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Non-linear transfers between sea waves

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Abstract- Hasselmann's equation, describing the non-linear interactions between sea waves, is studied numerically for a Pierson–Moskowitz spectrum The apparent creation of order, due to the non-linear enhancement of the peak of the spectrum, occurs as the by-product of a large amount of disorder, created at high wavenumbers

It is also found that the physical process dominating the non-linear transfer is the scattering of waves near the peak of the spectrum by short waves in the tail of the spectrum In this process the long waves become longer and the short waves shorter

INTRODUCTION

OBSERVATIONS of sea waves have always shown that the sea wave spectrum has its main peak just before a low frequency or low wavenumber cutoff. Recently BARNETT and WILKERSON (1967), HASSELMANN *et al.* (1973) in the JONSWAP (Joint North Sea Wave Project) experiments, and others, have shown that the peak is usually higher and narrower than had been thought previously. In fact, the peak was commonly two or three times higher than expected.

The JONSWAP experiments were also concerned with the evolution of the peak with time, in particular, the gradual movement of the low wavenumber cutoff to even lower wavenumbers. The work indicated that both the enhanced peak and the movement of the low wavenumber cutoff could be explained, at least qualitatively, in terms of the non-linear interactions between the sea waves as modeled by Hasselmann's equation (HASSELMANN, 1962, 1963a, 1963b)

The full set of equations describing the non-linear interactions between sea waves is too complicated to be of any practical use in describing the evolution of a sea state. Instead one has to make approximations, and the usual approach is to assume that the amplitude of a wave changes little in one period and that the higher order correlations between waves can be neglected For water waves, one of the simplest outcomes of this approach is Hasselmann's equation

Hasselmann's equation estimates the rate of change of the mean energy (or mean action density) at each wavenumber. The mean is defined as an average over an ensemble of representative oceans

Because when we use Hasselmann's equation we are not keeping a detailed record of the amplitude and phase of each wavenumber, the information we have about the ocean may become degraded with time As a result the entropy (or disorder) of the solution is either constant or increases with time.

But here we hit a snag With Hasselmann's equation disorder should increase with time,

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but when it is applied to a realistic sea wave spectrum, the non-linearities produce a high narrow peak—so apparently they are producing order Put in the terms of the JONSWAP paper, the effect of the non-linear transfer "runs counter to intuition"

The present paper arises from some work carried out to get a better understanding of the properties of Hasselmann's equation and in particular to understand how the highly peaked spectrum is produced. As one might expect, conservation of energy and momentum are important Equally important are the rapid increase with wavenumber in the strength of the non-linearities, the high wavenumber tail of the sea wave spectrum, and the extra phase space associated with high wavenumbers.

It is found that for a realistic sea wave spectrum, the non-linearities do cause a localised increase in order near the low wavenumber peak of the spectrum This, however, is the by-product of a great increase in disorder at high wavenumbers.

This creation of localised order, at the expense of disorder elsewhere, is not unique to sea waves Similar behaviour is also found in many of the other systems of classical thermodynamics, from refrigerators to biological cells, in which there is a flow of energy through the system.

HASSELMANN'S EQUATION

If n_i is the action density at wavenumber $\mathbf{k}_i$, then Hasselmann's equation may be written as

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = \int \mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3\,C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)$$

$$\delta(\omega_1+\omega_2-\omega_3-\omega_4)\left[n_1 n_3(n_4-n_2)+n_2 n_4(n_3-n_1)\right] \qquad (2.1)$$

ω_i is the angular velocity at wavenumber $\mathbf{k}_i$ and $C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ is the interaction coefficient (Appendix 1) In this paper we shall use the action density form of Hasselmann's equation, partly because it helps to simplify the equation. However, for weakly interacting waves action density has the useful property of behaving as particle density does in systems of colliding particles As a result action density is useful for getting a physical picture of the processes involved.*

Hasselmann's equation conserves action density. To be physically realistic it should also conserve energy and momentum. In fact the wavenumber delta function is there to ensure momentum conservation and the angular velocity delta function to ensure energy conservation (HASSELMANN, 1963a)

A further requirement of the equation is that the action density should never become negative (KRAICHNAN, 1961; ELSASSER, 1973). This is true for Hasselmann's equation and like the conservation of action density it arises from the adiabatic assumption

A useful consequence of the conservation of action, is that, if one considers just the interaction of four waves $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and $\mathbf{k}_4$, satisfying the delta functions, then,

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = \frac{\mathrm{d}n_2}{\mathrm{d}t} = -\frac{\mathrm{d}n_3}{\mathrm{d}t} = -\frac{\mathrm{d}n_4}{\mathrm{d}t}$$

As a result one may take Hasselmann's equation as representing the scattering of wave-

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* Although action density is a little understood quantity, for periodic systems it is often the most convenient to use This arises because action is an adiabatic invariant of periodic systems (LANDAU and LIFSHITZ, 1960) It is interesting to note that in quantum mechanics conservation of action density corresponds to conversation of the number of particles

numbers $\mathbf{k}_3$ and $\mathbf{k}_4$ to give $\mathbf{k}_1$ and $\mathbf{k}_2$—plus the reverse process. This analogy also shows up in the symmetry of the interaction coefficient Thus $C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ is invariant under interchanges of $\mathbf{k}_1$ with $\mathbf{k}_2$ or of $\mathbf{k}_3$ with $\mathbf{k}_4$, or of both $\mathbf{k}_1$ and $\mathbf{k}_2$ with $\mathbf{k}_3$ and $\mathbf{k}_4$

Unfortunately the detailed behaviour of the interaction coefficient is rather complicated However, its most important property is that it is a rapidly increasing function of wavenumber Thus for deep water waves, for which ω is proportional to $k^{\frac{1}{2}}$,

$$C(\alpha\mathbf{k}_1,\alpha\mathbf{k}_2,\alpha\mathbf{k}_3,\alpha\mathbf{k}_4) = \alpha^6 C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4).$$

This rapid increase in the coupling coefficient with wavenumber reflects the much more non-linear nature of short waves

THE TRANSFER FUNCTION

Figure 1 shows the calculated non-linear transfer function dn/dt for the Pierson–Moskowitz spectrum of Fig. 2. The results of the JONSWAP experiment indicate that the Pierson–Moskowitz spectrum may be typical of a sea in which the enhanced peak due to the non-linearities has not been developed.

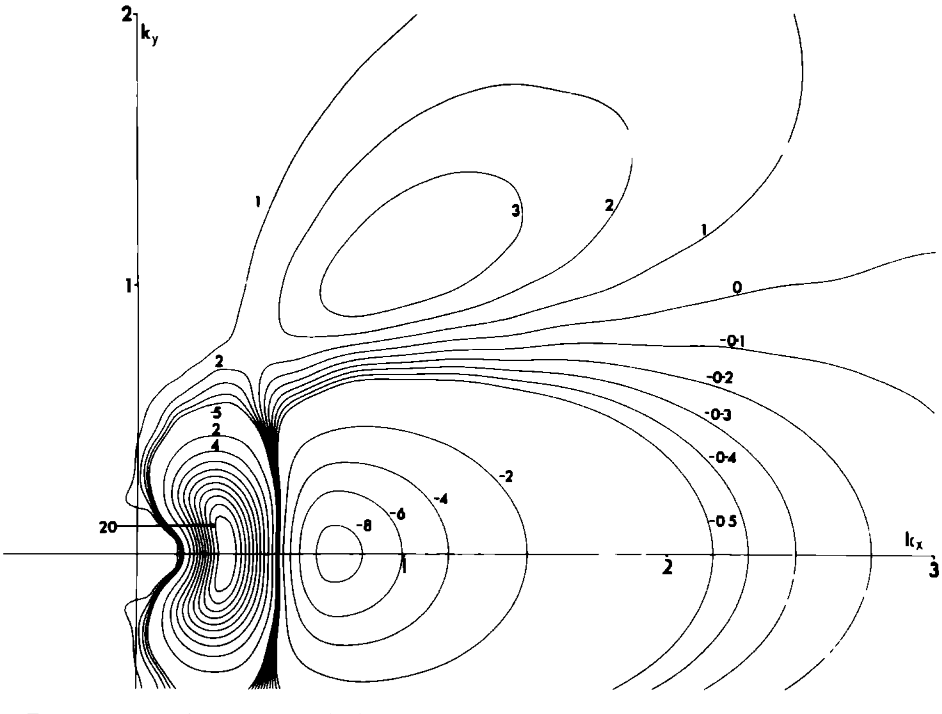

Fig 1 The non-linear transfer dn/dt as a function of wavenumber The contours are marked in units of 10^{-3} m k s units

The present calculation corresponds to case PNB in Sell and Hasselmann (1972). A smoothed version of their results is shown as Fig 2.21b in the JONSWAP paper (Hasselmann *et al*, 1973). Both their results and the present results were obtained by integrating equation (2.1) numerically. In the present case the relative error of each of the calculated values used to construct Fig. 1 should be less than a few per cent. Further details of the procedure are given in Appendix 3

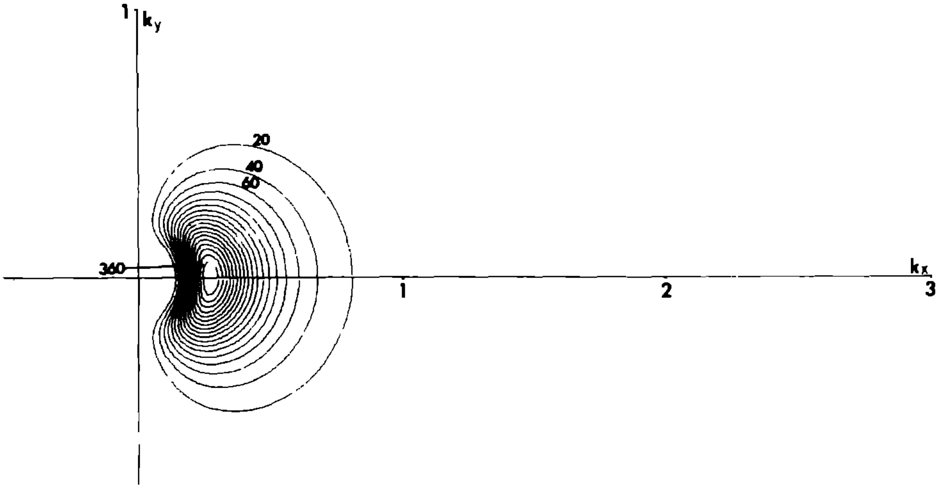

Fig 2. The sea wave spectrum $n(\mathbf{k})$ used in these calculations This is a Pierson–Moskowitz spectrum with a $\cos^2\theta$ spreading factor, a Phillips constant of 0 01 and a maximum variance at 0 3 Hz Case PWB of Sell and Hasselmann (1972) is similar but used a Phillips constant of 0 081

Figure 1 illustrates that the non-linearities transfer action density (and momentum and energy) away from intermediate and high wavenumber waves in the downwind direction This action density is transferred predominantly to a region near the peak of the spectrum and also to two regions at high wavenumbers in a direction approximately 45° to the downwind direction

In Fig. 1, the main peak of $\mathrm{d}n/\mathrm{d}t$ occurs at a wavenumber corresponding to 18-m waves. The minimum value of $\mathrm{d}n/\mathrm{d}t$ corresponds to 8-m waves, and the two secondary maxima correspond to wavelengths of 4 m.

Although the connection is not at first obvious, the high and low wavenumber peaks in $\mathrm{d}n/\mathrm{d}t$ do complement each other. The energy associated with a unit of action density is proportional to the angular velocity ω (Hasselmann, 1963a). As a result, if action density is being transferred somewhere in the spectrum to higher wavenumbers and higher ω, then for the overall energy of the system to be conserved there must be a corresponding transfer somewhere else in the spectrum to lower wavenumbers and lower ω.

DIFFUSING AND PUMPING

To analyse the behaviour further, it is useful to split equation (2.1) into a diffuse term and a pumping term Remembering that equation (2.1) describes the transfers between the pairs of wavenumbers $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$, $\mathbf{k}_4$, the term $n_2 n_4 (n_3 - n_1)$ can be looked on as describing a diffusive transfer between wavenumbers $\mathbf{k}_3$ and $\mathbf{k}_1$, this transfer being catalysed by wavenumbers $\mathbf{k}_2$ and $\mathbf{k}_4$.

The term 'diffusive' seems apt, because the transfer is proportional to the difference in action density at the two wavenumbers $\mathbf{k}_1$ and $\mathbf{k}_2$ and results in a flow from the higher action density to the lower action density. Conservation of energy and momentum then requires that the diffusive transfer be matched by a 'pumped' transfer between $\mathbf{k}_2$ and $\mathbf{k}_4$ The term 'pumped' is chosen to emphasize how the diffusive transfer between $\mathbf{k}_1$ and $\mathbf{k}_3$ forces or pumps the transfer between $\mathbf{k}_2$ and $\mathbf{k}_4$ Similarly the diffusive transfer between

$\mathbf{k}_2$ and $\mathbf{k}_4$ gives a pumped transfer between $\mathbf{k}_1$ and $\mathbf{k}_3$. In equation (2.1) this pumped transfer is represented by the term $n_1 n_3 (n_4 - n_2)$.

To make the definition of the pumped and diffusive terms precise, it is convenient to rewrite the integral so that $|\mathbf{k}_1 - \mathbf{k}_3|$ is less than $|\mathbf{k}_1 - \mathbf{k}_4|$. Thus if $(\mathrm{d}n/\mathrm{d}t)_p$ is the non-linear transfer due to the pumping term and $(\mathrm{d}n/\mathrm{d}t)$ is that due to the diffusive term, then,

$$\left(\frac{\mathrm{d}n}{\mathrm{d}t}\right)_p = 2\int \mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3\,\mathrm{d}\mathbf{k}_4 C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)$$
$$\delta(\omega_1+\omega_2-\omega_3-\omega_4)\,\theta(|\mathbf{k}_1-\mathbf{k}_4|-|\mathbf{k}_1-\mathbf{k}_3|)\,n_1 n_3 (n_4-n_2), \tag{4.1}$$

and

$$\left(\frac{\mathrm{d}n}{\mathrm{d}t}\right)_d = 2\int \mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3\,\mathrm{d}\mathbf{k}_4 C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)$$
$$\delta(\omega_1+\omega_2-\omega_3-\omega_4)\,\theta(|\mathbf{k}_1-\mathbf{k}_4|-|\mathbf{k}_1-\mathbf{k}_3|)\,n_2 n_4 (n_3-n_1) \tag{4.2}$$

Here

$$\begin{aligned}\theta(x) &= 1 \quad \text{if} \quad x>0,\\ &= 0 \quad \text{if} \quad x<0.\end{aligned}$$

As discussed in Appendix 2, the entropy H of the sea wave spectrum is given by,

$$H = \kappa \int \ln n(k)\,\mathrm{d}k, \tag{4.3}$$

where κ is Boltzmann's constant. Transfer of a unit of action density from a region where $n(k)$ is large to a region where it is small will increase the entropy of the spectrum. That is, it will decrease the amount of usable information we have about the system.

As shown by HASSELMANN (1966), the total rate of entropy production due to the non-linear transfer is,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \kappa\int \mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3\,\mathrm{d}\mathbf{k}_4\, C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\delta(\mathbf{k}_1+\ldots)\delta(\omega_1+\ldots)$$
$$\tfrac{3}{4}n_1 n_2 n_3 n_4\left[\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\right]^2, \tag{4.4}$$
$$\geqq 0$$

In the steady state case this is zero, otherwise the overall entropy increases. If one considers the entropy production due only to the diffusive term, then using the same method used do derive equation (4.4), one finds that,

$$\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_d = \kappa\int \mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_3\,\mathrm{d}\mathbf{k}_4\, C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\delta(\mathbf{k}_1+\ldots)\delta(\omega_1+\ldots)$$
$$\tfrac{3}{4}n_1 n_2 n_3 n_4\left[\left(\frac{1}{n_1}-\frac{1}{n_3}\right)^2+\left(\frac{1}{n_2}-\frac{1}{n_4}\right)^2\right], \tag{4.5}$$
$$\geqq 0.$$

Again this is positive or zero, exactly as one would expect from a diffusive like process

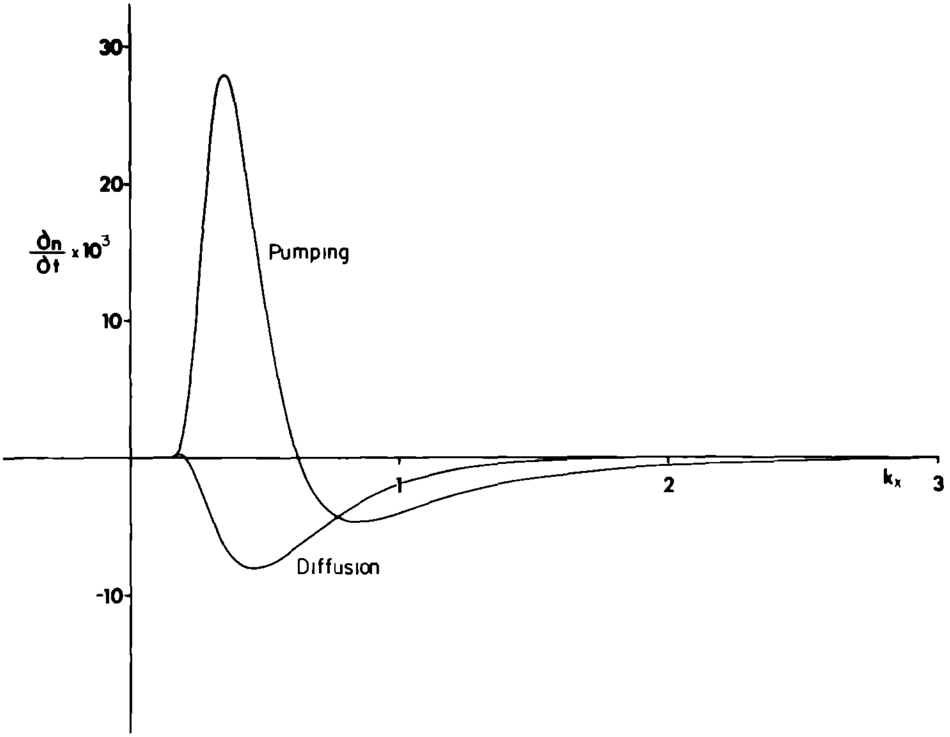

Fig 3 The pumping and diffusive contributions to dn/dt along the k_x axis

The entropy production due to the pumping term is the difference of the above quantities,

$$\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_p = \kappa \int \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3\, \mathrm{d}\mathbf{k}_4\, C(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)\delta(\mathbf{k}_1 + \;.\;)\delta(\omega_1 + \;..) \\ \tfrac{3}{2} n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} - \frac{1}{n_3}\right)\left(\frac{1}{n_2} - \frac{1}{n_4}\right). \tag{4.6}$$

This may have either sign, depending on the properties of $(1/n_1 - 1/n_3)(1/n_2 - 1/n_4)$

Thus the pumping term may create order by pumping action density from regions where the action density is small to regions where it is large However equations (4 4) to (4 6) show that the order created is always less than the disorder created by the diffusive term

The ordering behaviour of the pumping term is illustrated in Figs. 3 and 4. Figure 3 shows the pumping and diffusive contributions in the downwind direction. Figure 4 shows the behaviour of the same quantities at an angle of 45°. One sees that the transfer from intermediate wavenumbers to the main low wavenumber peak is due to the pumping term Similarly the diffusive term is transferring action density from intermediate wavenumbers to high wavenumbers at an angle to the downwind direction. As the entropy change depends on the relative* change in action density, the diffusive process, which transfers action to regions where it is initially small, increases the entropy by a large amount. In contrast although the pumping term produces an impressive peak at low wavenumbers, the relative change of action density is small and so the decrease in entropy is also small

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* $(\delta \ln n = \delta n/n)$

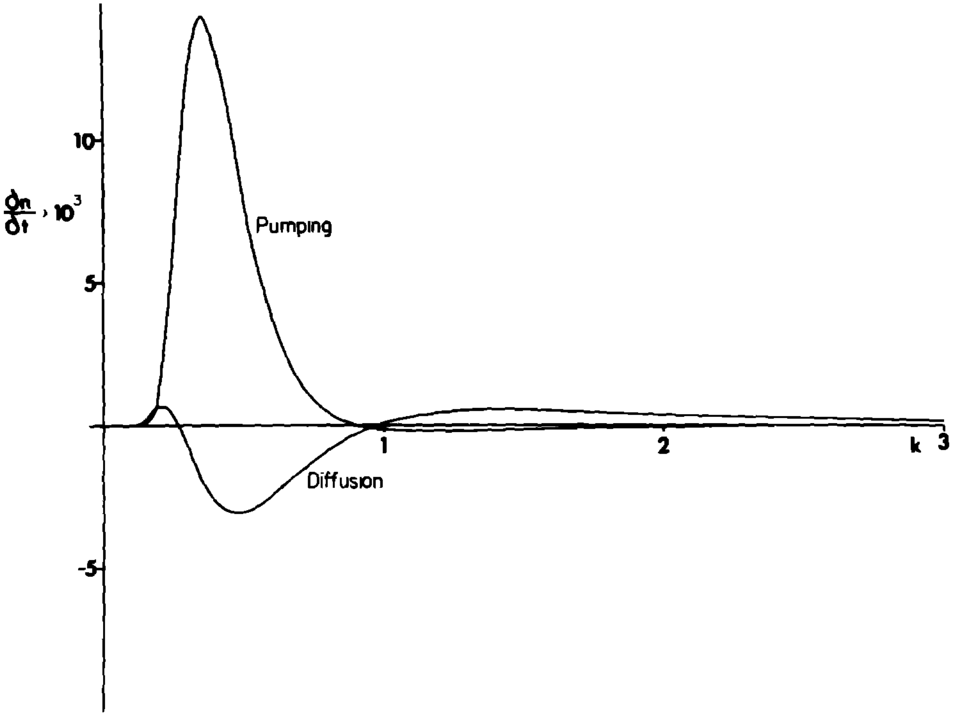

Fig 4. The pumping and diffusive contributions to dn/dt along a line at 45° to the k_x axis

Figures 3 and 4 and the above comments on entropy constitute the main results of this paper. The rest of this paper represents an attempt to describe qualitatively how the flows of action density arise.

Before leaving Figs. 3 and 4, it is interesting to note that the diffusive term also produces a transfer down the low wavenumber face of the spectrum. In fact one finds that at very low wavenumbers, wave growth due to diffusive transport is much more important than that due to pumped transport. This is partly due to the pumped term $(\mathrm{d}n/\mathrm{d}t)_p$ being proportional to n_1. It thus drops off sharply at low wavenumbers.

TRANSFER

In this section, we take advantage of the two-particle scattering nature of Hasselmann's equation and rewrite equation (2.1) as an integral of a transfer function $T(\mathbf{k}_1,\mathbf{k}_3)$, where $T(\mathbf{k}_1,\mathbf{k}_3)$ gives the rate at which wave $\mathbf{k}_3$ is scattered into wave $\mathbf{k}_1$. Again we shall use the convention that $\mathbf{k}_1$ is nearer to $\mathbf{k}_3$ than $\mathbf{k}_4$.

Thus from equation (2.1),

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = \int \mathrm{d}k_3\, T(\mathbf{k}_1,\mathbf{k}_3), \tag{5.1}$$

$$T(\mathbf{k}_1,\mathbf{k}_3) = 2\int \mathrm{d}\mathbf{k}_2\,\mathrm{d}\mathbf{k}_4\, C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\,\delta(\mathbf{k}_1 + \quad)\,\delta(\omega_1 + \ldots)$$
$$\theta(|\mathbf{k}_1-\mathbf{k}_4| - |\mathbf{k}_1-\mathbf{k}_3|)\,[n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)]. \tag{5.2}$$

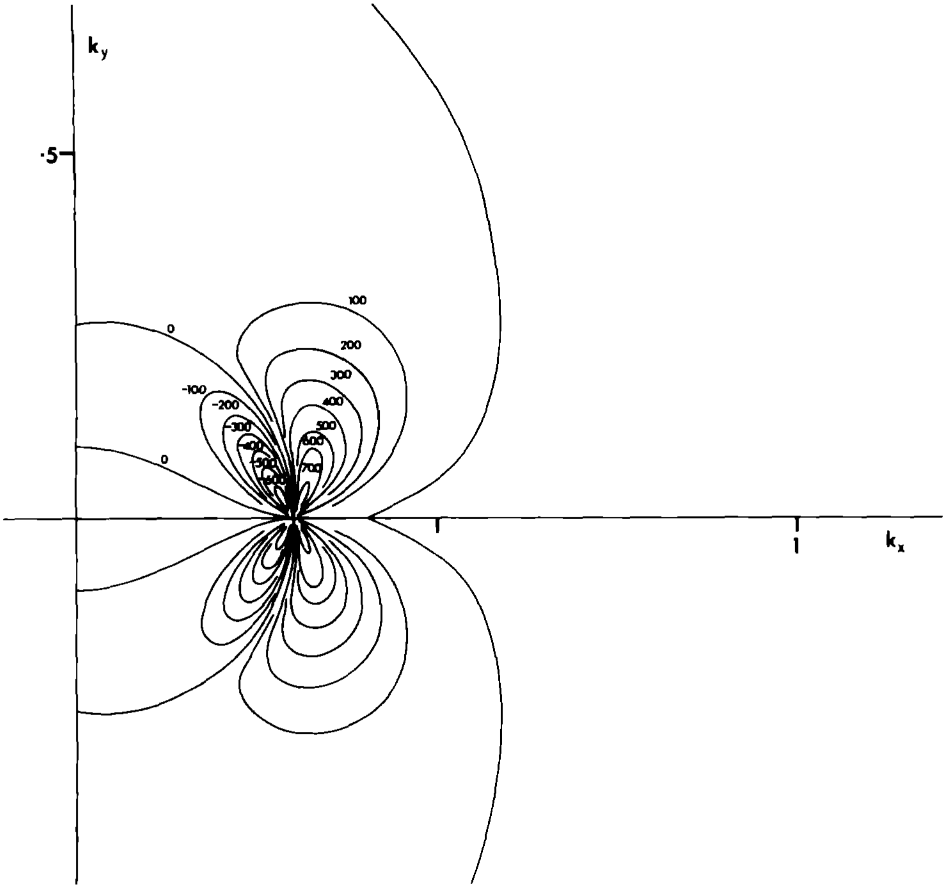

Fig 5a The pumped transfer $T_p(\mathbf{k}_1, \mathbf{k}_3)$ for $\mathbf{k}_1$ equal to (0 3, 0 0), plotted as a function of $\mathbf{k}_3$ Contour values in units of 10^{-3}

As before this can be split into a pumped transfer and a diffusive transfer.

The pumped and diffusive transfers for three values of $\mathbf{k}_1$ are shown in Figs. 5, 6 and 7. Figure 6b especially emphasizes that we are dealing with a flux of action density, transfer towards $\mathbf{k}_1$ from one direction usually being matched by a transfer away from $\mathbf{k}_1$ in the opposite direction. The figures also show that for the pumping terms, the flux is usually towards lower wavenumbers and that for the diffusive terms, the flux is towards high wavenumbers.

However, to understand the reasons for this behaviour, we have to look at the integral (5.2) in more detail.

INTEGRAL

The delta functions on $\mathbf{k}$ and ω in equation (5 2) limit the integral to one around a contour. If $\mathbf{s}$ is the unit vector along the contour and $\mathbf{n}$ the normal vector, then (5.2)

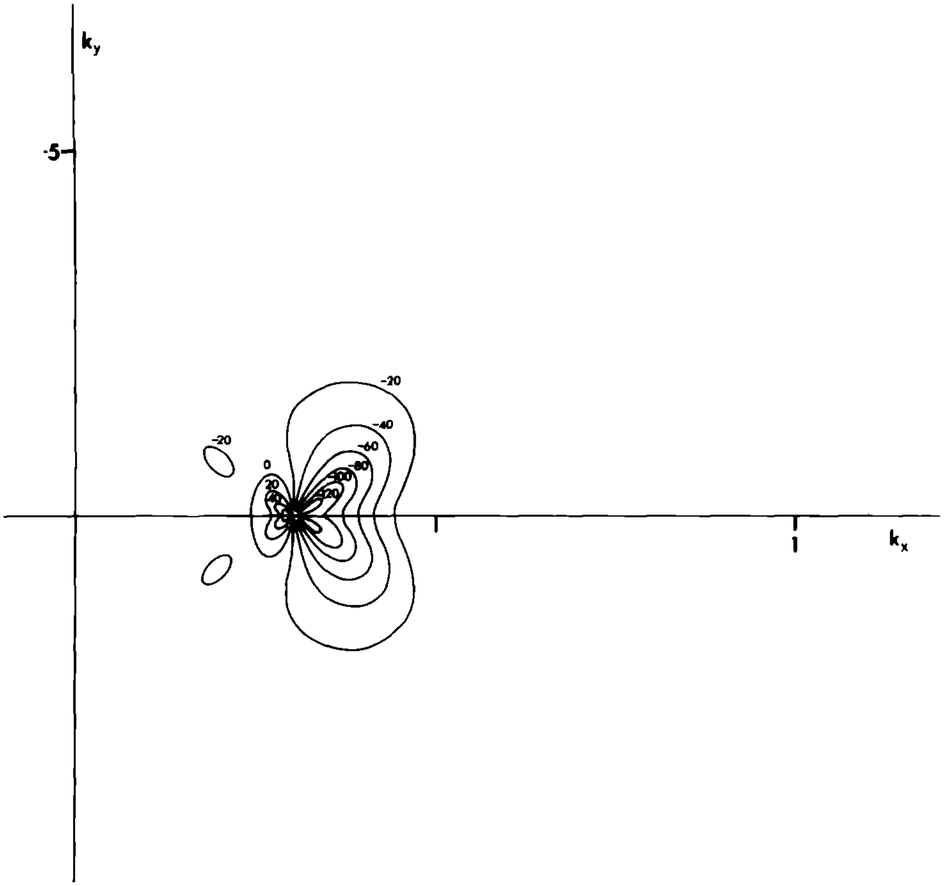

Fig 5b The diffusive transfer $T_d(\mathbf{k}_1,\mathbf{k}_3)$ for $\mathbf{k}_1$ equal to (0 3,0 0) Contour values in units of 10^{-3}

becomes,

$$T(\mathbf{k}_1,\mathbf{k}_3) = 2\oint \mathrm{ds}\, C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\left|\frac{\partial\omega}{\partial\mathbf{n}}\right|^{-1}$$

$$\theta(|\mathbf{k}_1-\mathbf{k}_4|-|\mathbf{k}_1-\mathbf{k}_3|)\left[n_1 n_3(n_4-n_2)+n_2 n_4(n_3-n_1)\right], \qquad (6.1)$$

where $\mathbf{k}_4 = \mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3$ and $\mathbf{k}_2 = \mathbf{k}_2(\mathbf{s},\mathbf{k}_1,\mathbf{k}_3)$. The equation for the locus of $\mathbf{k}_2$ is,

$$\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)-\omega(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3) = 0.$$

For deep water, for which ω is proportional to $k^{\frac{1}{2}}$, this becomes,

$$Q+k_2^{\frac{1}{2}}-(|\mathbf{P}-\mathbf{k}_2|)^{\frac{1}{2}} = 0, \qquad (6.2)$$

where $Q = k_1^{\frac{1}{2}}-k_3^{\frac{1}{2}}$ and $\mathbf{P} = \mathbf{k}_1-\mathbf{k}_3$.

A typical locus for $\mathbf{k}_2$ for given values of $\mathbf{k}_1$ and $\mathbf{k}_3$ is shown in Fig. 8 It is roughly

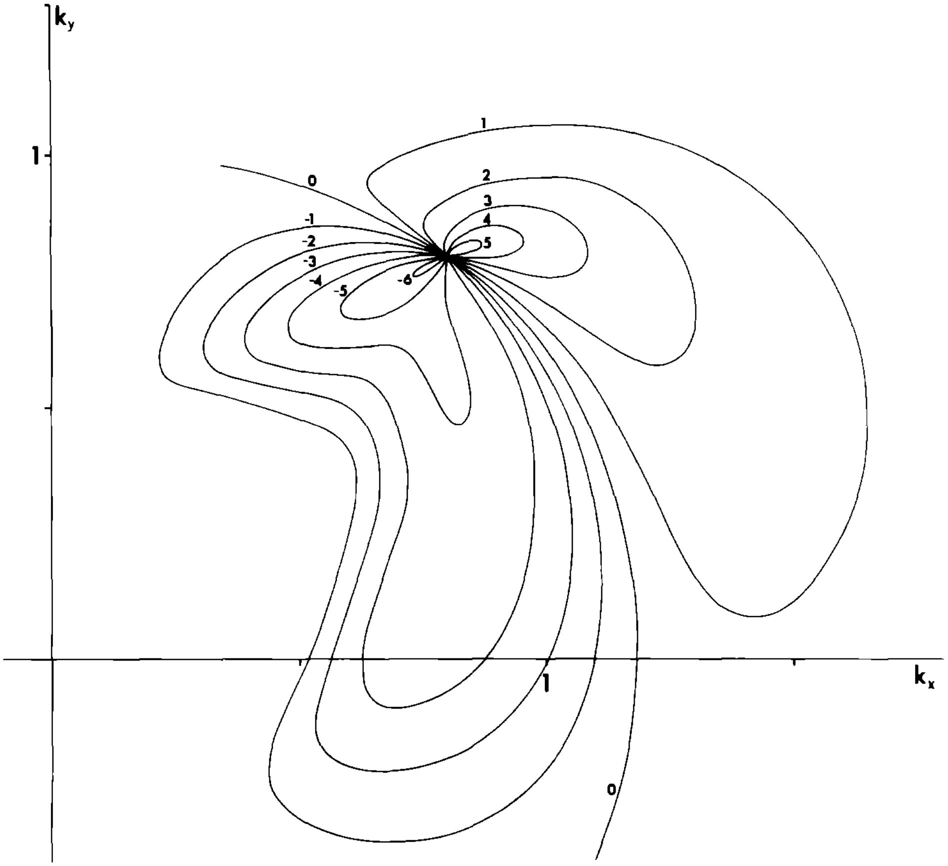

Fig. 6a The pumped transfer $T_p(\mathbf{k}_1,\mathbf{k}_3)$, for $\mathbf{k}_1$ equal to (0 8, 0 8) Contour values in units of 10^{-3}

egg shape, becoming circular when $(P-Q)/Q^2$ is very small or very large The diameter of the path of integration is approximately equal to $(P/Q-Q)^2/4$. So in the limit, as Q tends to zero with P fixed, the length of the path of integration tends to infinity.

As mentioned previously the interaction coefficient is a rapidly increasing function of wavenumber. The term $|\partial\omega/\partial\mathbf{n}|^{-1}$ also rapidly increases with wavenumber In particular it is straightforward to show that when k_2 is large compared with P, $|\partial\omega/\partial\mathbf{n}|^{-1}$ is pro-proportional to $k_2^{3/2}$.

This sharp increase in $|\partial\omega/\partial\mathbf{n}|^{-1}$ is a phase space effect and reflects the fact that when k_2 is large compared with k_1 and k_3, a small change in $\mathbf{k}_1$ or $\mathbf{k}_3$ will produce a large change in the positions of $\mathbf{k}_2$ and $\mathbf{k}_4$. As a result a small group of waves in the region of $\mathbf{k}_1$ and $\mathbf{k}_3$ will interact with many more waves in the region of $\mathbf{k}_2$ and $\mathbf{k}_4$, enhancing the effect of the non-linearities

The behaviour of the various terms in the intergrand along a typical contour are illustrated in Fig. 9. The contour used is that of Fig. 8 Figure 9 illustrates how both the coefficient C and the $|\partial\omega/\partial\mathbf{n}|^{-1}$ term increase with wavenumber However, as mentioned by

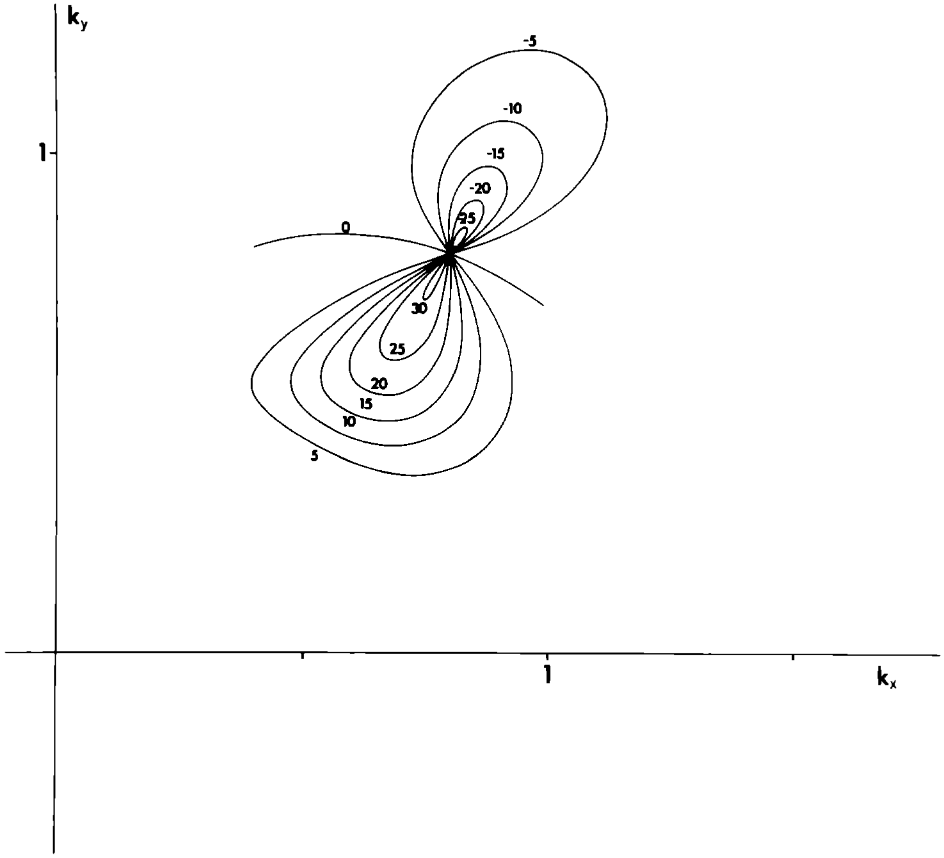

Fig 6b The diffusive transfer $T_d(\mathbf{k}_1, \mathbf{k}_3)$ for $\mathbf{k}_1$ equal to (0.8, 0.8) Contour values in units of 10^{-3}

HASSELMANN (1963b), the coefficient C drops off sharply when the angle between the wavenumbers becomes large.

The integrand of the diffusive term $n_2 n_4(n_3 - n_1)$ is largest when both n_2 and n_4 lie within the peak of the spectrum. The greater value of the product $n_2 n_4$ thus more than compensates for the relatively small values of C and $|\partial\omega/\partial\mathbf{n}|^{-1}$ This was a general property of the diffusive term, with the result that the direction of maximum transfer $T_d(\mathbf{k}_1, \mathbf{k}_3)$ occurred when the separation of $\mathbf{k}_1$ and $\mathbf{k}_3$ was such that the path of integration passed through the peak. This direction shows up clearly in Fig. 6b

A simple calculation shows that, for a narrow peak at $(A, 0)$ and a small value of P, the maximum diffusive transfer will occur when the direction of P is such that.*

$$\frac{P_x}{P_y} = \frac{A^{-\frac{1}{2}}(x^2 + y^2)^{\frac{1}{4}} - x}{y}, \tag{6.3}$$

where $(x, y) = k_1$

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*This corresponds to flow along streamlines defined by the stream-function $\Psi(x, y) = x - 2A^{1/2}(x^2 + y^2)^{1/4}$

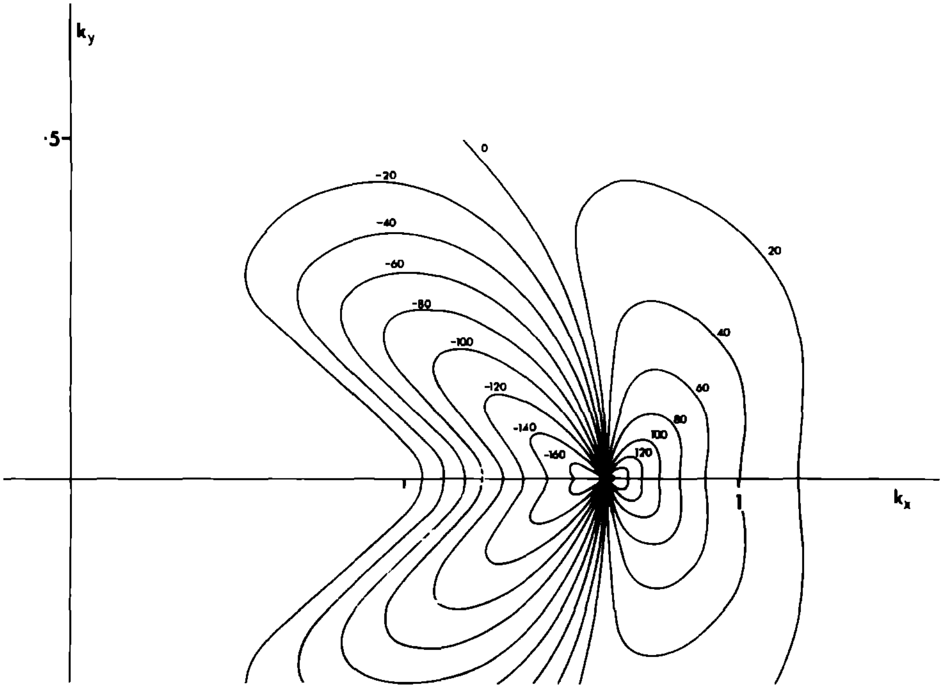

Fig 7a The pumped transfer $T_p(\mathbf{k}_1,\mathbf{k}_3)$ for $\mathbf{k}_1$ equal to (0.8,0.0) Contour values in units of 10^{-3}

Near the downwind axis, y in equation (6.3) is small and so the diffusive transfer is predicted to be at right angles to the axis As shown in Figs. 5b and 7b, in reality this is not quite the case, essentially because of the broadness of the main peak in $n(\mathbf{k})$ and because, for diffusion at right angles to the axis, the term (n_3-n_1) becomes very small. However, as shown in Fig. 3, there is still a net diffusion away from the downwind axis.

Turning to the pumping term, $n_1 n_3 (n_4-n_2)$; as we are now concerned with the difference (n_4-n_2) there is no longer such an advantage in having n_4 and n_2 near the peak. Instead the largest integrals are obtained where the length of the contour of integration is large Then large values of k_2 and k_4 are involved, so that both C and $|\partial\omega/\partial\mathbf{n}|^{-1}$ are large However, there is a compromise involved, for when k_2 and k_4 are very large compared with k_1 and k_3, C increases relatively slowly (HASSELMANN, 1963b) and so increases in the contour length no longer compensate for the reduction in (n_4-n_2)

These effects can be seen in Figs. 5a, 6a and 7a. The maximum pumped transfer is never directly towards the origin, because the path of integration is then shortest. Instead it tends to be at an angle of about 45° to the origin, the action density being pumped to lower wavenumbers in a crabwise fashion. The transfer is always to lower wavenumbers as it is produced predominantly by diffusive transfer in the tail of the spectrum, the direction of which is towards even higher wavenumbers.

The behaviour of the integral can be illustrated further by neglecting the geometric

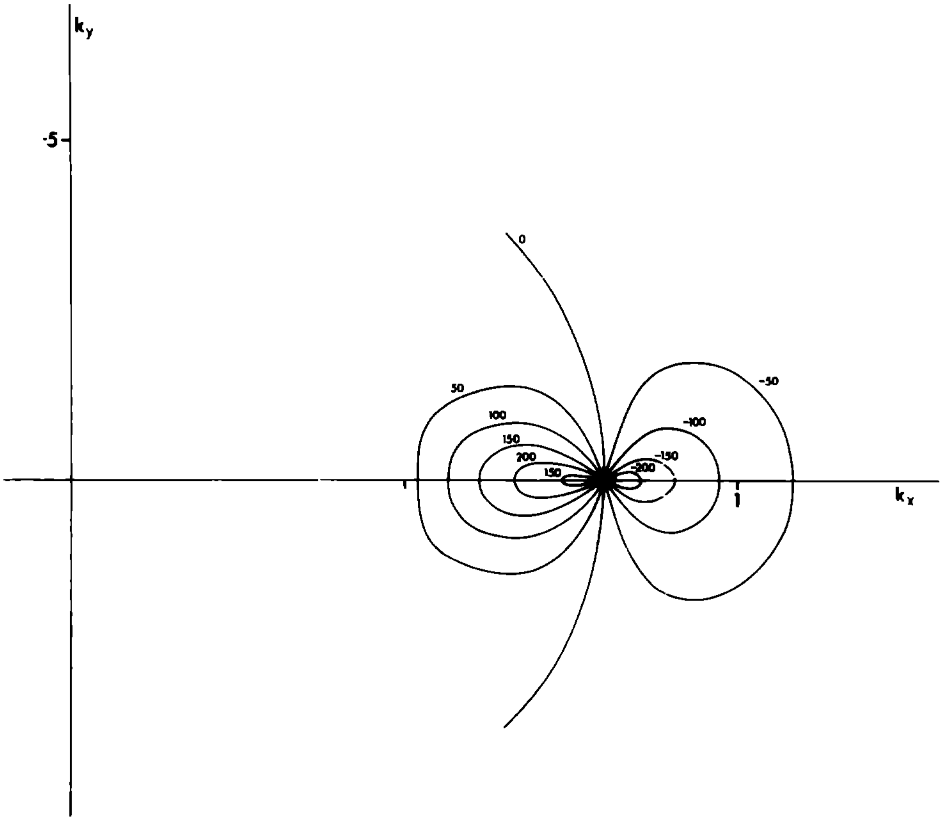

Fig 7b The diffusive transfer $T_d(\mathbf{k}_1, \mathbf{k}_3)$ for $\mathbf{k}_1$ equal to (0 8,0 0) Contour values in units of 10^{-3}

factors and writing the integral (6 1) as an integral over $k_2 (= |k_2|)$ Thus,

$$T(\mathbf{k}_1, \mathbf{k}_3) \sim \int \mathrm{d}k_2\, C(k_1, k_2, k_3, k_4) \left|\frac{\partial \omega}{\partial n}\right|^{-1} [n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)]. \tag{6.4}$$

At high wavenumbers,

$$n(k) \sim k^{-9/2},$$

If P is small,

$$\left|\frac{\partial \omega}{\partial n}\right|^{-1} \sim k^{3/2},$$

and we will take

$$C(k_1, k_2, k_3, k_4) \sim k_1^3 k_2^3$$

for small values of P. Substituting in (6.4)

$$T(k_1, k_3) = \int \mathrm{d}k_2\, k_1^3 k_2^3 k_2^{3/2} [k_1^{-9} k_2^{-11/2} + k_2^{-9} k_1^{-11/2}], \tag{6.5}$$

$$= \int \mathrm{d}k_2\, (k_1^{-6} k_2^{-1} + k_1^{-5/2} k_2^{-9/2}) \tag{6.6}$$

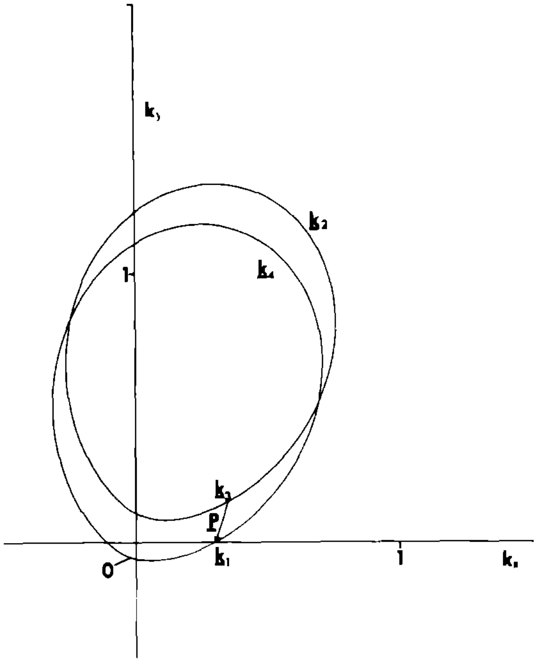

Fig 8 The contour of integration for $T(\mathbf{k}_1,\mathbf{k}_3)$ where $\mathbf{k}_1$ equals (0 3,0 0) and $\mathbf{k}_3$ equals (0 35,0 15) The curve marked $\mathbf{k}_4$ shows the locus of $\mathbf{k}_4$ during the integration At the same time $\mathbf{k}_2$ follows the curve marked $\mathbf{k}_2$, but with $\mathbf{k}_4 - \mathbf{k}_2$ equal to $\mathbf{P}$

The diffusive process gives the second term. This is dominated by the behaviour when k_2 is small, that is to say by the peak of the spectrum The pumping process gives the first term. This drops off much more slowly and could in fact give a logarithmic singularity. However, for very large values of k_2, the coupling coefficient increases more slowly and so the integral is finite.

To sum up, the diffusive scattering is dominated by scattering off waves near the peak of the sea wave spectrum. Diffusive scattering is always towards regions of lower action density, but the position of the spectral peak can influence the ease at which transfer occurs in various directions.

On the other hand, the pumped transfer is dominated by scattering off much shorter waves. As a result the pumping mechanism transfers action density to the lower wavenumbers near the origin in a crabwise fashion.

GROWTH AND DECAY

So far we have considered the quantities $T(\mathbf{k}_1,\mathbf{k}_3)$. From Figs. 5 to 7 they are seen to behave much like fluxes, with the transfer towards $\mathbf{k}_1$ from one direction being roughly balanced by a transfer away from $\mathbf{k}_1$ in the opposite direction. The rate of change of the spectrum depends on the difference between the two fluxes.

The flux of pumped action density is most important and appears to behave most simply, so we shall consider it first At high wavenumbers, the pumped flux is towards lower wavenumbers, and as indicated by equation (6.6), it increases as $\mathbf{k}_1$ is reduced. There is thus a positive divergence of the flux and so $\mathrm{d}n/\mathrm{d}t$ is negative At lower wavenumbers,

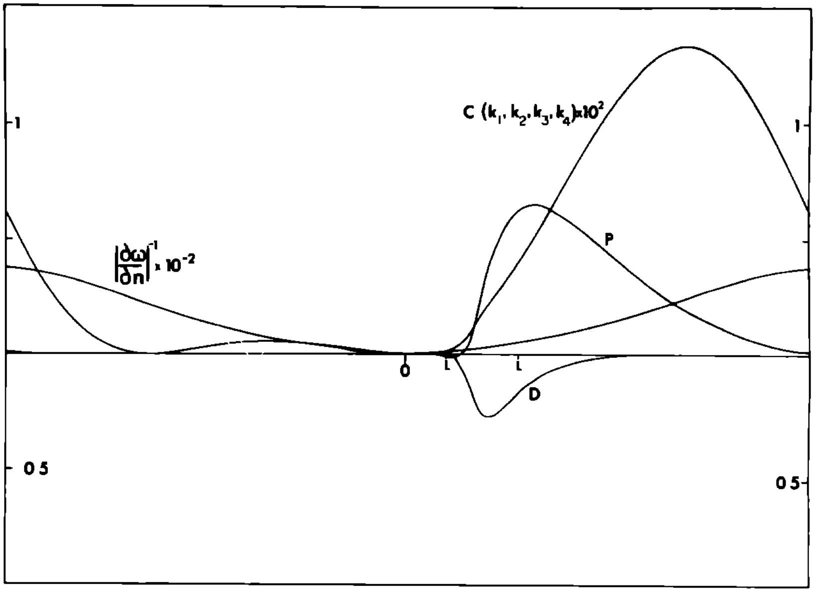

Fig 9 This shows the behaviour of various terms in the integrand around the contour of integration shown in Fig 8 The curves marked P and D are the total pumped and diffusive parts of the integrand The point O corresponds to the similar point marked in Fig 8 The region marked L–L is not included in the integration because of the constraint $|k_1 - k_3| < |k_1 - k_4|$

the flux of action density increases rapidly and $\mathrm{d}n/\mathrm{d}t$ becomes even more negative (see Fig. 3)

However, as the peak of the spectrum is reached, the transfer to lower wavenumbers is reduced. This is because the product $n_1 n_3$ is no longer increasing rapidly enough to counteract the fall in the coefficient C and the phase space terms (i.e. the $|\partial\omega/\partial\mathbf{n}|^{-1}$ term and the length of the contour of integration).

This sudden reduction in the transfer to lower wavenumbers means that $\mathrm{d}n/\mathrm{d}t$ becomes very large and positive. But on the low wavenumber face of the spectrum both the transfer term $T(\mathbf{k}_1,\mathbf{k}_3)$ and $\mathrm{d}n/\mathrm{d}t$ include a term proportional to n, which drops off very rapidly in this region. As a result the pumped part of $\mathrm{d}n/\mathrm{d}t$ has its maximum value near the peak of the sea wave spectrum

For the diffusive transfer, at high wavenumbers the flux is directed outwards and it increases rapidly as the wavenumber is lowered. Thus $\mathrm{d}n/\mathrm{d}t$ is positive and increases towards lower wavenumbers. As seen in Fig 6b, the width of the lobes of $T(\mathbf{k}_1,\mathbf{k}_3)$ contributing to the diffusive flux is approximately equal to the width of the mean peak From what was said earlier this is to be expected for it means that in the integral (6.1) both $\mathbf{k}_2$ and $\mathbf{k}_4$ can lie within the main peak at the same time.

As $\mathbf{k}_1$ moves to lower wavenumbers, the positive lobe of $T(\mathbf{k}_1,\mathbf{k}_3)$ starts to intersect the spectral peak. At this point the gradient $(n_3 - n_1)$ is not increasing rapidly enough as the wavenumbers are reduced to compensate for the reduced coupling coefficient and phase space effects. As a result the diffusive contribution to $\mathrm{d}n/\mathrm{d}t$ stops increasing and eventually becomes negative under the main peak

Physically, the diffusive term is behaving as one would expect, with a negative lobe under the main spectral peak and two positive lobes on either side. However, an important property of sea waves is that for a realistic spectrum the high wavenumber positive lobe is enhanced and the low wavenumber positive lobe is suppressed. This anisotropy is then also reflected in the pumped transfer.

DISCUSSION

In this paper we have seen how the relatively non-linear short waves in the ocean scatter off the waves near the peak of the sea-wave spectrum to become even shorter waves. Conversely the waves near the peak of the spectrum are scattered by the short waves and become even longer. As a result of this process one might expect to see short breaking waves at an angle of 45° to the mean sea, because of the saturation of the spectrum there. One may also see some longer breaking waves arising from the over-enhanced peak of the main spectrum.

One of the reasons why the pumped flux of action density stops near the peak of the spectrum is that it cannot be pumped to shorter wavenumbers. This is because the pumping process requires energy or action already to exist at a given wavenumber before it is effective. The question then arises as to what happens if there is a swell wave at a wavenumber lower than that of the low wavenumber cutoff of the main sea. Admittedly the coupling coefficient and the phase space terms will be small, but there would be some effect.

From the discussions earlier in this paper one would expect the presence of the main sea to enhance the diffusive decay of the swell peak, making it broader. In addition diffusion between sea waves coupled with pumping between two waves within the swell peak would lead to a movement of the swell peak to even lower wavenumbers. Finally, diffusion between sea waves coupled with pumping between one sea wave and one swell wave would lead to growth of the swell peak.

We can also consider qualitatively what would happen with one of the sharply peaked spectra, found during the JONSWAP (Joint North Sea Wave Project) experiment, considering it to be composed of a Pierson–Moskowitz spectrum plus a very narrow spectrum placed at the peak. We already know what the Pierson–Moskowitz spectrum does, and a very narrow spectrum tends to broaden out (Fox, 1976). There remains the interaction between the two

One would expect that the main effect of the narrow peak on the Pierson–Moskowitz spectrum would be to enhance the diffusive process amongst the high wavenumbers This might mean, for example, more short-breaking waves.

In the other direction, the effects of the Pierson–Moskowitz spectrum would increase the rate at which the narrow peak spreads out diffusively The scattering of the narrow peak off high wavenumbers in the Pierson–Moskowitz spectrum would also tend to pump the narrow peak to lower wavenumbers.

As a result of this one might expect the low wavenumber maximum in dn/dt to be moved to shorter wavenumbers, that is, towards the low wavenumbers face of the main spectrum. In the calculations of Hasselmann and Sell (HASSELMANN *et al.*, 1973, Fig. 2.21) such an effect can be observed.

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APPENDIX 1 THE COUPLING COEFFICIENT

Hasselmann (1962, equations 4.3, 4.9, 4 10) gives rather involved formulae for the coupling coefficients. For deep water waves, when the delta functions are satisfied (i e. $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$ and $\omega_1 + \omega_2 = \omega_3 + \omega_4$), these equations simplify a little to give,

$$\begin{aligned} D(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = {} & \frac{2(w_1+w_2)^2(k_1k_2-\mathbf{k}_1\cdot\mathbf{k}_2)(k_3k_4-\mathbf{k}_3\cdot\mathbf{k}_4)}{w_{1+2}^2-(w_1+w_2)^2} \\ & + \frac{2(w_1-w_3)^2(k_1k_3+\mathbf{k}_1\cdot\mathbf{k}_3)(k_2k_4+\mathbf{k}_2\cdot\mathbf{k}_4)}{w_{1-3}^2-(w_1-w_3)^2} \\ & + \frac{2(w_1-w_4)^2(k_1k_4+\mathbf{k}_1\cdot\mathbf{k}_4)(k_2k_3+\mathbf{k}_2\cdot\mathbf{k}_3)}{w_{1-4}^2-(w_1-w_4)^2} \\ & + \tfrac{1}{2}[\mathbf{k}_1\cdot\mathbf{k}_2\mathbf{k}_3\;\mathbf{k}_4+\mathbf{k}_1\cdot\mathbf{k}_3\mathbf{k}_2\cdot\mathbf{k}_4+\mathbf{k}_1\cdot\mathbf{k}_4\mathbf{k}_2\cdot\mathbf{k}_3] \\ & - \tfrac{1}{4}(\mathbf{k}_1\cdot\mathbf{k}_2+\mathbf{k}_3\;\mathbf{k}_4)(w_1+w_2)^4+\tfrac{1}{4}(\mathbf{k}_1\;\mathbf{k}_3+\mathbf{k}_2\cdot\mathbf{k}_4)(w_1-w_2)^4 \\ & + \tfrac{1}{4}(\mathbf{k}_1\cdot\mathbf{k}_4+\mathbf{k}_2\cdot\mathbf{k}_3)(w_1-w_4)^4+\tfrac{5}{2}k_1k_2k_3k_4 \\ & + (w_1+w_2)^2(w_1-w_3)^2(w_1-w_4)^2(k_1+k_2+k_3+k_4). \quad \text{(A1)} \end{aligned}$$

Here,

$$k_1 = |\mathbf{k}_1|, \quad w_1 = k_1{}^{\frac{1}{2}}$$

and similarly for k_2, w_2, etc.

$$w_{1+2} = (|\mathbf{k}_1+\mathbf{k}_2|)^{\frac{1}{2}}, \quad w_{1-3} = (|\mathbf{k}_1-\mathbf{k}_3|)^{\frac{1}{2}}, \quad w_{1-4} = (|\mathbf{k}_1-\mathbf{k}_4|)^{\frac{1}{2}}.$$

The coefficient C of Hasselmann's equation (2.1) is

$$C(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = \frac{\pi g^2 D(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)}{4\rho^2\omega_1\omega_2\omega_3\omega_4}. \tag{A2}$$

g is the acceleration of gravity, ρ the density of water

Equation (A1) was obtained with the help of an algebraic manipulator (BARTON and FITCH, 1972) and has been checked numerically against Hasselmann's original equations The algebraic manipulator was also used to obtain its asymptotic form and this agreed with a related asymptotic expansion given by HASSELMANN (1963b) Equation (A1) is probably not in its physically most transparent form, but it does give some indication of the physical processes that are occurring.

Where wavenumbers $\mathbf{k}_1$ and $\mathbf{k}_2$ interact non-linearly, they produce a perturbation with wavenumber $\mathbf{k}_1+\mathbf{k}_2$. If the frequencies match, that is ω_{1+2} equals $\omega_1+\omega_2$, there will be a transfer of energy and momentum to wavenumber $\mathbf{k}_1+\mathbf{k}_2$ and this wave can then propagate away as a free wave With surface waves this process is not possible. However, the forced wave can decay to two other free waves k_3 and k_4 with overall conservation of energy and momentum This process corresponds to the first term of equation (A1), which shows that the rate at which it happens depends on the mismatch of energy, i.e. $[\omega_{1+2}^2-(\omega_1+\omega_2)^2]^{-1}$. The first term also includes the effect of transfer in the opposite direction, that is from $\mathbf{k}_3$ and $\mathbf{k}_4$ to $\mathbf{k}_1$ and $\mathbf{k}_2$ The second and third terms correspond to similar transfers among the other combinations of wavenumbers.

Some rearrangement among the first three terms and the remaining terms is possible, but the remaining terms must mainly represent the direct process, that is, the direct non-linear interaction of two waves to give two new waves without the intermediate forced wave

APPENDIX 2 ENTROPY

HASSELMANN (1963a) used mathematical arguments to show that the function H, defined by,

$$H = \text{Const} + \int \ln n(k)\,\mathrm{d}k \tag{B1}$$

behaves like entropy This appendix shows how the above equation may be related to the more conventional definitions of entropy.

In statistical mechanics (HUANG, 1965; LANDAU and LIFSHITZ, 1969), entropy is defined in terms of the probability $f(p)$ that the system is in a particular part of the phase space p

$$H = -\kappa \int f(p)\ln f(p)\,\mathrm{d}p \tag{B2}$$

κ is Boltzmann's constant and the integral is over all the phase space. For a system of waves, if $\mathbf{p}_i$ defines the amplitude and phase the wave $\mathbf{k}_i$ say, then,

$$\mathrm{d}p = \prod_i \mathrm{d}\mathbf{p}_i$$

Hasselmann's equation gives the ensemble average action density at each wavenumber. The corresponding probability distribution will be the one that has maximum entropy, given the constraints that the total probability is one,

$$\int f \, \mathrm{d}p = 1, \tag{B3}$$

and that the mean action density at wavenumber $\mathbf{k}_1$ is that given by Hasselmann's equation,

$$\frac{g}{2\omega_i}\int \mathbf{p}_i^2 f \, \mathrm{d}p = n(\mathbf{k}_i)/N,$$
$$= n_i \text{ say}$$

Here N is the density of states. It is convenient to rewrite this equation in the form,

$$\int \mathbf{p}_i^2 f \, \mathrm{d}p = q_i, \tag{B4}$$

where $q_i = 2w_i n_i/g$

This problem can be solved using the calculus of variations (see for example MATTHEWS and WALKER, 1964) The derivation is essentially the one used to derive the Maxwell–Boltzmann distribution, except that we do not have equal energy per degree of freedom If α and β_i are Lagrange multipliers, we maximize

$$-\kappa\int f \ln f \, \mathrm{d}p + \alpha\left(\int f \, \mathrm{d}p - 1\right) + \sum_i \beta_i\left(\int \mathbf{p}_i^2 f \, \mathrm{d}p - q_i\right),$$

for all small variations δf of the probability f This gives

$$\int \delta f\left(-\kappa \ln f - \kappa + \alpha + \sum_i \beta_i \mathbf{p}_i{}^2\right)\mathrm{d}p = 0$$

Thus,

$$f = A \exp\left(\sum_i \gamma_i \mathbf{p}_i^2\right) \tag{B5}$$

The constants A and γ_i are found by substituting (B5) into (B3) and (B4) This gives,

$$f = \prod_i \frac{1}{2q_i}\exp(-\mathbf{p}_i^2/q_i)$$

Substituting into equation (B3), one finds,

$$H = -\kappa\int \prod_i \frac{1}{2q_i}\exp(-\mathbf{p}_i^2/q_i)\ln\left[\prod_j \frac{1}{2q_j}\exp(-\mathbf{p}_j^2/q_j)\right]\mathrm{d}p,$$
$$= \kappa\int \sum_i \frac{1}{2q_i}\exp(-\mathbf{p}_i^2/q_i)(\ln 2 + \ln q_i - \mathbf{p}_i^2/q_i)\,\mathrm{d}\mathbf{p}_i$$

Thus,

$$H = \text{Const} + \kappa \sum_i \ln q_i,$$

$$= \text{Const} + \kappa \sum \ln n_i.$$

If the density of states is large, the equation can be transformed into an integral over k

$$H = \text{Const} + \kappa \int \ln n(k)\,\mathrm{d}k. \tag{B6}$$

Thus the statistical mechanical (and thermodynamical) definition of entropy is obtained by multiplying equation (B1) by Boltzmann's constant.

A possible drawback of the above derivation is that it does not include any information on the correlations between waves implicit in the derivation of Hasselmann's equation.

APPENDIX 3 COMPUTING NOTES

Hasselmann's equation was integrated in the form of the two integrals of equations (5.1) and (5.2). The first of these was carried out using polar coordinates. For the radial integral a transformation was used of the form

$$r = y/(1-y),$$

the integral of r from zero to infinity being changed into an integral of y from zero to one. For the contour integral (5.2), the length of which could be very large, a similar technique was used, but it was made into an angular integral about a point near the centre of the contour. For each value of the integrating variable, equation (6.2) was solved for $\mathbf{k}_2$ by iteration.

A number of methods for carrying out the integrals were tried (KAHANER, 1971). An adaptive Simpson's rule technique was more efficient at following the vagaries of the integrand than the more sophisticated methods. Some trouble was experienced at the higher levels when the strategy at one of the lower levels of integration changed, as this introduced small steps in the integrand. This was overcome by requiring a relative accuracy of 1% at the highest level, 0.3% at next and 0.1% at the lowest level of integration. Each evaluation of $\mathrm{d}n/\mathrm{d}t$ took about 10 s on an IBM 360/195.

About 200 evaluations were made, interpolated using splines, and then contoured to give Fig. 1.