymbol{U}_0 \cdot \nabla\eta_0\, \phi_{,z} - \boldsymbol{U}_1 \cdot \nabla\phi\right)\bar{\boldsymbol{u}}_0 + \nabla\eta_0\, \phi_{,z}\bar{p}_0 = 0, \tag{30}$$

where $\phi_{,z}$ is the derivative of ϕ with respect to z. Also, from the $O(\epsilon)e^{i\psi_0/\epsilon}$ terms

$$(\omega - \boldsymbol{U}_0 \cdot \nabla\psi_0)\bar{\boldsymbol{v}}_1 - \nabla\psi_0\, \bar{q}_1 = (\omega - \boldsymbol{U}_0 \cdot \nabla\psi_0)\bar{\zeta}_0 \boldsymbol{U}_{0,z} - \nabla\psi_0\, \bar{\zeta}_0 P_{0,z}. \tag{31}$$

Insertion of the ansatz into (23) yields, from the $O(1/\epsilon)$ terms

$$\bar{\boldsymbol{u}}_0 \cdot \nabla\phi = 0, \tag{32}$$

and from $O(1)e^{i\phi/\epsilon}$

$$\bar{\boldsymbol{u}}_1 \cdot \nabla\phi = i\nabla \cdot \bar{\boldsymbol{u}}_0 + \nabla\eta_0 \cdot \bar{\boldsymbol{u}}_0\phi_{,z}, \tag{33}$$

and from $O(1)e^{i\psi_0/\epsilon}$

$$\bar{\boldsymbol{v}}_1 \cdot \nabla\psi_0 = \bar{\zeta}_0 \left(\nabla\psi_0 \cdot \boldsymbol{U}_{0,z}\right). \tag{34}$$

Boundary conditions are obtained by putting the ansatz into (24) to obtain, from $O(1)$ and $O(\epsilon)$ respectively,

$$\bar{p}_0 = \bar{\zeta}_0 \quad \text{on} \quad z = 0, \tag{35}$$

$$\bar{p}_1 + \bar{q}_1 = \bar{\zeta}_1 \quad \text{on} \quad z = 0. \tag{36}$$

Similarly, putting the ansatz into (25) yields

$$\bar{u}_0^3 = -i(\omega - \boldsymbol{U}_0 \cdot \nabla\psi_0)\bar{\zeta}_0 \quad \text{on} \quad z = 0, \tag{37}$$

$$\bar{u}_1^3 + \bar{v}_1^3 = -i(\omega - \boldsymbol{U}_0 \cdot \nabla\psi_0)\bar{\zeta}_1 + \boldsymbol{U}_0 \cdot \nabla\bar{\zeta}_0 + i\boldsymbol{U}_1 \cdot \nabla\psi_0\bar{\zeta}_0 + \bar{u}_0 \cdot \nabla\eta_0 \quad \text{on} \quad z = 0\,. \tag{38}$$

Equations (29)–(38) determine the leading-order terms in the WKB expansion.

4. The eikonal equations

In this section we will determine eikonal equations for the WKB phases, and relate ψ_0 to the physical phase ψ. Dotting (29) with $\nabla\phi$, and using (32) gives the underwater eikonal equation

$$(\nabla\phi)^2 = 0. \tag{39}$$

Equivalently,

$$\phi_{,z} = -\mathrm{i}[(\phi_{,x})^2 + (\phi_{,y})^2]^{1/2}, \tag{40}$$

where the sign has been chosen to give decay of the wave with depth.

To get the surface phase, equate $\bar{u}_0^3$ as determined by (37) to that determined by (29) restricted to $z = 0$. The quantity $\phi_{,z}$ is determined by (40) on $z = 0$, where $\phi_{,x} = \psi_{0,x}, \phi_{,y} = \psi_{0,y}$ because of (26). Also, $\boldsymbol{U}_0 \cdot \nabla\phi = \boldsymbol{U}_0 \cdot \nabla\psi_0$ on $z = 0$ because $U_0^3 = 0$ there (see (17)). Use of (35) then gives an equation for ψ_0 alone

$$(\omega - \boldsymbol{U}_0 \cdot \nabla\psi_0)^2 = [(\psi_{0,x})^2 + (\psi_{0,y})^2]^{1/2}. \tag{41}$$

Equation (41) is the surface eikonal equation. Identifying $\tilde{\nabla} = (\partial/\partial x, \partial/\partial y)^T$ and $\tilde{\boldsymbol{U}}_0 = (U_0^1, U_0^2)^T|_{z=0}$, equation (41) becomes

$$\omega = \Omega_0(\boldsymbol{x}, \tilde{\nabla}\psi_{\boldsymbol{0}}), \tag{42}$$

where

$$\Omega_0(\boldsymbol{x}, \boldsymbol{k}) = \pm|\boldsymbol{k}|^{1/2} + \boldsymbol{k} \cdot \tilde{\boldsymbol{U}}_0. \tag{43}$$

In non-dimensional terms ($g = 1$), equation (42) for ψ_0 is almost the same as equation (9) for the physical phase ψ, with the dispersion relations (1), (2). The only difference is that $\tilde{\boldsymbol{U}}_0$ is used in the equation for ψ_0, while $\tilde{\boldsymbol{U}}$ is used in the equation for ψ. This difference is to be expected, since the WKB eikonal equation cannot depend on ϵ. Therefore equation (42) determines the WKB phase ψ_0 as the leading-order term in the regular perturbation expansion of the physical phase ψ

$$\psi = \psi_0 + \epsilon\psi_1 + O(\epsilon^2), \tag{44}$$

where the first-order term, ψ_1, satisfies the equation

$$\Omega_{0,\boldsymbol{k}}(\boldsymbol{x}, \tilde{\nabla}\psi_0) \cdot \tilde{\nabla}\psi_1 = \left(\pm\frac{1}{2}\frac{\tilde{\nabla}\psi_0}{|\tilde{\nabla}\psi_0|^{3/2}} + \tilde{\boldsymbol{U}}_0\right) \cdot \tilde{\nabla}\psi_1 = -\tilde{\boldsymbol{U}}_1 \cdot \tilde{\nabla}\psi_0. \tag{45}$$

The term ψ_1 contributes to the leading-order expression for the waves, since

$$\hat{\zeta} \sim \bar{\zeta}_0 \mathrm{e}^{\mathrm{i}\psi_0/\epsilon} \sim \bar{\zeta}_0 \mathrm{e}^{-\mathrm{i}\psi_1} \mathrm{e}^{\mathrm{i}\psi/\epsilon}. \tag{46}$$

5. The pressure equation

In this section, an equation will be derived for the pressure. First dot (31) with $\nabla\psi_0$ and use (34) to get

$$\bar{q}_1 = \bar{\zeta}_0 P_{0,z}. \tag{47}$$

Substitution of (47) into (31) gives

$$\bar{\boldsymbol{v}}_1 = \bar{\zeta}_0 \boldsymbol{U}_{0,z}. \tag{48}$$

Next, dot (30) with $\nabla\phi$ and use (32), (33) and (39) to get

$$\begin{aligned}(\omega - \boldsymbol{U}_0 \cdot \nabla\phi)[\mathrm{i}\nabla \cdot \bar{\boldsymbol{u}}_0 + \nabla\eta_0 \cdot \bar{\boldsymbol{u}}_0\phi_{,z}] + \mathrm{i}\,[(\boldsymbol{U}_0 \cdot \nabla)\bar{\boldsymbol{u}}_0] \cdot \nabla\phi \\ +\mathrm{i}\,[(\bar{\boldsymbol{u}}_0 \cdot \nabla)\boldsymbol{U}_0] \cdot \nabla\phi + i\nabla\phi \cdot \nabla\bar{p}_0 + (\nabla\eta_0 \cdot \nabla\phi)\phi_{,z}\bar{p}_0 = 0.\end{aligned} \tag{49}$$

However, from (29)

$$[(\boldsymbol{U}_0 \cdot \nabla)\bar{\boldsymbol{u}}_0] \cdot \nabla\phi = 0, \tag{50}$$

as can be shown by differentiation of (29), and use of (39) and the derivative of (39),

$$\nabla\nabla\phi \cdot \nabla\phi = \tfrac{1}{2}\nabla(\nabla\phi \cdot \nabla\phi) = 0. \tag{51}$$

Also, from (29)

$$\nabla \cdot \bar{\boldsymbol{u}}_0 = \frac{\nabla^2\phi}{(\omega - \boldsymbol{U}_0 \cdot \nabla\phi)}\bar{p}_0 + \frac{\nabla\phi \cdot \nabla\bar{p}_0}{(\omega - \boldsymbol{U}_0 \cdot \nabla\phi)} + \frac{\nabla\phi^T \cdot \nabla\boldsymbol{U}_0 \cdot \nabla\phi}{(\omega - \boldsymbol{U}_0 \cdot \nabla\phi)^2}\bar{p}_0. \quad (52)$$

Here $\nabla\boldsymbol{U}_0$ is the matrix, and the quadratic form in $\nabla\phi$ is equivalent to

$$\nabla\phi^T \cdot \nabla\boldsymbol{U}_0 \cdot \nabla\phi = -\nabla\phi \cdot \nabla(\omega - \boldsymbol{U}_0 \cdot \nabla\phi) \quad (53)$$

through use of (51). Substitution of (50) and (52) into (49) yields the pressure equation:

$$\nabla\phi \cdot \nabla\bar{p}_0 + \left\{ \tfrac{1}{2}\nabla^2\phi + \frac{\nabla\phi^T \cdot \nabla\boldsymbol{U}_0 \cdot \nabla\phi}{(\omega - \boldsymbol{U}_0 \cdot \nabla\phi)} - \mathrm{i}\phi_{,z}\nabla\eta_0 \cdot \nabla\phi \right\} \bar{p}_0 = 0. \quad (54)$$

6. The surface transport equation

The underwater pressure equation must be restricted to the surface $z = 0$ to get the wave action equation. Inspection of (54) suggests that expressions must be obtained for $\phi_{,z}, \nabla^2\phi, \nabla\phi^T \cdot \nabla\boldsymbol{U}_0 \cdot \nabla\phi$, and $\bar{p}_{0,z}$ on $z = 0$.

Let

$$\Lambda = \omega - \boldsymbol{U}_0 \cdot \nabla\psi_0. \quad (55)$$

First, recall that $\boldsymbol{U}_0 \cdot \nabla\phi = \boldsymbol{U}_0 \cdot \nabla\psi_0$ on $z = 0$, to obtain from (40), (41) that

$$\phi_{,z} = -\mathrm{i}\Lambda^2, \qquad z = 0. \quad (56)$$

Second, we obtain $\nabla^2\phi = \nabla^2\psi_0 + \phi_{,zz}$. Differentiation of (40) and use of (56) and (41) yields

$$\phi_{,zz} = -2\nabla\psi_0 \cdot \frac{\nabla\Lambda}{\Lambda}, \qquad z = 0\,. \quad (57)$$

Third,

$$\nabla\phi^T \cdot \nabla\boldsymbol{U}_0 \cdot \nabla\phi = -\nabla\psi_0 \cdot \nabla\Lambda - 2\Lambda^3\boldsymbol{U}_0 \cdot \nabla\Lambda + \phi_{,z}\nabla\psi_0 \cdot \boldsymbol{U}_{0,z} + (\phi_{,z})^2 U^3_{0,z} \quad \text{for} \quad z = 0, \quad (58)$$

where $\phi_{,z}$ may be substituted from (56). Equation (58) may be derived by writing $\nabla\phi = \nabla\psi_0 + \phi_{,z}\boldsymbol{e}_3, z = 0$ (for $\boldsymbol{e}_3 = \nabla z$), and then obtaining $\nabla\psi_0^T \cdot \nabla\boldsymbol{U}_0 \cdot \nabla\psi_0$ from the identity

$$\nabla\psi_0 \cdot \nabla\Lambda = -\nabla\psi_0^T \cdot \nabla\boldsymbol{U}_0 \cdot \nabla\psi_0 - \nabla\psi_0^T \cdot \nabla\nabla\psi_0 \cdot \boldsymbol{U}_0 \quad (59)$$

where $\nabla\nabla\psi_0 \cdot \nabla\psi_0 = \frac{1}{2}\nabla(\nabla\psi_0)^2$ and (41) are also used.

Fourth,

$$\bar{p}_{0,z} = 2\mathrm{i}\Lambda\boldsymbol{U}_0 \cdot \nabla\bar{\zeta}_0 + \mathrm{i}\boldsymbol{U}_0 \cdot \nabla\Lambda\bar{\zeta}_0 + \mathrm{i}\nabla\psi_0 \cdot \nabla\eta_0\bar{\zeta}_0 - 2\Lambda\boldsymbol{U}_1 \cdot \nabla\psi_0\bar{\zeta}_0 \quad \text{for} \quad z = 0. \quad (60)$$

To obtain equation (60), first note that from (47) and (18)

$$\bar{q}_1 = 0 \quad \text{for} \quad z = 0. \quad (61)$$

Next, substitute (61) into (36) to get

$$\bar{p}_1 = \bar{\zeta}_1 \quad \text{for} \quad z = 0. \quad (62)$$

Now equations (29), (26), (35), (48) and (62) can be substituted into (38) to get

$$\bar{u}^3_1 = -\bar{\zeta}_0 U^3_{0,z} - \mathrm{i}\Lambda\bar{p}_1 + \boldsymbol{U}_0 \cdot \nabla\bar{\zeta}_0 + \mathrm{i}\boldsymbol{U}_1 \cdot \nabla\psi_0\bar{\zeta}_0 + \nabla\eta_0 \cdot \frac{\nabla\psi_0}{\Lambda}\bar{\zeta}_0 \quad \text{for} \quad z = 0. \quad (63)$$

Finally, this last expression is put into the z-component of (30) to obtain (60). The expression has been simplified using that $U_1^3 = \boldsymbol{U}_0 \cdot \nabla \eta_0$ (from (19)), and using (56).

Putting equations (57), (58) and (60) into (54) now yields

$$\left\{\nabla\psi_0 + 2\Lambda^3 \boldsymbol{U}_0\right\} \cdot \nabla\bar{\zeta}_0 + \left\{\tfrac{1}{2}\,\nabla^2\psi_0 - \Lambda^2 \boldsymbol{U}_0 \cdot \nabla\Lambda - \mathrm{i}\Lambda\nabla\psi_0 \cdot \boldsymbol{U}_{0,z} - \Lambda^3 U^3_{0,z} + 2\mathrm{i}\Lambda^3 \boldsymbol{U}_1 \cdot \nabla\psi_0 - 2\nabla\psi_0 \cdot \frac{\nabla\Lambda}{\Lambda}\right\}\bar{\zeta}_0 = 0 \quad \text{for} \quad z = 0. \tag{64}$$

Let

$$\nabla_2 = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\nabla} \\ 0 \end{pmatrix} \tag{65}$$

be the two-dimensional gradient. Note that $\nabla \cdot \boldsymbol{U}_0 = 0$ (from (17)) implies that $\nabla_2 \cdot \boldsymbol{U}_0 = -U^3_{0,z}$. Then from (64) it follows that

$$\nabla_2 \cdot \left\{ \frac{[\frac{1}{2}\nabla\psi_0 + \Lambda^3 \boldsymbol{U}_0]}{\Lambda^4} (\bar{\zeta}_0)^2 \right\} = \mathrm{i}\left\{ \frac{\nabla\psi_0 \cdot \boldsymbol{U}_{0,z}}{\Lambda^3} - 2\frac{\nabla\psi_0 \cdot \boldsymbol{U}_1}{\Lambda} \right\} \bar{\zeta}_0^2 \quad \text{on} \quad z = 0. \tag{66}$$

Let $\tilde{\boldsymbol{U}}_0 = (U_0^1, U_0^2)^T|_{z=0}$, $\tilde{\boldsymbol{U}}_{0,z} = (U^1_{0,z}, U^2_{0,z})^T|_{z=0}$ and $\tilde{\boldsymbol{U}}_1 = (U_1^1, U_1^2)^T|_{z=0}$. Then use of equation (43) in (66) yields a surface equation

$$\pm\tilde{\nabla} \cdot \left\{ \frac{\bar{\zeta}_0^2}{\bar{\Omega}(\tilde{\nabla}\psi_0)} \Omega_{0,\boldsymbol{k}}(\boldsymbol{x}, \tilde{\nabla}\psi_0) \right\} = \tilde{\nabla} \cdot \left\{ \frac{\bar{\zeta}_0^2}{|\tilde{\nabla}\psi_0|^{\frac{1}{2}}} \left(\pm \frac{\frac{1}{2}\tilde{\nabla}\psi_0}{|\tilde{\nabla}\psi_0|^{\frac{3}{2}}} + \tilde{\boldsymbol{U}}_0 \right) \right\}$$
$$= \mathrm{i}\left\{ \frac{(\tilde{\nabla}\psi_0 \cdot \tilde{\boldsymbol{U}}_{0,z})}{|\tilde{\nabla}\psi_0|^{3/2}} - 2\frac{(\tilde{\nabla}\psi_0 \cdot \tilde{\boldsymbol{U}}_1)}{|\tilde{\nabla}\psi_0|^{1/2}} \right\} \bar{\zeta}_0^2. \tag{67}$$

7. Conservation of wave action, and a phase shift

The waves may be written in complex polar form

$$\hat{\zeta} \sim A\mathrm{e}^{\mathrm{i}(\psi/\epsilon + \chi)}, \tag{68}$$

where the real quantity A is the amplitude and the real phase shift χ is a correction to the physical phase ψ. Thus the physical phase is truly the phase of the wave if and only if $\chi = 0$, which, we will show, holds for irrotational currents.

Since ψ is already determined, to $O(\epsilon)$, by equation (9) it remains to determine equations for A and χ. Note that to determine $\hat{\zeta}$ to leading order, i.e. $O(1)$, A and χ need only be determined to $O(1)$. This is in contrast to ψ, which must be determined to $O(\epsilon)$, because of the occurrence of ψ/ϵ in equation (68). This fact was noted in the derivation of equation (46), which, when compared with equation (68) yields

$$\bar{\zeta}_0 = A\mathrm{e}^{\mathrm{i}(\psi_1 + \chi)}. \tag{69}$$

Substitution of equation (69) into equation (67), and separation of real and imaginary parts (after cancellation of phase factors), yields, for the real part

$$\tilde{\nabla} \cdot \left\{ \frac{A^2}{|\tilde{\nabla}\psi_0|^{1/2}} \left(\pm\frac{1}{2} \frac{\tilde{\nabla}\psi_0}{|\tilde{\nabla}\psi_0|^{3/2}} + \tilde{\boldsymbol{U}}_0 \right) \right\} = 0. \tag{70}$$

Equation (70) determines the amplitude A. However $|\psi - \psi_0|$ and $|\boldsymbol{U} - \boldsymbol{U}_0|$ are of

$O(\epsilon)$. Thus we may replace $\psi_0 \to \psi$, $\tilde{\boldsymbol{U}}_0 \to \tilde{\boldsymbol{U}}$ in this equation without altering the solution for A by more than $O(\epsilon)$. That is, we may make this substitution without affecting A to leading order, $O(1)$. The resulting equation for A is CWA:

$$\tilde{\nabla} \cdot \left\{ \frac{A^2}{|\tilde{\nabla}\psi|^{1/2}} \left(\pm \frac{1}{2} \frac{\tilde{\nabla}\psi}{|\tilde{\nabla}\psi|^{3/2}} + \tilde{\boldsymbol{U}} \right) \right\} = 0. \tag{71}$$

To verify that this equation is CWA with non-dimensional $g = 1$, compare it to equation (10), with the dispersion (1) and expression (5).

Equating imaginary parts in the substitution of equation (69) into equation (67) yields

$$\left(\pm \frac{1}{2} \frac{\tilde{\nabla}\psi_0}{|\tilde{\nabla}\psi_0|^{3/2}} + \tilde{\boldsymbol{U}}_0 \right) \cdot \tilde{\nabla}\chi = \frac{1}{2} \frac{\tilde{\nabla}\psi}{|\tilde{\nabla}\psi_0|} \cdot \tilde{\boldsymbol{U}}_{0,z}, \tag{72}$$

where $\tilde{\boldsymbol{U}}_{0,z} = (U^1_{0,z}, U^2_{0,z})^T|_{z=0}$. In deriving equation (72), ψ_1 and $\boldsymbol{U}_1$ were eliminated because of equation (45). The resulting simplification shows that the physical phase is indeed more physical than the WKB phase.

Equation (72) determines the phase shift χ. Equivalently, we may replace $\psi_0 \to \psi$, $\tilde{\boldsymbol{U}}_0 \to \tilde{\boldsymbol{U}}$, $\tilde{\boldsymbol{U}}_{0,z} \to \tilde{\boldsymbol{U}}_{,z}$, since, as these quantities differ by only $O(\epsilon)$ the corresponding solution for χ will differ only by $O(\epsilon)$, and so χ will still be determined correctly to $O(1)$. The resulting equation for the phase shift is

$$\Omega_{,\boldsymbol{k}}(\boldsymbol{x}, \tilde{\nabla}\psi) \cdot \tilde{\nabla}\chi = \left(\pm \frac{1}{2} \frac{\tilde{\nabla}\psi}{|\tilde{\nabla}\psi|^{3/2}} + \tilde{\boldsymbol{U}} \right) \cdot \tilde{\nabla}\chi = \frac{1}{2} \frac{\tilde{\nabla}\psi}{|\tilde{\nabla}\psi|} \cdot \tilde{\boldsymbol{U}}_{,z}. \tag{73}$$

The phase shift χ is easily obtained by appending one additional ordinary differential equation to an existing ray tracing code. Compare equation (73) to the ray equations (4). Then χ is given along a ray by the function $\bar{\chi}$ which satisfies the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\chi} = \frac{1}{2} \frac{\bar{\boldsymbol{k}}}{|\bar{\boldsymbol{k}}|} \cdot \tilde{\boldsymbol{U}}_{,z}. \tag{74}$$

Therefore the phase shift accumulates along a ray in proportion to half of the vertical derivative of the horizontal component of velocity, in the direction that the waves are travelling.

Finally, we show that $\chi = 0$ for irrotational currents, in accord with the usual irrotational theory. Assuming the existence of a velocity potential,

$$\tilde{\boldsymbol{U}}_{,z} = \left(\begin{array}{c} U^1_{,z} \\ U^2_{,z} \end{array} \right)_{z=0} = \left(\begin{array}{c} U^3_{,x} \\ U^3_{,y} \end{array} \right)_{z=0} + O(\epsilon) = \left(\begin{array}{c} U^3_{0,x} \\ U^3_{0,y} \end{array} \right)_{z=0} + O(\epsilon) = O(\epsilon), \tag{75}$$

since $U^3_0 = 0$ on $z = 0$ by equation (17). The $O(\epsilon)$ terms arise in (75) first, because the coordinate transformation (14) slightly skews the direction of the derivatives, and second, because $\boldsymbol{U}$ and $\boldsymbol{U}_0$ differ by $O(\epsilon)$. So for irrotational currents, the right-hand side of equation (74) vanishes, to leading order, and no phase correction can accumulate along a ray.

8. A physical interpretation of the phase shift

It remains to give a physical interpretation of the new phase shift χ. First, note that all of our final equations have been written in terms of $\tilde{\boldsymbol{U}}$ and $\tilde{\boldsymbol{U}}_{,z}$. That is, we have used the horizontal components of velocity and their vertical derivatives evaluated *on* *the surface*. However the wave is not confined to the surface itself, but extends below to a depth on the order of a wavelength.

We will show that one interpretation of our results is that the waves really respond to the current not at the surface, but at a non-zero *depth of influence*, ϵD. The small parameter ϵ is inserted in this expression because the depth of influence is expected to be on the order of a wavelength. More specifically, it is expected to vary in space with the local wavelength, so that

$$D = D(\lambda), \tag{76}$$

where

$$\lambda = \frac{2\pi}{|\nabla\psi|} = \frac{2\pi}{k} \tag{77}$$

and $\epsilon\lambda$ is the local wavelength. Let

$$\psi_T = \psi + \epsilon\chi = \psi_0 + \epsilon\psi_1 + \epsilon\chi \tag{78}$$

be the total phase, so that the wave is of the form $\hat{\zeta} = A\mathrm{e}^{\mathrm{i}\psi_T/\epsilon}$. We suppose that the total phase ψ_T satisfies the surface eikonal, equation (9) (with non-dimensional $g = 1$), but with the current evaluated at the depth of influence $z = -\epsilon D$. Then

$$\pm|\tilde{\nabla}\psi_T|^{1/2} + (U^1(x, y, -\epsilon D), U^2(x, y, -\epsilon D)) \cdot \tilde{\nabla}\psi_T = \omega. \tag{79}$$

We substitute equation (78) into (79) and expand in powers of ϵ. Use of (42) and (45) then eliminates the phases ψ_0 and ψ_1 to give an equation for χ in terms of D:

$$\left(\pm\frac{1}{2}\frac{\tilde{\nabla}\psi_0}{|\tilde{\nabla}\psi_0|^{3/2}} + \tilde{\boldsymbol{U}}_0\right) \cdot \tilde{\nabla}\chi = D\tilde{\boldsymbol{U}}_{0,z} \cdot \tilde{\nabla}\psi_0 + O(\epsilon). \tag{80}$$

Equation (80) agrees, to leading order, with the previously derived equation (72), provided that we identify

$$D = \frac{1}{2k} = \frac{\lambda}{4\pi}. \tag{81}$$

Thus the response of the wave to the current at a depth of influence ϵD, with D determined by (81), accounts for the phase shift up to the order of accuracy of the perturbation expansion.

The expression (81) agrees with the depth of influence obtained from other considerations by Teles da Silva & Peregrine (1988, see remarks after their equation (2.11)), for the special case of two-dimensional steady waves on a current of constant vorticity, over a finite bottom.

9. Conclusions

As discussed in the Introduction, methods based on ray theory and wave action conservation are often used to describe the interaction of waves on deep water with a spatially varying current, whether or not the current has vorticity. The present theory gives theoretical support to the use of these methods. Conversely, the success of these methods in explaining experimental data gives experimental support to the main results of the present theory.

However, at the present time there is no supporting evidence, theoretical, numerical or experimental, for the newly-derived phase shift χ. For a single-frequency wave, measurement of χ would entail measurement of the precise positions of the wave crests to an accuracy smaller than a wavelength. The current velocity must also be known accurately, to $O(\epsilon)$. It is predicted that the positions of the crests will be displaced by a distance on the order of a wavelength from the positions calculated by ray theory, and that this displacement can be accounted for by the phase shift χ.

In view of this interpretation, it is not surprising that χ has not been observed. Calculations and measurements based solely on modulations of the wave amplitude, direction and period over large spatial scales are not sufficient for observing it. Conversely, if one is only interested in modulations of the waves over large spatial scales, i.e. scales much larger than a wavelength, then χ is of no importance.

The phase shift χ is zero in two important special cases – when the current is irrotational and when the current does not vary with depth. Othewise, calculation of χ may be accomplished by appending a single additional ordinary differential equation to a standard ray tracing code, the equation (74). However, implementation of this equation requires input data on the vertical derivative of the horizontal components of current velocity, at the water's surface. This is a two-dimensional vector field $\tilde{\boldsymbol{U}}_{,z}$ over the water's surface, that must be specified in addition to the usual vector field $\tilde{\boldsymbol{U}}$ which specifies the current velocity itself. Then according to equation (74) the phase correction accumulates along a ray in proportion to half of the projection of $\tilde{\boldsymbol{U}}_{,z}$ onto the direction of the waves.

Appendix. General time dependence

Using Fourier superposition, the time-dependent wave $\hat{\zeta}(t,\boldsymbol{x})$ may be written to leading order as

$$\hat{\zeta}(t,\boldsymbol{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\zeta}_0(\omega,\boldsymbol{x})\exp^{(\mathrm{i}[\psi_0(\omega,\boldsymbol{x})-\omega t]/\epsilon)}\,\mathrm{d}\omega, \tag{A 1}$$

where the frequency dependence in $\bar{\zeta}_0, \psi_0$, which was suppressed in the previous notation, is explicitly displayed. By the method of stationary phase (Whitham 1974)

$$\hat{\zeta}(t,\boldsymbol{x}) \sim \bar{\zeta}_0(\bar{\omega},\boldsymbol{x})\exp^{(\mathrm{i}[\psi_0(\bar{\omega},\boldsymbol{x})-\bar{\omega}t]/\epsilon)}\left(\frac{\epsilon}{2\pi\mathrm{i}\psi_{0,\omega\omega}(\bar{\omega},\boldsymbol{x})}\right)^{1/2}, \tag{A 2}$$

where $\psi_{0,\omega}, \psi_{0,\omega\omega}$ are derivatives with respect to ω, and $\bar{\omega} = \bar{\omega}(t,\boldsymbol{x})$ satisfies

$$\psi_{0,\omega}(\bar{\omega}(t,\boldsymbol{x}),\boldsymbol{x}) = t. \tag{A 3}$$

Let

$$\theta_0(t,\boldsymbol{x}) = \psi_0(\bar{\omega}(t,\boldsymbol{x}),\boldsymbol{x}) - t\,\bar{\omega}(t,\boldsymbol{x}). \tag{A 4}$$

By differentiation of equation (A 3) we obtain

$$\psi_{0,\omega\omega}(\bar{\omega},\boldsymbol{x})\,\bar{\omega}_{,t} = 1. \tag{A 5}$$

By differentiation of (A 4) we obtain

$$\theta_{0,t} = -\bar{\omega}, \tag{A 6}$$

$$\tilde{\nabla}\theta_0 = \tilde{\nabla}\psi_0(\bar{\omega},\boldsymbol{x}). \tag{A 7}$$

Putting (A 6) and (A 7) into the surface eikonal equation (42) yields the time-varying surface eikonal equation

$$\frac{\partial}{\partial t}\theta_0 + \Omega_0(\boldsymbol{x},\tilde{\nabla}\theta_0) = 0. \tag{A 8}$$

Time differentiation of equation (A 8) and use of equation (A 6) yields

$$\bar{\varpi}_{,t} + \Omega_{0,\boldsymbol{k}}(\boldsymbol{x}, \tilde{\nabla}\theta_0)\tilde{\nabla}\bar{\varpi} = 0. \tag{A 9}$$

Equation (A 8) is almost the same as the equation for the physical phase, θ, which satisfies (3). As in the single-frequency case, the WKB phase θ_0 is the leading-order term in the expansion of the physical phase θ

$$\theta = \theta_0 + \epsilon\theta_1 + O(\epsilon^2), \tag{A 10}$$

where θ_1 satisfies

$$\theta_{1,t} + \Omega_{0,\boldsymbol{k}}(\boldsymbol{x}, \tilde{\nabla}\theta_0)\tilde{\nabla}\theta_1 + \tilde{\nabla}\theta_0 \cdot U_1 = 0. \tag{A 11}$$

From equations (A 2), (A 4) and (A 5) we obtain, to $O(\epsilon)$

$$\hat{\zeta}(t, \boldsymbol{x}) = \left(\frac{\epsilon}{2\pi \mathrm{i}}\right)^{1/2} \left\{\bar{\varpi}_{,t}(t, \boldsymbol{x})^{1/2}\bar{\zeta}_0(\bar{\varpi}(t, \boldsymbol{x}), \boldsymbol{x})\right\} \exp^{(\mathrm{i}\theta_0(t,\boldsymbol{x})/\epsilon)}. \tag{A 12}$$

Now implicit differentiation and use of equations (A 6), (A 7) (A 9), (42) and (67) yields that

$$\tilde{\nabla} \cdot \left\{ \frac{\left(\bar{\varpi}_{,t}^{1/2}\bar{\zeta}_0(\bar{\varpi}, \boldsymbol{x})\right)^2}{\bar{\Omega}(\tilde{\nabla}\theta_0)} \Omega_{0,\boldsymbol{k}}(\boldsymbol{x}, \tilde{\nabla}\theta_0) \right\}$$

$$= -\frac{\partial}{\partial t}\left\{ \frac{(\bar{\varpi}_{,t}^{1/2}\bar{\zeta}_0(\bar{\varpi}, \boldsymbol{x}))^2}{\bar{\Omega}(\tilde{\nabla}\theta_0)} \right\} \pm \mathrm{i}\left(\bar{\varpi}_{,t}^{1/2}\bar{\zeta}_0(\bar{\varpi}, \boldsymbol{x})\right)^2 \left\{ \frac{\tilde{\boldsymbol{U}}_{0,z} \cdot \tilde{\nabla}\theta_0}{|\tilde{\nabla}\theta_0|^{3/2}} - 2\frac{\tilde{\boldsymbol{U}}_1 \cdot \tilde{\nabla}\theta_0}{|\tilde{\nabla}\theta_0|^{1/2}} \right\}. \tag{A 13}$$

We may represent $\hat{\zeta}$ in terms of amplitude and phase by the complex polar form

$$\hat{\zeta} = A\mathrm{e}^{\mathrm{i}(\theta/\epsilon + \chi)} = A\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}(\theta_0/\epsilon + \chi)}, \tag{A 14}$$

where the amplitude A and the phase shift χ are real. Equating (A 14) and (A 12) yields an expression for $\bar{\varpi}_{,t}^{1/2}\bar{\zeta}_0(\bar{\varpi}, \boldsymbol{x})$, which is then substituted into (A 13), using (A 11). The resulting equation yields two real equations after separating real and imaginary parts. After replacing $\boldsymbol{U}_0 \to \boldsymbol{U}$, which is permissible to $O(\epsilon)$, the first of these equations is identical to equation (8), showing that wave action is conserved. The second of the real equations gives an equation for the phase shift

$$\frac{\partial}{\partial t}\chi + \Omega_{,\boldsymbol{k}}(\boldsymbol{x}, \tilde{\nabla}\theta) \cdot \tilde{\nabla}\chi = \frac{1}{2}\frac{\tilde{\nabla}\theta}{|\tilde{\nabla}\theta|} \cdot \tilde{\boldsymbol{U}}_{,z}. \tag{A 15}$$

Again, in deriving equation (A 15), we have replaced $\tilde{\boldsymbol{U}}_0 \to \tilde{\boldsymbol{U}}, \theta_0 \to \theta$, which does not change the leading-order term in the solution for χ.

Writing equation (A 15) as an equation along a ray results in an equation identical to that in the time-harmonic case, i.e. equation (74). Thus the phase correction vanishes in general for irrotational currents, by the same argument as in the single-frequency case.

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