Contents lists available at ScienceDirect

Continental Shelf Research

journal homepage: www.elsevier.com/locate/csr

Wave-induced setup of the mean surface over a sloping beach

Jan Erik H. Weber [a,*], Kai H. Christensen [b], Cléa Denamiel [c]

[a] Department of Geosciences, University of Oslo, 0315 Oslo, Norway
[b] Norwegian Meteorological Institute, 0313 Oslo, Norway
[c] Laboratoire Geosciences, University of Montpellier, Montpellier 2, France

ARTICLE INFO

Article history:
Received 1 August 2008
Received in revised form
10 February 2009
Accepted 23 March 2009
Available online 5 April 2009

Keywords:
Wave-induced setup
Sloping beach

ABSTRACT

A new theoretical approach for the wave-induced setup over a sloping beach is presented that takes into consideration the explicit variations of the surface waves due to bottom slope and viscosity. In this way, the wave forcing of the mean Lagrangian volume fluxes is calculated without assuming that the local depth is constant. The analysis is valid in the region outside the surf zone and is based on the shallow-water assumption. A novel approach for separating the viscous damping of the waves from the frictional damping of the mean flow is introduced, where the mean Eulerian velocity is applied in the bottom stress for the mean fluxes. In the case where the onshore Lagrangian mean transport is zero, a new formula is derived for the Eulerian mean free surface slope, in which the effects of bottom slope, viscous wave damping and frictional bottom drag on the mean flow are clearly identified. The analysis suggests that viscous damping of the waves and frictional dissipation of the Eulerian near-bed return flow could lead to setup outside the surf zone.

1. Introduction

The mean mass transport due to swell on a sloping beach has been studied extensively in the past. This problem has direct application to the near-shore transport of bottom sediments in suspension, and is therefore of great practical importance. From theoretical point of view, the inclusion of radiation stresses (Longuet-Higgins and Stewart, 1962) in the transport equations is crucial as a driving mechanism. With waves along the x_1-axis, the divergence of the radiation stress component S_{11} of Longuet-Higgins and Stewart appears as a forcing term for the mean flow. Dolata and Rosenthal (1984) claim that the radiation stress in the momentum equation for shallow water must be different from that derived by Longuet-Higgins and Stewart (1962). In their analysis, the value of S_{11} for deep water appears as forcing in shallow water although most studies (e.g., Lentz and Raubenheimer, 1999; Raubenheimer et al., 2001; Longuet-Higgins, 2005; Dean and Bender, 2006; Apotsos et al., 2007) apply Longuet-Higgins and Stewart's shallow-water version of S_{11}.

In this paper, we examine wave-induced transports by integrating the governing equations from the variable bottom to the undulating surface for a fluid of constant density. This is a traditional approach which yields the Lagrangian volume fluxes. The primary wave motion is obtained for a gently sloping bottom and a constant eddy viscosity by a two-scale analysis (e.g., Mei et al. (2005)). We here introduce a new approach for the wave field that enables us to separate the viscous boundary-layer part from the barotropic part, in order to obtain the viscous damping of the latter. We then derive explicitly the wave-induced forcing of the mean flow using wave solutions that are valid for a sloping bottom. This is not the conventional approach. In fact, most authors have applied Longuet-Higgins and Stewart's concept of radiation stress in the momentum flux balance using an expression for S_{11} which is actually based on wave solutions valid for constant depth. By a more rigorous approach, we determine the radiation stress more accurately, which enables us to check the controversial result of Dolata and Rosenthal (1984) for shallow water.

In the equations governing the mean transport we model frictional effects by a parameterized bottom drag based on the Eulerian mean velocity. This in contrast to previous studies (e.g., Longuet-Higgins (2005), Dean and Bender (2006)) who calculate this stress from the vertically varying streaming solution of Longuet-Higgins (1953). It is shown that the parameterization of the bottom stress has important consequences for the change of the mean sea level (setup/setdown).

This paper is organized as follows: in Section 2, we state the governing Eulerian equations and in Section 3, we discuss the effect of viscous bottom boundary layers on the linear wave dynamics. In Section 4, we present a two-scale approach and obtain solutions for the linear barotropic wave field in the presence of friction and a sloping bottom. In Sections 5 and 6,

* Corresponding author.
E-mail address: j.e.weber@geo.uio.no (J.E.H. Weber).

0278-4343/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.csr.2009.03.010

we calculate and discuss the non-linear wave-forcing terms for the vertically integrated Lagrangian mean momentum and in Section 7, we discuss the bottom stress acting on the mean flow. Section 8 contains an explicit formulation for the gradient of the mean surface level outside the surf zone. In this formulation, for known wave amplitude and wave number at the beginning of our domain, all quantities are functions only of the local depth or the local bottom slope. Section 9 contains a discussion of the results. Finally, a summary and some concluding remarks are found in Section 10.

2. Governing equations and method of solution

Consider shallow-water motion in a rotating ocean with constant density ρ. A Cartesian coordinate system is chosen such that the x and y axes are situated at the undisturbed sea surface, and the z-axis is positive upwards. The velocity components in these directions are (u,v,w), respectively. The bottom topography is generally given by $z=-H(x,y)$. We assume that the horizontal scale is large enough to justify the hydrostatic approximation, i.e., we take for the pressure p that

$$p=-\rho g(z-\zeta)+P_0. \tag{1}$$

Here $\zeta(x,y,t)$, where t is time, denotes the deviation of the free surface from the equilibrium position, g is the acceleration owing to gravity, and P_0 is the constant atmospheric surface pressure. For simplicity, we take that $H=H(x)$ in this analysis. Integrating the continuity equation in the vertical, and applying the appropriate boundary conditions at the surface and at the bottom, we obtain for the conservation of horizontal momentum and mass (e.g., Phillips, 1977)

$$\begin{aligned} &u_t+uu_x+vu_y+wu_z-fv=-g\zeta_x+\tau_z^{(x)},\\ &v_t+uv_x+vv_y+wv_z+fu=-g\zeta_y+\tau_z^{(y)},\\ &\zeta_t=-\left(\int_{-H(x)}^{\zeta}u\,dz\right)_x-\left(\int_{-H(x)}^{\zeta}v\,dz\right)_y, \end{aligned} \tag{2}$$

where subscripts denote partial differentiation. Here $(\tau^{(x)},\tau^{(y)})$ are the frictional stresses in the fluid, and f is the constant Coriolis parameter (we do not consider planetary flows here).

This problem is solved by a series expansion after the wave amplitude (or formally the wave steepness) as a small parameter. The first-order problem, proportional to the wave amplitude, yields the linear gravity wave solutions. This problem is non-trivial, since the waves propagate in a viscous ocean of variable depth. To second order in wave amplitude, we only consider the average volume fluxes and the mean change of the position of the sea level. Higher-order effects, proportional to the third or fourth power of the wave amplitude, are neglected in the present analysis. Hence our results are valid in the region outside the surf zone. Inside this zone the waves become very steep and finally break. We assume that breaking effectively inhibits reflection of wave momentum from the surf or swash zones.

3. Linear wave dynamics; the bottom boundary layer

We let the waves propagate in the x-direction, i.e., perpendicular to the shoreline. To include frictional effects in shallow-water waves in the simplest way, we consider a horizontal velocity that consists of two parts:

$$u=\tilde{u}(x,t)+\hat{u}(x,z,t), \tag{3}$$

i.e., one barotropic part $\tilde{u}$ which is independent of depth and a boundary-layer part $\hat{u}$ which is different from zero when the viscosity has a non-zero value. For plane waves along the x-axis, the continuity equation in (2) becomes

$$\zeta_t=-\frac{\partial}{\partial x}\left[(H+\zeta)\tilde{u}+\int_{-H}^{\zeta}\hat{u}\,dz\right]. \tag{4}$$

Linearizing (4), we obtain

$$\zeta_t=-(H\tilde{u})_x-\frac{\partial}{\partial x}\int_{-H}^{0}\hat{u}dz. \tag{5}$$

For waves of constant frequency ω, we may write $\zeta_t=-i\omega\zeta$ and $\tilde{u}_t=-i\omega\tilde{u}$, such that the barotropic pressure gradient in (2) can be written as

$$-g\zeta_x=-\frac{g}{\omega^2}(H\tilde{u})_{xxt}-B(\hat{u}), \tag{6}$$

where

$$B(\hat{u})=-\frac{ig}{\omega}\frac{\partial^2}{\partial x^2}\int_{-H}^{0}\hat{u}\,dz. \tag{7}$$

As discussed in the literature (Jenkins, 1989; Weber and Melsom, 1993; Ardhuin and Jenkins, 2006), one should use different parameterizations for the frictional effect on the waves and on the mean flow. This is because the timescale of the waves is so short that, e.g., eddies on the spatial scale of the waves have no influence on the oscillatory motion. The opposite is true for the mean flow, and in the depth-integrated equations governing the mean motion we will use a bottom drag formulation commonly used for shallow-water flows. For the periodic motion, however, we assume a constant eddy viscosity v. Furthermore, we assume that our surface gravity waves have much higher frequency than the inertial frequency f. Hence, we can neglect the effect of the earth's rotation on the wave field (but not on the much slower wave drift). The linearized horizontal momentum equation in (2) then becomes

$$\tilde{u}_t+\hat{u}_t-v(\tilde{u}_{xx}+\hat{u}_{zz})=-\frac{g}{\omega^2}(H\tilde{u})_{xxt}-B(\hat{u}), \tag{8}$$

where we have utilized that $|\hat{u}_{xx}|\ll|\hat{u}_{zz}|$ in the viscous boundary layer. Since the barotropic terms in (8) are linearly independent of the z-dependent terms, the equation governing the boundary-layer motion is thus

$$\hat{u}_t-v\hat{u}_{zz}=0. \tag{9}$$

At the bottom we require a no-slip condition. Hence, $\hat{u}=-\tilde{u}$ at $z=-H$. Furthermore, we require that $\hat{u}$ is confined to the bottom boundary layer, i.e., we neglect the much weaker viscous effects at the surface. Taking that $\hat{u}=g(z)\tilde{u}(x,t)$, the solution of (9) becomes

$$\hat{u}=-\tilde{u}\exp[-(1-i)\gamma(z+H)]. \tag{10}$$

Here $\gamma^{-1}=\delta=(2v/\omega)^{1/2}$ is the thickness of the bottom boundary layer, e.g., Longuet-Higgins (1953). The present analysis rests on the assumption that the boundary-layer thickness is much smaller than the local water depth, i.e., $\delta/H\ll 1$. We may now evaluate the contribution from the boundary-layer solution in the equation governing the barotropic part of the motion. By vertical integration of (10), we find from the definition (7) that

$$B(\hat{u})=(1-i)r\tilde{u}, \tag{11}$$

where

$$r=\frac{\omega}{2\gamma H}=\frac{(\omega v)^{1/2}}{2^{1/2}H}. \tag{12}$$

We realize that $r/\omega=\delta/(2H)$ is a small parameter. This will be utilized in the forthcoming calculations. Defining a local wave number k, we have that $v\tilde{u}_{xx}\approx -vk^2\tilde{u}$. Because $(vk^2)/r=O(kH\,k\delta)\ll 1$, we neglect the barotropic viscous term on the left of (8). With these modifications in mind, we obtain, by subtracting (9) from (8), an equation for the barotropic part of the linear wave field

$$\tilde{u}_t + (1-i)r\tilde{u} = -g\tilde{\zeta}_x. \tag{13}$$

Here we have defined a "non-viscous" surface elevation $\tilde{\zeta}$ by

$$\tilde{\zeta}_t = -(H\tilde{u})_x \tag{14}$$

and utilized that $\tilde{\zeta}_{tt} = -\omega^2\tilde{\zeta}$. We realize that in this novel approach the barotropic part of the wave field attenuates according to a linear friction law or a so-called Rayleigh friction. However, the linear friction coefficient r is related to the eddy viscosity, the frequency, and the depth through (12). In this problem $|dr/dx|$ is a very small quantity, and will be neglected in the following analysis, i.e., we take r to be constant.

Finally, we are applying our results to the case with a rough sea bottom. For a rippled bed, the eddy viscosity, causing the dissipation of the wave field, will depend on the wave motion. From Longuet-Higgins (2005) we obtain that

$$\nu = K^2 \frac{ga^2}{2H\omega}, \tag{15}$$

where a is a typical wave amplitude and $K = 0.16$. Accordingly, our friction parameter r in (12) for the damping of linear waves becomes

$$r = K\frac{g^{1/2}a}{2H^{3/2}}. \tag{16}$$

4. Linear wave dynamics; the barotropic wave field

We assume for the surface wave that

$$\tilde{\zeta} = F(x)\exp(-i\omega t). \tag{17}$$

In this problem, we consider wave amplitudes that change in space. Accordingly, ω is real. Utilizing that $r/\omega \ll 1$, we obtain from (13) and (14) that

$$\frac{d}{dx}\left(gH\frac{dF}{dx}\right) + (\omega^2 + i\omega r)F = 0. \tag{18}$$

The part of the first term in (18), proportional to dH/dx, contributes to the dependence of the wave amplitude with depth, while the last term, proportional to $i\omega r$, contributes to the spatial change of wave amplitude owing to the effect of viscosity. The corresponding horizontal wave velocities can be written to this approximation

$$\tilde{u} = -g\left(\frac{i\omega + r}{\omega^2}\right)\frac{dF}{dx}\exp(-i\omega t),$$
$$\tilde{v} = 0. \tag{19}$$

We take that the ocean bottom varies gently. That means that $|dH/dx| \ll 1$ in this problem. Accordingly, we can neglect any wave reflection from the slope in this case. Furthermore, we assume that the effect of friction is small. More precisely, we take that the frictional decay of the waves occurs on a length scale L that is large compared to the wavelength λ, i.e., $\lambda/L \ll 1$. By inspecting (18), we see that this is equivalent to require $r/\omega \ll 1$. But since $r/\omega = \delta/(2H)$ from (12), we realize that $\lambda/L \ll 1$ is fulfilled when the viscous boundary layer is much thinner than the local depth.

Following Mei et al. (2005), we apply a two-scale approach in solving (18), where we assume a rapidly varying wave phase, and a slowly varying wave amplitude A. If the typical dimensionless scale of the bottom slope is μ, we define a slow variable ξ by

$$\xi = \mu x. \tag{20}$$

We then write

$$F = A(\xi)\exp(iS(\xi)/\mu), \tag{21}$$

where S is the phase function. The wave number k in the x-direction is then given by

$$k(\xi) = \frac{1}{\mu}\frac{dS}{dx} = \frac{dS}{d\xi}. \tag{22}$$

Inserting (21) into (18), carrying out the differentiations, and neglecting small terms of $O(\mu^2, \mu r/\omega, r^2/\omega^2)$, the real part of (18) yields to the lowest order

$$\omega = (gH)^{1/2}k. \tag{23}$$

Furthermore, multiplying the imaginary part of (18) by the amplitude A, we find

$$\frac{dA^2}{dx} = \left(-\frac{d(\ln(kH))}{dx} - \frac{kr}{\omega}\right)A^2. \tag{24}$$

In the term that involves the product of k and the small parameter r/ω, we can assume that k is constant. Hence, the solution becomes

$$A = A_0\left(\frac{k_0H_0}{kH}\right)^{1/2}\exp\left(-\frac{r}{2C}x\right). \tag{25}$$

Here the sub-zeros denote the values at the start of our domain $x = 0$ and $C = (gH)^{1/2}$ is the local phase speed.

We assume that the wave number is constant in time. From the kinematical conservation equation for the density of waves (Whitham, 1962), it follows that the frequency must be constant in space. Since $\omega = \omega(k,H)$, we can write

$$\frac{d\omega}{dx} = \frac{\partial\omega}{\partial k}\frac{dk}{dx} + \frac{\partial\omega}{\partial H}\frac{dH}{dx} = 0. \tag{26}$$

Utilizing (23), we readily obtain (e.g., Longuet-Higgins and Stewart, 1962)

$$k^2 = \frac{H_0}{H}k_0^2. \tag{27}$$

We define the part of our wave amplitude that depends on the bottom topography as

$$a = A_0\left(\frac{k_0H_0}{kH}\right)^{1/2}. \tag{28}$$

If we introduce a new phase function φ by

$$\varphi = \frac{1}{\mu}\int k\,d\xi - \omega t, \tag{29}$$

the real parts of $\tilde{\zeta}$ and $\tilde{u}$ can be written to sufficient accuracy as

$$\tilde{\zeta} = a\exp\left(-\frac{r}{2C}x\right)\cos\varphi,$$
$$\tilde{u} = \frac{g}{\omega}\exp\left(-\frac{r}{2C}x\right)\left(ka\cos\varphi + \left(\mu\frac{da}{d\xi} + \frac{r}{2C}a\right)\sin\varphi\right). \tag{30}$$

Dolata and Rosenthal (1984) neglect the effect of viscosity on the linear wave field in their analysis of wave setup, and introduce spatial damping in their analysis through an empirical damping coefficient in the expression for the wave energy density. This friction coefficient is not related to the eddy viscosity as in our case, e.g., (12).

5. Non-linear analysis; the volume flux equations

We integrate the momentum balance in (2) from the bottom to the undulating surface. Averaging over one wave period, we define the Lagrangian mean volume fluxes (e.g., Phillips, 1977)

$$U^L = \overline{\int_{-H}^{\zeta} u dz}, \quad V^L = \overline{\int_{-H}^{\zeta} v dz}, \tag{31}$$

where the over-bar denotes the averaging process. Applying the appropriate kinematic boundary conditions at the free surface and the bottom, we obtain from (2) for the wave-induced Lagrangian volume fluxes in the absence of surface forcing

$$\begin{aligned} U_t^L - fV^L &= -\overline{\left(\int_{-H}^{\zeta} uu\,dz\right)_x} - \overline{\left(\int_{-H}^{\zeta} vu\,dz\right)_y} - g\overline{(\zeta+H)\zeta_x} - \tau_B^{(x)}/\rho, \\ V_t^L + fU^L &= -\overline{\left(\int_{-H}^{\zeta} uv\,dz\right)_x} - \overline{\left(\int_{-H}^{\zeta} vv\,dz\right)_y} - g\overline{(\zeta+H)\zeta_y} - \tau_B^{(y)}/\rho, \\ \bar{\zeta}_t &= -U_x^L - V_y^L. \end{aligned} \tag{32}$$

Here $(\tau_B{}^{(x)},\tau_B{}^{(y)})$ are the frictional bottom stress components. Neglecting the small contribution from the viscous bottom layers to the integrals on the right-hand side of (32), we obtain to second order in wave amplitude

$$\begin{aligned} U_t^L - fV^L + gHh_x &= \tau_w^{(x)}/\rho - \tau_B^{(x)}/\rho, \\ V_t^L + fU^L &= -\tau_B^{(y)}/\rho, \\ h_t &= -U_x^L, \end{aligned} \tag{33}$$

where we have defined the Eulerian mean sea level $h \equiv \bar{\zeta}$. The wave-induced part of the forcing $\tau_w{}^{(x)}$ has been defined as

$$\tau_w^{(x)}/\rho = -\left(H\overline{\tilde{u}^2} + \frac{1}{2} g\overline{\zeta^2}\right)_x. \tag{34}$$

The equations in (33) are the conventional surge equations, here forced by waves along the x-axis.

It is a simple exercise to show that for constant depth, the right-hand sides of (33) are zero when we neglect the effect of friction. For horizontally uniform conditions, we thus reproduce Hasselmann's (1970) result for an inviscid fluid that individual particles (here vertical fluid columns) in a wave field on average move in closed inertial circles. Accordingly, there is no net mass transport when averaged over the inertial period. When friction and bottom slope are taken into account, the forcing terms are generally non-zero and we may have a wave-induced mean drift.

6. Wave forcing

Inserting from (30) into (34), we obtain to $O(a^2)$

$$\begin{aligned} \tau_w^{(x)}/\rho = &-\frac{d}{dx}\left(\frac{3}{4} ga^2 \exp\left(-\frac{r}{C}x\right)\right) \\ &-\frac{1}{2} ga^2 \exp\left(-\frac{r}{C}x\right)\left(\frac{1}{H}\frac{dH}{dx} + \frac{1}{k^2}\frac{dk^2}{dx}\right). \end{aligned} \tag{35}$$

Since the frequency here is constant (e.g., (27)), we realize that the last term on the right-hand side is identically zero. For surface gravity waves, the wave energy density E can be written in the present notation as

$$E = \frac{1}{2}\rho g a^2 \exp\left(-\frac{r}{C}x\right). \tag{36}$$

The radiation stress component S_{11} of Longuet-Higgins and Stewart (1962) for shallow-water waves is

$$S_{11} = \frac{3}{2}E = \frac{3}{4}\rho g a^2 \exp\left(-\frac{r}{C}x\right). \tag{37}$$

We note that the divergence of this quantity is just the non-zero term on the right-hand side of the wave forcing (35). Longuet-Higgins and Stewart's result that $-\partial(S_{11}/\rho)/\partial x$ constitutes the wave forcing of the volume fluxes was derived using wave solutions valid locally for constant depth. Here we have derived this result more rigorously by applying wave solutions that are valid for a sloping bottom.

By expressing (35) in terms of the wave energy density, (33) can be written as

$$\begin{aligned} U_t^L - fV^L + gHh_x &= -\frac{3}{2\rho}E_x - \frac{\tau_B^{(x)}}{\rho}, \\ V_t^L + fU^L &= -\frac{\tau_B^{(y)}}{\rho}, \\ h_t + U_x^L &= 0. \end{aligned} \tag{38}$$

7. The bottom stress

In any vertically integrated approach, the assessment of the principally unknown bottom stress $(\tau_B^{(x)},\tau_B^{(y)})$ in (38) poses a problem. For the setup in the non-rotating case, Longuet-Higgins (2005) calculated the bottom stress from his vertically varying streaming solution (Longuet-Higgins, 1953). The same approach was used by Dean and Bender (2006). This yields an on-shore-directed bottom stress when the waves propagate towards the shore.

However, the steady viscosity-modified wave-drift considered by Longuet-Higgins (1953) is, strictly speaking, valid only for laminar flow above a smooth bottom. The typical diffusion time t_s to establish a steady value in the fluid layer is $t_s \sim H^2/\nu_m$, where ν_m is the molecular viscosity ($\nu_m \approx 10^{-6}\,\mathrm{m^2\,s^{-1}}$). Depending on the fluid depth, t_s ranges from weeks to many months. Therefore, the stress associated with Longuet-Higgins' solution cannot be applied in this problem. More principally, even by assuming that the wave eddy viscosity could be applied to Longuet-Higgins' streaming solution, this would imply that the redistribution of mean momentum in turbulent flows was related to the viscous damping of the wave field, which has been opposed from a physical point of view (Jenkins, 1989; Weber and Melsom, 1993; Ardhuin and Jenkins, 2006). Since the Stokes drift has negligible shear in shallow water, we assume that the mean stresses on the water column depend on the shear of the mean Eulerian flow. This in principle is unknown. According to the usual procedure in ocean modelling, we take that these stresses are given by the square of the mean Eulerian velocity with a drag coefficient that depends on the bottom conditions.

In the present problem, the Lagrangian mean transport can be written as a sum of an Eulerian flux (U^E,V^E) plus a Stokes transport (U^S,V^S)

$$\begin{aligned} U^L &= U^E + U^S, \\ V^L &= V^E + V^S. \end{aligned} \tag{39}$$

Here, for shallow-water waves along the x-axis the Stokes transport (Stokes, 1847) is

$$U^S = \frac{\omega a^2}{2kH}\exp\left(-\frac{r}{C}x\right), \quad V^S = 0. \tag{40}$$

According to the discussion above, we parameterize the bottom stresses as

$$\tau_B^{(x)} = \rho c_B W U^E/H^2, \quad \tau_B^{(y)} = \rho c_B W V^E/H^2, \tag{41}$$

where $W = ((U^E)^2 + (V^E)^2)^{1/2}$ and c_B is the bottom drag coefficient. This coefficient depends on the bottom conditions. Very close to the bottom, the mean horizontal stresses are partly used to accelerate sediment particles that are kept in suspension by the oscillating wave motion. This part of the mean stress is not felt by the water column just above the rippled bed, and the effect of sediment transport must be reflected in the value of bottom drag coefficient. For a corrugated bed an appropriate value appears to be $c_B = 0.1$ (Longuet-Higgins, 2005). However, values of c_B (and ν) reported in the literature varies by more than one order of magnitude. For a more thorough discussion of friction coefficients and eddy viscosities relevant for this problem we refer to Apotsos et al. (2007) and references therein.

Utilizing that the Stokes flux is independent of time, the horizontal flux equation reduces in the non-rotating case ($V^E = 0$) to

$$U_t^E = -gHh_x - \frac{3}{2\rho}E_x - \frac{c_B}{H^2}\left|U^E\right|U^E. \tag{42}$$

This is the traditional surge equation for flow in a channel, except that the wind stress is replaced by the divergence of the wave-induced radiation stress. This result lends support to our formulation (41) of the bottom stress in terms of the Eulerian mean velocity.

In Dean and Bender (2006), the mean stress due to vegetation elements is accounted for by introducing a stress term proportional to $-\int_{-H}^{\zeta}\tilde{u}|\tilde{u}|dz$ in our notation, where the linear, non-damped solution for $\tilde{u}$ is applied. To $O(a^2)$ in wave amplitude, this expression is zero, and Dean and Bender conclude that linear waves do not induce any setup. This is clearly wrong, since the damping coefficient (12) of linear waves must be a part of E as shown here (see (36)), and hence affect the setup. This is seen straight away from the steady part of (42).

8. Wave-induced setup

The solutions to the homogenous part of (38) yield the transient part of this problem. It consists of long, damped surface waves in a rotating ocean of variable depth. This fact is important for the numerical solution of (38). However, in our simple study we shall not discuss the early, time-dependent development of the solutions. We focus on the steady, adjusted state, so we take the time derivative in (38) to be zero. Hence, for momentum

$$gH\frac{dh}{dx} = -\frac{3}{2\rho}\frac{dE}{dx} - \frac{\tau_B^{(x)}}{\rho} + fV^L,$$
$$fU^L = -\frac{\tau_B^{(y)}}{\rho}, \tag{43}$$

where $(\tau_B{}^{(x)}, \tau_B{}^{(y)})$ is given by (41).

The steady solution depends on the lateral boundary conditions. This is seen from the continuity equation in (38), which yields

$$\frac{dU^L}{dx} = 0. \tag{44}$$

Dolata and Rosenthal (1984) point out from (44), by using (39), that a spatially decaying wave field acts as a source for the divergence of the Eulerian fluxes.

In our case, the slope extends to the beach but we must end our domain at the beginning of the surf zone, where our regular waves cease to exist. Inside the surf zone, the waves finally break and disappear according to our assumption. From a mass transport point of view, we must have $U^L = 0$ just outside this zone. Since we have a Stokes flux U^S in the x-direction given by (40), the result $U^L = 0$ implies that we have a compensating Eulerian flux $U^E = -U^S$ outside the surf zone (e.g., Dolata and Rosenthal, 1984). It now follows from (43) that $V^E = 0$ and hence $V^L = 0$. Then (43) reduces to

$$gH\frac{dh}{dx} = -\frac{3}{2\rho}\frac{dE}{dx} + \frac{c_B(U^S)^2}{H^2}. \tag{45}$$

As mentioned earlier, for steady flow in the absence of rotation Longuet-Higgins (2005) calculated the bottom stress in the momentum balance (45) from the viscous no-slip streaming velocity (Longuet-Higgins, 1953). This would yield an opposite sign in the last term of (45), implying that frictional effects on the mean flow should promote a setdown, instead of a setup. As recalled, we have in Section 7 discarded this bottom stress as unrealistic.

Inserting our previous results for the wave number variation (27) and the wave amplitude variation (28) with depth, (45) finally reduces to

$$\frac{dh}{dx} = \frac{3H_0^{1/2}A_0^2}{8H^{5/2}}\left[\frac{dH}{dx} + \frac{2Hr}{C} + \frac{2c_BH_0^{1/2}A_0^2}{3H^{5/2}}\exp\left(-\frac{r}{C}x\right)\right]\exp\left(-\frac{r}{C}x\right), \tag{46}$$

where again $C = (gH)^{1/2}$.

9. Discussion of results

Since the steady mean water level associated with non-linear water waves is not uniquely determined, but has an arbitrariness of $O(a^2)$, (e.g., Whitham, 1962), we shall define wave setup by the sign of the surface slope. By this definition, we realize from (46) that the question of whether we have a setdown ($dh/dx<0$) or a setup ($dh/dx>0$) is not easily answered. We note that an up-sloping beach ($dH/dx<0$) always favours a wave setdown (e.g., Longuet-Higgins and Stewart, 1962). Since $r>0$, the effect of frictional damping of the wave field itself favours a wave setup. From a different approach, this was shown by Dolata and Rosenthal (1984). We also note that the bottom drag on the mean flow (last term on the right-hand side) promote a setup. Utilizing (16) for the wave damping, we find that in (46) $2Hr/C = Ka/H$ which increases in shallow water. Accordingly, the frictional damping of the waves, promoting a setup, will be increasingly important when the mean depth becomes smaller.

From the observations by Herbers et al. (2000), their station B on the shelf of North Carolina, we find a typical mean gradient of the sea bed $|dH/dx| \sim 1.3\times10^{-3}$. With a mean depth of 20 m, a typical swell amplitude $a \sim 0.65$ m, and a wave frequency $\omega \sim 0.44\,\mathrm{s}^{-1}$, the wave damping term in (46) becomes $2Hr/C \sim 5\times10^{-3}$, while for the bottom friction acting on the mean flow we find $2c_Ba^2/3H^2 \sim 7\times10^{-5}$. Hence, in this particular case, the strong damping of the swell overrides the effect of bottom slope and causes a setup of the mean water level.

It should be noted that in the present example, we have $kH \approx 0.63$, which only approximately fulfils the hydrostatic approximation. This is due to the lack of reliable field measurements satisfying $kH \ll 1$. Since our calculations are based on the hydrostatic assumption, the numbers derived from the observations by Herbers et al. (2000) should only be regarded in a qualitative sense. Returning to our previous assumptions, the eddy viscosity (15) related to corrugations on the sea bed is quite high in this example ($6\times10^{-3}\,\mathrm{m^2\,s^{-1}}$). Although the result that wave damping here leads to setup is basically owing to the large eddy viscosity value, the effects of wave damping increase with decreasing depth, and similar waves in shallower water would lead to setup for smaller values of the viscosity.

Field observations in the Mediterranean sea (Gulf of Lions) are reported in Denamiel (2006). In the south-west of this region (the Têt inner shelf) sea-bed slopes are about 0.01. On 6th March 2004 there is virtually no wind, and the swell is propagating normal to the isobaths. We compare wave measurements at two stations (SODAT, SOPAT), which are separated by a cross-isobath distance of about 1520 m. The local depths at the two stations are 11 and 28 m, respectively. The typical wave amplitude is 0.5 m, the wave number is $0.15\,\mathrm{m}^{-1}$, and the wave period is about 5.2 s. Hence these waves are deep water waves ($kH = 2.9$ at mean depth). From observed swell amplitude decay between the two stations, we estimate a spatial attenuation coefficient $\alpha \sim 4 \times 10^{-5}\,\mathrm{m}^{-1}$. In deep water, the spatial decay coefficient is related to the bulk eddy viscosity by $\nu = \omega\alpha/(4k^3)$ (Jenkins, 1986), which here yields $\nu = 3.5 \times 10^{-3}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. This underestimates the eddy viscosity in shallow water, but we use it in our calculation of the amplitude (25) as the waves propagate into shallow water. When $H = 3$ m, (27) and (25) yield that $k = 0.29\,\mathrm{m}^{-1}$ and $A = 0.6$ m in this case. From Longuet-Higgins formula (15) we then obtain for the eddy viscosity in shallow water $\nu = 1.2 \times 10^{-2}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. Finally, the dimensionless wave damping parameter in (46) then becomes $2Hr/C \sim 4.2 \times 10^{-2}$. Since here typically $|dH/dx| \sim 10^{-2}$, we realize that the effect of the bottom slope in the Gulf of Lions may override the wave damping and promote a set-down.

In both the examples discussed above, the viscous boundary-layer thickness δ in (10) is much smaller than the fluid depth. We find that $\delta/H \sim 0.01$ for the first case and $\delta/H \sim 0.05$ for the second case, which renders our bottom boundary-layer considerations in Section 3 valid. The real problem with waves in shoaling water is, as pointed out already, the lack of observations outside the surf zone where locally $kH \ll 1$.

10. Summary and concluding remarks

For shallow-water waves over a gently sloping bottom, the wave amplitude and the wave number have been determined as slowly varying functions of the coordinate in the direction of wave propagation (here toward the shore). We have separated the slow variation due to depth changes from the slow spatial variation due to viscous dissipation in the calculation of the linear wave field. In that way, for given wave amplitude, wave number and depth at the start of our domain, all variables can be expressed in terms of the local water depth.

From these results, we have computed the radiation stress component for a sloping bottom that forces the mean Lagrangian fluxes. We confirm Longuet-Higgins and Stewart's (1962) result that $-\partial(S_{11}/\rho)\partial x$ constitutes the wave forcing, where S_{11} takes the shallow-water value (37). An objection to this wave forcing in shallow water is found in Dolata and Rosenthal (1984). They argue that in the analysis of momentum transport in shallow water one must use the form of the radiation stress for deep water. However, as shown here, Longuet-Higgins and Stewart's result for shallow water is indeed correct.

In the present investigation, we have introduced a novel separation of the viscous effect on the wave field from the frictional effect on the mean fluxes. This means that the redistribution of mean momentum is allowed to occur at a different rate than the damping rate of the waves (Jenkins, 1989; Weber and Melsom, 1993; Ardhuin and Jenkins , 2006). In this way, we find that both the wave dissipation and the bottom drag on the mean flow promote a wave setup.

The next natural step in this type of investigation is to solve numerically the surge Eq. (38) for obliquely propagating swell, with the additional along-shore wave-forcing terms. We leave this task for future research.

Acknowledgements

The authors would like to thank Dr. F. Bouchette for his help with data from the Gulf of Lions. K. H. Christensen gratefully acknowledges financial support from the Research Council of Norway through Grant no. 171215.

References

Apotsos, A., Raubenheimer, B., Elgar, S., Guza, R.T., Smith, J.A., 2007. Effects of wave rollers and bottom stress on wave setup. Journal of Geophysical Research 112, 2003–2013.
Ardhuin, F., Jenkins, A.D., 2006. On the interaction of surface waves and upper ocean turbulence. Journal of Physical Oceanography 36, 551–557.
Dean, R.G., Bender, C.J., 2006. Static wave setup with emphasis on damping effects by vegetation and bottom friction. Coastal Engineering 53, 149–156.
Denamiel, C., 2006. Modélisation hydrodynamique 3D en zone pré-littorale: charactérisation des effects des houles de tempête sur la circulation océanique. PhD-Thesis, University of Montpellier II, France, 424 pp.
Dolata, L.F., Rosenthal, W., 1984. Wave setup and wave-induced currents in coastal zones. Journal of Geophysical Research 89, 1973–1982.
Hasselmann, K., 1970. Wave-driven inertial oscillations. Geophysical Fluid Dynamics 1, 463–502.
Herbers, T.H.C., Hendrickson, E.J., O'Reilly, W.C., 2000. Propagation of swell across a wide continental shelf. Journal of Geophysical Research 105, 19,729–19,737.
Jenkins, A.D., 1986. A theory for steady and variable wind- and wave-induced currents. Journal of Physical Oceanography 16, 1370–1377.
Jenkins, A.D., 1989. The use of a wave prediction model for driving a near-surface current model. Deutsche Hydrografische Zeitschrift 43, 133–149.
Lentz, S., Raubenheimer, B., 1999. Field observations of wave setup. Journal of Geophysical Research 104, 25869–25875.
Longuet-Higgins, M.S., 1953. Mass transport in water waves. Philosophical Transactions of the Royal Society of London A 245, 535–581.
Longuet-Higgins, M.S., 2005. On wave setup in shoaling water with a rough sea bed. Journal of Fluid Mechanics 527, 217–235.
Longuet-Higgins, M.S., Stewart, R.W., 1962. Radiation stress and mass transport in gravity waves, with application to "surf beats". Journal of Fluid Mechanics 13, 485–502.
Mei, C.C., Stiassnie, M., Yue, D.K.-P., 2005. Theory and Applications of Ocean Surface Waves. Part 1: Linear aspects. Advanced Series on Ocean Engineering 23. World Scientific Publishing Co., Singapore.
Phillips, O.M., 1977. The Dynamics of the Upper Ocean, 2d ed. Cambridge University Press, Cambridge.
Raubenheimer, B., Guza, R.T., Elgar, S., 2001. Field observation of wave-driven setdown and setup. Journal of Geophysical Research 106, 4629–4638.
Stokes, G.G., 1847. On the theory of oscillatory waves. Transactions of Cambridge Philosophical Society 8, 441–455.
Weber, J.E., Melsom, A., 1993. Transient ocean currents induced by wind and growing waves. Journal of Physical Oceanography 23, 193–206.
Whitham, G.B., 1962. Mass momentum and energy flux in water waves. Journal of Fluid Mechanics 12, 135–147.