The radiation of atmospheric microbaroms by ocean waves

Roger Waxler[a] and Kenneth E. Gilbert
National Center for Physical Acoustics, University of Mississippi, University, Mississippi 38677

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A two-fluid model, air over seawater, is used to investigate the radiation of infrasound by ocean waves. The acoustic radiation which results from the motion of the air/water interface is known to be a nonlinear effect. The second-order nonlinear contribution to the acoustic radiation is computed and the statistical properties of the received microbarom signals are related to the statistical properties of the ocean wave system. The physical mechanisms and source strengths for radiation into the atmosphere and ocean are compared. The observed ratio of atmospheric to oceanic microbarom peak pressure levels (approximately 1 to 1000) is explained. *© 2006 Acoustical Society of America.* [DOI: 10.1121/1.2191607]

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I. INTRODUCTION

Energetic systems of ocean waves, such as those produced by storms at sea, radiate detectable levels of infrasound into both the atmosphere and the ocean and, through the ocean, generate seismic waves at the sea floor which radiate into the earth.[1–6] The radiation is in a frequency band approximately 0.1 Hz wide centered at about 0.2 Hz. This radiation is referred to as microbaroms in the atmospheric case and as microseisms in the seismic case. Although the oceanic signals are sometimes referred to as microseisms, in this paper the terminology oceanic microbaroms will be used.

The microbarom and microseism peak is a permanent feature of the oceanic and seismic noise floors, respectively.[1,7,8] Spectra measured on the floor of the Pacific Ocean show peak levels up to 100 $\mathrm{Pa}/\sqrt{\mathrm{Hz}}$.[7,8] Atmospheric microbaroms are detected at large distances (up to thousands of kilometers) from the waves which produced them. Their propagation is highly dependent on the direction of the atmospheric winds.[9–12] Atmospheric microbarom spectra measured downwind from the wave system have typical peak levels of about 0.1 $\mathrm{Pa}/\sqrt{\mathrm{Hz}}$.[13] Note that oceanic microbarom signals are a thousand times greater than atmospheric signals.

In this paper a detailed calculation of the microbarom source strength is presented. It is shown that the physical mechanism for the radiation of microbaroms into the atmosphere is different from that for the radiation of microbaroms into the ocean. The predicted radiation into the atmosphere is three orders of magnitude less than the radiation into the ocean, in agreement with observation. Further, in order to relate sea states to observed microbarom levels, a direct connection is obtained between the stochastic models used to describe ocean waves[14,15] and the statistical properties of the received microbarom signal.

The problem of the generation of oceanic microbaroms was solved by Longuet-Higgens.[2] He showed that microbaroms and, through their interaction with the ocean floor, microseisms are produced through a second-order non-linear effect by the interaction of gravity waves of nearly equal frequency and nearly opposite propagation direction. He found that to obtain the source strength for the radiation of oceanic microbaroms it is sufficient to assume that the ocean is incompressible and that the atmosphere is a vacuum. In particular, once the ocean surface is in motion, the atmosphere's effect on the radiation of microbaroms into the ocean is negligible. Hasselmann[3] extended Longuet-Higgens' result to general sea states and made a direct connection between the stochastic models used to describe ocean waves[14,15] and the statistics of the observed microseism signals.

It was pointed out by Brekhovskikh *et al.* in Ref. 5 that the ocean's effect on the radiation of atmospheric microbaroms is significant. A large part of the atmospheric microbarom signal is due to sound radiated into the atmosphere from pressure fluctuations produced in the water by the motion of the ocean surface. It will be seen in this paper that the rest of the signal is due to the compression of the air by the motion of the ocean surface.

It follows that the study of atmospheric microbaroms requires a two-fluid model, consisting of a rare fluid, the atmosphere, over a dense fluid, the ocean. The interface between atmosphere and ocean must be allowed to deviate from its equilibrium position, assumed in this paper to be at $z=0$. For simplicity, in this paper the ocean will be assumed to be infinitely deep.

Of interest is the case in which there is a region of the ocean surface in which the sea state contains energetic counter-propagating waves. The acoustic signal radiated from this region to a distant sensor is considered. The mechanism through which the sea surface has been excited (presumably through strong winds produced by storms[14,15]) is not considered here. As in Refs. 3 and 5, the statistical properties of the microbarom signal are related to those of the sea state.

The paper is organized as follows. In Sec. II the notion of a source region, the region of the ocean surface from which the infrasound is radiated, is defined. The equations of fluid mechanics are then solved in the source region to sec-

[a]Electronic mail: rwax@olemiss.edu

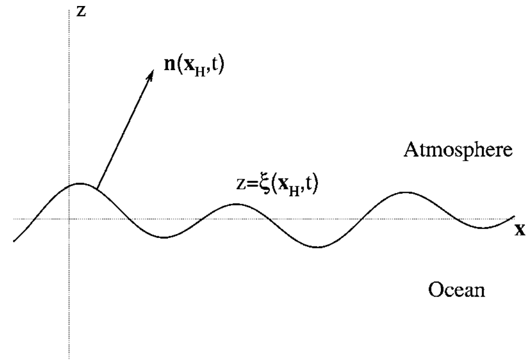

FIG. 1. A two-dimensional cross section of a portion of the source region. The air/water interface and an upwardly pointing normal vector to the interface are shown.

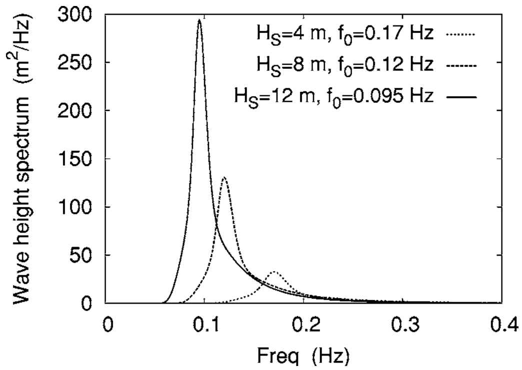

FIG. 2. The JONSWAP model for the frequency spectrum $\bar{F}(f) = \int_0^{2\pi} F(f,\theta)\, d\theta$ for significant wave heights H_S equal to 4, 8, and 12 m and peak frequencies f_0 equal to 0.17, 0.12, and 0.095 Hz, respectively.

ond order in the ratio of wave height to acoustic wavelength. In Sec. III the statistical properties of the received microbarom signal are related to those of the sea state. The source strength spectra for both atmospheric and oceanic microbaroms are obtained and compared. An explanation for the nearly three order of magnitude difference between the atmospheric and the oceanic signal strength is given. In Sec. IV an example is presented: a simple model is considered for the atmospheric microbarom signal received from storms over the deep ocean far from land masses. Section V contains our conclusions.

II. THE SOURCE REGION

Let S be a region of the ocean surface in which the sea is extremely active. Let z represent altitude relative to the undisturbed air/water interface at $z=0$. Assume that the surface of the water has been disturbed in such a way that the actual air/water interface is at $z=\xi(\mathbf{x}_H,t)$. Here, and throughout, $\mathbf{x}_H$ denotes the two-dimensional horizontal coordinate vector with components x and y. The function $\xi(\mathbf{x}_H,t)$ will be said to specify the sea state. A two-dimensional cross section of a portion of S is depicted in Fig. 1.

A. The sea state

Sea states are usually treated statistically[14,15] in the sense that $\xi(\mathbf{x}_H,t)$ is taken to be a stochastic process, commonly assumed to be Gaussian with mean zero. Let ξ_S be this Gaussian process and let $\langle\cdot\rangle_S$ be the expectation value of functions of ξ_S. If one writes

$$\xi_S(\mathbf{x}_H,t) = \mathrm{Re} \int \hat{\xi}_S(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}_H - \omega(\mathbf{k})t)}\, d^2k, \tag{1}$$

then, since it is Gaussian, the expectation value $\langle\cdot\rangle_S$ is completely specified by

$$\langle \hat{\xi}_S(\mathbf{k}) \hat{\xi}_S(\mathbf{q}) \rangle_S = \langle \hat{\xi}_S(\mathbf{k})^* \hat{\xi}_S(\mathbf{q})^* \rangle_S = 0$$

and

$$\langle \hat{\xi}_S(\mathbf{k}) \hat{\xi}_S(\mathbf{q})^* \rangle_S = \mathcal{F}(\mathbf{k})\, \delta(\mathbf{k}-\mathbf{q}). \tag{2}$$

Here $\mathbf{k}$ in (1) is the two dimensional wave vector and $\mathcal{F}(\mathbf{k})$ is the wave number spectral density function;[14,15] $\mathcal{F}(\mathbf{k})$ is real valued. The physical picture is that, under the action of a fairly steady wind over long times and large areas of the sea surface, an approximate equilibrium is reached between the energy being deposited in the sea by the wind and the various linear and nonlinear loss mechanisms. The resulting steady state may be described as a superposition of linear waves, (1), whose statistics are specified by (2).[14,15]

Given a sea state dispersion relation $2\pi f = \omega(\mathbf{k})$, one can relate the wave vector density function $\mathcal{F}(\mathbf{k})$ to the directional spectral density function $F(f,\theta)$ by[14,15]

$$\mathcal{F}(\mathbf{k}) d^2k = F(f,\theta) df d\theta. \tag{3}$$

Here f is frequency in Hz and θ indicates direction of propagation relative to some fixed direction. As a function of θ, $F(f,\theta)$ is generally strongly peaked at angles near those of the direction of the prevailing winds. The integrated spectrum,

$$\bar{F}(f) = \int_0^{2\pi} F(f,\theta)\, d\theta, \tag{4}$$

is known as the frequency spectrum. The "significant wave height"

$$H_S = 4\sqrt{\int \mathcal{F}(\mathbf{k})\, d^2k} = 4\sqrt{\int_0^{\infty} \bar{F}(f)\, df} \tag{5}$$

is commonly used as a measure of how excited the sea state is.

Ocean wave spectra generally have sharp low-frequency cutoffs and high-frequency tails. These features are due to nonlinear effects which cause energy to cascade from longer to shorter wavelengths. For energetic seas the frequency spectrum generally saturates at some limiting form strongly peaked in a narrow frequency band centered at around 0.1 Hz.[14,15] A quasi-empirical form for the frequency spectrum is given by the two-parameter JONSWAP[15,16] model for highly excited seas. JONSWAP spectra are shown in Fig. 2 for several significant wave heights and peak frequencies. Note that while observed microbarom spectra are peaked at about 0.2 Hz, the spectra of the ocean waves that produced them are peaked at about 0.1 Hz. The frequency doubling observed in the microbarom spectra is a consequence of the nonlinearity of the radiation mechanism.[2]

Assuming that the source region is over deep water one has[17] $\omega(\mathbf{k})=\sqrt{g|\mathbf{k}|}$, where g is the acceleration due to gravity at the earth's surface, $g=9.8\ \mathrm{m/s^2}$. At the peak frequency, 0.1 Hz, one has an ocean surface wavelength of $2\pi/|\mathbf{k}| \approx 156$ m. At the microbarom peak frequency of 0.2 Hz the acoustic wavelengths are about 1700 m in the air and about 7500 m in the water. Thus, in the frequency band in which microbarom radiation is significant, the ocean surface wavelengths are much shorter than the acoustic wavelengths. It follows that, if c_{a} and c_{w} are the speeds of sound in the atmosphere and ocean, respectively, the values of $\mathbf{k}$ for which $\mathcal{F}(\mathbf{k})$ is significant satisfy

$$\frac{\omega}{c_{\mathrm{w}}} < \frac{\omega}{c_{\mathrm{a}}} \ll |\mathbf{k}|. \tag{6}$$

Note as well that the effect of gravity is small in the sense that

$$\frac{g}{\omega c_{\mathrm{w}}} < \frac{g}{\omega c_{\mathrm{a}}} \ll 1. \tag{7}$$

The pointlike correlation between sea state wave vector components indicated by the δ function in (2) is equivalent to the translation invariance of the stochastic process. In reality the statistical properties of the sea state are neither temporally nor translationally invariant. However, these changes are negligible over times of several ocean periods (tens of seconds) and distances of several ocean wavelengths (hundreds of meters). It will be assumed that (2) remains valid if $\mathcal{F}$ is allowed to vary slowly with both time and horizontal position.

B. The equations of motion in the source region

The highly active region S of the ocean surface and a shallow layer of the ocean and atmosphere surrounding S will be referred to as the source region. The vertical extent of the source region, above and below the ocean surface, is greater than the significant ocean wavelengths; however, the height to which it extends in the atmosphere is much less than the acoustic wavelength in air and the depth to which it extends in the water is much less than the acoustic wavelength in water. In this region the effects of viscosity, thermal conduction and molecular relaxation can be ignored. Thus, in the source region the air and the water obey the equations of lossless fluid mechanics.[18] One has the equation of continuity

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{v}) = 0, \tag{8}$$

the Euler equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right) + \nabla P = -\rho g \hat{\mathbf{z}}, \tag{9}$$

where g is the acceleration due to gravity, and the thermodynamic equation of state,

$$\rho = f_\sigma(P) \tag{10}$$

where f_σ, for $\sigma=a,w$, are the adiabatic equations of state for air, a, and water, w, respectively. Note that this notation will be used throughout for quantities which are discontinuous across the air/water interface: the subscript σ will be assumed to be a for air and w for water.

The pressure and normal components of the velocity field must be continuous at the air/water interface. Thus one has

$$P(\mathbf{x}_H, \xi+0^+, t) = P(\mathbf{x}_H, \xi-0^+, t), \tag{11}$$

and

$$\mathbf{n}(\mathbf{x}_H,t)\cdot(\mathbf{v}(\mathbf{x}_H,\xi+0^+,t) - \mathbf{v}(\mathbf{x}_H,\xi-0^+,t)) = 0 \tag{12}$$

where $\mathbf{n}$ is an upwardly pointing normal vector to the interface (see Fig. 1) chosen here to be

$$\mathbf{n}(\mathbf{x}_H,t) = \begin{pmatrix} -\boldsymbol{\nabla}_H \xi \\ 1 \end{pmatrix}$$

with

$$\nabla_H = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}.$$

Further, the vertical component of the acceleration of the interface must equal that of the adjoining fluid elements,

$$v_z(\mathbf{x}_H, \xi\pm 0^+, t) = \frac{\partial \xi}{\partial t} + \mathbf{v}(\mathbf{x}_H, \xi \pm 0^+, t)\cdot\boldsymbol{\nabla}\xi. \tag{13}$$

C. The ambient state

A solution to (11)–(13) with $\xi=0$ will be called an ambient state. The corresponding pressure, density, and velocity will be denoted P_0, ρ_0, and $\mathbf{v}_0$, respectively. We will assume that P_0, ρ_0, and $\mathbf{v}_0$ depend only on height/depth z and that $\hat{\mathbf{z}}\cdot\mathbf{v}_0=0$. With these assumptions the equations of fluid mechanics reduce to

$$\frac{dP_0}{dz} = -\rho_0 g \tag{14}$$

for which the general solution is given implicitly by

$$\int_{P_0(0)}^{P_0(z)} \frac{1}{f(P_0)}\, dP_0 = -gz.$$

Letting

$$\rho_\sigma = f_\sigma(P_0(0))$$

and introducing the small-signal sound speeds at the interface

$$c_\sigma = \sqrt{\frac{1}{f'_\sigma(P_0(0))}}$$

one has, for small z,

$$P_0(z) = P_0(0) - \rho_\sigma g z + \frac{\rho_\sigma g^2}{2c_\sigma^2} z^2 + \dots . \tag{15}$$

It will be assumed here that

$$\mathbf{v}_0 = 0. \tag{16}$$

Note that the ambient density and sound speed profiles, $\rho_0 = f_\sigma(P_0(z))$ and $c_0(z) = 1/\sqrt{f'_\sigma(P_0(z))}$, are discontinuous at the air/water interface. By (7) and (15) both are approximately piecewise constant,

$$\rho_0 \approx \begin{cases} \rho_{\mathrm{a}} & \text{if } z > 0, \\ \rho_{\mathrm{w}} & \text{if } z < 0, \end{cases}$$

and

$$c_0 \approx \begin{cases} c_{\mathrm{a}} & \text{if } z > 0, \\ c_{\mathrm{w}} & \text{if } z < 0, \end{cases}$$

in the source region. For later reference note that $f''_\sigma(P_0)$ is related to the "B/A" parameter of nonlinearity[19] through

$$\frac{B}{A} = -\rho_0 c_0^4 f''_\sigma(P_0) \approx -\rho_\sigma c_\sigma^4 f''_\sigma(P_0(0)). \tag{17}$$

The parameter B/A has been tabulated for many gases and fluids.[19,20] For air one has $B/A \approx 0.4$ and for sea water $B/A \approx 5.25$.

D. The order expansion

Solving (11) and (13) in the case in which $\xi \neq 0$ requires some approximation method. It was pointed out in Ref. 2 that the nonlinearity of the equations can be treated using regular perturbation theory to second order.[20,21] The procedure is to perform an expansion about the ambient state,

$$P = P_0 + P_1 + P_2 + \cdots,$$

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots,$$

$$\rho = \rho_0 + \rho_1 + \rho_2 + \cdots,$$

and

$$\xi = \xi_1 + \xi_2 + \cdots,$$

where the subscript 1 indicates solutions of the linear approximation and the subscript 2 indicates terms quadratic in the linear solutions. In the expansion of the sea state ξ it will be assumed[3,5] that the linear approximation ξ_1 is given by the Gaussian process ξ_S specified by (1) and (2). One then substitutes the order expansions into the equations of fluid mechanics and interface conditions and expands. The solution is obtained order by order, beginning with the ambient state at zeroth order.

In expanding the interface conditions some care is required since the interface is in motion. The interface conditions for the full problem are imposed at $z = \xi(\mathbf{x}_H, t)$, however, for the ambient state they are imposed at $z=0$. Thus the order expansion for the interface conditions involves both expanding in powers of ξ about $\xi=0$ as well as in the order expanded variables.

The order parameter, or Mach number, for such a perturbation expansion is the ratio of a typical fluid velocity to the small-signal sound speed.[20] For this problem the order parameter can be taken to be $\omega_0 H_S / c_0$ where $\omega_0 = 2\pi f_0$ and f_0 is the peak frequency of the sea state, about 0.1 Hz. Note that this order parameter is in fact the Mach number at the interface times 2π and that the Mach numbers for air and water are different. The subscript n in the perturbation expansion indicates a term whose magnitude is proportional to $(\omega_0 H_S / c_0)^n$. In addition to Mach number one may exploit the small parameters $\omega_0 / c_\sigma k_0$ and $g/\omega_0 c_\sigma$; here k_0 is equal to $|\mathbf{k}|$ evaluated at the peak frequency f_0. The second-order solutions are required, but only to lowest nonzeroth order in these small parameters.

E. The linear response

In this section the linear approximation will be obtained[17] and put into a suitable form. It has been emphasized[2,4–6] that in the linear approximation the sea does not radiate. Indeed, the first-order acoustic fields are vertically evanescent and are negligible outside of the source region itself.

Substituting the order expansions into the equations of fluid mechanics at first order one has the familiar equations of lossless acoustics. The equation of state, (10), becomes

$$\rho_1 = \frac{1}{c_\sigma^2} P_1,$$

the equation of continuity, (8), becomes

$$\frac{\partial P_1}{\partial t} + \rho_\sigma c_\sigma^2 \boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0,$$

and the Euler equation (9) becomes

$$\boldsymbol{\nabla} P_1 + \rho_\sigma \frac{\partial \mathbf{v}_1}{\partial t} + g \rho_1 \hat{\mathbf{z}} = 0.$$

The interface conditions must be expanded about $\xi=0$ to first order in ξ_1 as well as in the first-order variables. The first-order part of the pressure interface condition (11) is

$$P_1\Big|_{z=-0^+}^{0^+} + \xi_1 \left. \frac{\partial P_0}{\partial z} \right|_{z=-0^+}^{0^+} = 0,$$

which, with (14) gives

$$(P_1 - \rho_\sigma g \xi_1)\Big|_{z=-0^+}^{0^+} = 0. \tag{18}$$

The velocity interface condition, (12), together with the assumption, (16), of no mean flow, gives

$$v_{1z}\Big|_{z=-0^+}^{0^+} = 0 \tag{19}$$

and (13) gives

$$\frac{\partial \xi_1}{\partial t} = v_{1z}(\mathbf{x}_H, 0^+, t) = v_{1z}(\mathbf{x}_H, -0^+, t). \tag{20}$$

Introduce a velocity potential ϕ_1 with $\mathbf{v}_1 = \boldsymbol{\nabla}\phi_1 + \mathbf{w}_1$, with $\boldsymbol{\nabla} \cdot \mathbf{w}_1 = 0$ and with

$$P_1(\mathbf{x}_H, z, t) = -\rho_\sigma \frac{\partial \phi_1}{\partial t}. \tag{21}$$

Choosing

$$\rho_\sigma \frac{\partial \mathbf{w}_1}{\partial t} = -\frac{g}{c_\sigma^2} P_1 \hat{\mathbf{z}} \tag{22}$$

the first-order Euler equation is solved. Substituting into the first-order equation of continuity one obtains the wave equation for the velocity potential,

$$\left(\nabla^2 - \frac{1}{c_\sigma^2}\frac{\partial^2}{\partial t^2}\right)\phi_1 = 0. \tag{23}$$

The system is driven by the interface motion through (18). As in (1) one has the horizontal wave vector expansion

$$\xi_1(\mathbf{x}_H,t) = \mathrm{Re}\int \hat{\xi}_1(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}_H - \omega(\mathbf{k})t)}\, d^2k \tag{24}$$

for the interface. The solutions to the first-order equations can be expanded similarly. Substituting in (23) one obtains

$$\phi_1(\mathbf{x},t) = \mathrm{Re}\int \hat{\phi}_1^{(\sigma)}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}_H - \omega(\mathbf{k})t) - \sqrt{|\mathbf{k}|^2 - \omega^2/c_\sigma^2}|z|}\, d^2k. \tag{25}$$

It follows from (22) that $w_{1z} \sim (g/\omega_0 c_\sigma)(1/\rho_\sigma c_\sigma)P_1$. Substituting (25) into (21) one finds that $\partial\phi_1/\partial z \sim (k_0 c_\sigma/\omega_0)(1/\rho_\sigma c_\sigma)P_1$. It follows that $w_{1z} \sim (g/\omega_0 c_\sigma) \times (\omega_0/k_0 c_\sigma)(\partial\phi_1/\partial z)$ so that, by (6) and (7), $\mathbf{w}_1$ is negligible as compared to $\nabla\phi_1$. It follows that one may write $\mathbf{v}_1 = \nabla\phi_1$.

Expanding with respect to the horizontal wave vector one has

$$P_1(\mathbf{x},t) = \mathrm{Re}\int \hat{P}_1^{(\sigma)}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}_H - \omega(\mathbf{k})t) - \sqrt{|\mathbf{k}|^2 - \omega^2/c_\sigma^2}|z|}\, d^2k \tag{26}$$

and

$$\mathbf{v}_1(\mathbf{x},t) = \mathrm{Re}\int \hat{\mathbf{v}}_1^{(\sigma)}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}_H - \omega(\mathbf{k})t) - \sqrt{|\mathbf{k}|^2 - \omega^2/c_\sigma^2}|z|}\, d^2k. \tag{27}$$

Substituting into (18)–(20) and letting $(-1)^\sigma$ be 1 in the air, $\sigma=a$, and -1 in the water, $\sigma=w$, one obtains

$$\hat{\phi}_1^{(\sigma)}(\mathbf{k}) = (-1)^\sigma \frac{i\omega}{\sqrt{|\mathbf{k}|^2 - \omega^2/c_\sigma^2}}\hat{\xi}_1(\mathbf{k}), \tag{28}$$

$$\hat{P}_1^{(\sigma)}(\mathbf{k}) = -(-1)^\sigma \frac{\rho_\sigma\omega^2}{\sqrt{|\mathbf{k}|^2 - \omega^2/c_\sigma^2}}\hat{\xi}_1(\mathbf{k}), \tag{29}$$

and

$$\hat{\mathbf{v}}_1^{(\sigma)}(\mathbf{k}) = \omega\left(-(-1)^\sigma \frac{\mathbf{k}}{\sqrt{|\mathbf{k}|^2 - \omega^2/c_\sigma^2}} - i\hat{\mathbf{z}}\right)\hat{\xi}_1(\mathbf{k}). \tag{30}$$

By (6) the first-order solutions decrease exponentially with either altitude or depth. At distances large compared to $1/|\mathbf{k}|$ from the interface they become negligible. In particular, the linear solutions do not radiate. The continuity of pressure (18) gives the surf dispersion relation

$$0 = (\rho_\mathrm{w} - \rho_\mathrm{a})g + \frac{\omega^2}{|\mathbf{k}|}\left(\frac{\rho_\mathrm{w}}{\sqrt{1 - \frac{\omega^2}{|\mathbf{k}|^2 c_\mathrm{w}^2}}} + \frac{\rho_\mathrm{a}}{\sqrt{1 - \frac{\omega^2}{|\mathbf{k}|^2 c_\mathrm{a}^2}}}\right)$$

which gives

$$|\mathbf{k}| \approx \frac{\omega^2}{g}$$

so that

$$\omega(\mathbf{k}) \approx \sqrt{g|\mathbf{k}|}. \tag{31}$$

Note that with this dispersion relation the two order parameters $\omega_0/k_0 c_\sigma$ and $g/\omega_0 c_\sigma$ are identical.

Checking the orders of the first-order solutions one finds that

$$\mathbf{v}_1 \sim c_\sigma \frac{\omega_0 H_S}{c_\sigma}$$

while

$$P_1 \sim \rho_\sigma c_\sigma^2 \frac{\omega_0}{k_0 c_\sigma}\frac{\omega_0 H_S}{c_\sigma}.$$

Both are first order in Mach number as expected, however, the pressure P_1 is also first order in the small parameter $\omega_0/k_0 c_\sigma$.

The first-order solutions given by Eqs. (28) and (30) are well approximated by setting $\omega/|\mathbf{k}|c_\sigma = 0$. In the linear approximation this is equivalent to assuming both air and water to be incompressible. In this approximation the acoustic pressure fields, $\hat{P}_1^{(\sigma)}$, are zero and the velocity fields, $\hat{\mathbf{v}}_1^{(\sigma)}$, differ in the water and in the air only by phase: the vertical components are equal while the horizontal components have equal magnitudes but opposite signs. In Ref. 2 it was shown that to obtain the oceanic microbaroms it is sufficient to replace the first-order solutions with their incompressible approximations. It will be seen below that to obtain the atmospheric microbaroms the incompressible approximation to the first-order solutions is not sufficient.

F. The second-order acoustics

It is known[2,5] that in the second-order approximation the sea does radiate. In this section the leading-order corrections to the linear approximation, the terms of second order in Mach number, are obtained. The solution is simplified somewhat by obtaining only the leading-order terms in the small parameter $\omega_0/k_0 c_\sigma$. The general form of the second-order solution is quite complicated, even to leading order in $\omega_0/k_0 c_\sigma$, and will not be given here. Rather, in this section the solution will be presented as a superposition of pairs of plane waves. The explicit determination of the solution is postponed to Sec. III where it is shown that, to determine the statistical properties of the microbarom signal, only the terms corresponding to counter-propagating ocean waves of equal frequency are required.[3]

Continuing the order expansion, at second order one finds[21] that the equation of state, (10), can be written

$$\rho_2 - \frac{1}{c_\sigma^2}P_2 = \frac{1}{2}f''(P_0)P_1^2,$$

the equation of continuity, (8), can be written

$$\frac{\partial P_2}{\partial t} + \rho_\sigma c_\sigma^2 \nabla \cdot \mathbf{v}_2 = \frac{(1-\rho_\sigma c_\sigma^4 f''(P_0))}{\rho_\sigma c_\sigma^2} P_1 \frac{\partial P_1}{\partial t} - \mathbf{v}_1 \cdot \nabla P_1$$

and the Euler equation, (9), can be written

$$\nabla P_2 + \rho_\sigma \frac{\partial \mathbf{v}_2}{\partial t} + g\rho_2 \hat{\mathbf{z}} = -\frac{1}{c_\sigma^2} P_1 \frac{\partial \mathbf{v}_1}{\partial t} - \frac{1}{2}\rho_\sigma \nabla(\mathbf{v}_1 \cdot \mathbf{v}_1).$$

Consider the orders of the various terms in the second-order equations of state and continuity. Note that, recalling (17),

$$\frac{1}{2} f''(P_0) P_1^2 \sim \frac{B}{A}\rho_\sigma \left(\frac{\omega_0}{k_0 c_\sigma}\right)^2 \left(\frac{\omega_0 H_S}{c_\sigma}\right)^2$$

and

$$\frac{(1-\rho_\sigma c_\sigma^4 f''(P_0))}{\rho_\sigma c_\sigma^2} P_1 \frac{\partial P_1}{\partial t} \sim \left(1+\frac{B}{A}\right)\rho_\sigma c_\sigma^2 \omega_0 \left(\frac{\omega_0}{k_0 c_\sigma}\right)^2 \left(\frac{\omega_0 H_S}{c_\sigma}\right)^2.$$

while

$$\mathbf{v}_1 \cdot \nabla P_1 \sim \rho_\sigma c_\sigma^2 \omega_0 \left(\frac{\omega_0 H_S}{c_\sigma}\right)^2.$$

All of these terms are second order in Mach number $\omega_0 H_S / c_\sigma$. However, the two terms involving the second derivative $f''(P_0)$ are second-order in the small parameter $\omega_0 / k_0 c_\sigma$ as well and thus can be dropped. Similarly, on the right side of the second-order Euler equation one has the terms

$$\frac{1}{c_\sigma^2} P_1 \frac{\partial \mathbf{v}_1}{\partial t} \sim \rho_\sigma c_\sigma \omega_0 \frac{\omega_0}{k_0 c_\sigma}\left(\frac{\omega_0 H_S}{c_\sigma}\right)^2$$

and

$$\frac{1}{2}\rho_\sigma \nabla(\mathbf{v}_1 \cdot \mathbf{v}_1) \sim \rho_\sigma c_\sigma^2 k_0 \left(\frac{\omega_0 H_S}{c_\sigma}\right)^2.$$

The first of these terms is again smaller by two orders of the small parameter $\omega_0 / k_0 c_\sigma$ than the second and can be dropped. One thus obtains the simpler set of equations

$$\rho_2 - \frac{1}{c_\sigma^2} P_2 = 0, \tag{32}$$

$$\frac{\partial P_2}{\partial t} + \rho_\sigma c_\sigma^2 \nabla \cdot \mathbf{v}_2 = -\mathbf{v}_1 \cdot \nabla P_1 \tag{33}$$

and

$$\nabla P_2 + \rho_\sigma \frac{\partial \mathbf{v}_2}{\partial t} + g\rho_2 \hat{\mathbf{z}} = -\frac{1}{2}\rho_\sigma \nabla(\mathbf{v}_1 \cdot \mathbf{v}_1). \tag{34}$$

The second-order part of the pressure interface condition (11) is

$$\left(P_2 + \xi_2 \frac{\partial P_0}{\partial z} + \xi_1 \frac{\partial P_1}{\partial z} + \frac{1}{2}\xi_1^2 \frac{\partial^2 P_0}{\partial z^2}\right)\Bigg|_{z=-0^+}^{0^+} = 0.$$

Using the first-order Euler equation one finds

$$\xi_1 \frac{\partial P_1}{\partial z} = -\rho_\sigma \xi_1 \frac{\partial v_{1z}}{\partial t} \sim \rho_\sigma c_\sigma^2 \left(\frac{\omega_0 H_S}{c_\sigma}\right)^2$$

and using (15) one finds

$$\frac{1}{2}\xi_1^2 \frac{\partial^2 P_0}{\partial z^2} = \frac{1}{2}\frac{\rho_\sigma g^2}{c_\sigma^2}\xi_1^2 \sim \rho_\sigma c_\sigma^2 \left(\frac{g}{\omega_0 c_\sigma}\right)^2 \left(\frac{\omega_0 H_S}{c_\sigma}\right)^2.$$

The last term is second order in $g / \omega_0 c_\sigma$ and can be dropped. Using (15), the second-order part of the pressure interface condition can be written

$$\left(P_2 - \rho_\sigma g \xi_2\right)\Big|_{z=-0^+}^{0^+} = \rho_\sigma \xi_1 \frac{\partial v_{1z}}{\partial t}\Bigg|_{z=-0^+}^{0^+}. \tag{35}$$

The second order part of the velocity interface condition (12) is

$$v_{2z}\Big|_{z=-0^+}^{0^+} = \left(-\xi_1 \frac{\partial v_{1z}}{\partial z} + \mathbf{v}_1 \cdot \nabla_H \xi_1\right)\Bigg|_{z=-0^+}^{0^+} \tag{36}$$

and the second-order part of (13) is

$$\frac{\partial \xi_2}{\partial t} = \left(v_{2z} + \xi_1 \frac{\partial v_{1z}}{\partial z} - \mathbf{v}_1 \cdot \nabla_H \xi_1\right)\Bigg|_{z=\pm 0^+}. \tag{37}$$

The procedure is to substitute the linear solutions (25)–(30) into the second-order wave equations (32)–(34) and find the outgoing solution which satisfies the interface conditions (35)–(37). The solutions are simplified somewhat by using the inequality (6) to conclude that compressibility is insignificant in the linear approximation so that $\sqrt{|\mathbf{k}|^2 - \omega(\mathbf{k})^2 / c_\sigma^2} \approx |\mathbf{k}|$. This approximation is valid as long as it gives a nonzero result.

The source terms in (36) and (37), however, can be written

$$-\xi_1 \frac{\partial v_{1z}}{\partial z} + \mathbf{v}_1 \cdot \nabla_H \xi_1 = -\xi_1 \nabla \cdot \mathbf{v}_1 + \nabla_H(\xi_1 \mathbf{v}_1). \tag{38}$$

The term $\nabla \cdot \mathbf{v}_1$ is zero in the incompressible approximation. In the product $\xi_1 \mathbf{v}_1$ the terms which are responsible for the microbarom radiation, those with wave numbers of equal magnitude (and thus equal frequency) but opposite direction,[2,3] are constant so that for these components the term $\nabla_H \cdot (\xi_1 \mathbf{v}_1)$ is zero as well. Thus, while the right side of (36) appears to be large, of order $(\omega_0 / k_0 c_\sigma)^{-1} (\omega_0 H_S / c_\sigma)^2$, in fact its contribution to the microbarom radiation will be seen to be small. For this reason some care must be taken in evaluating the inhomogeneous part of the velocity interface condition (36). In particular, the incompressible approximation cannot be used.

To solve the second-order Euler equation (34) one can write $\mathbf{v}_2 = \nabla \phi_2 + \mathbf{w}_2$ where $\nabla \cdot \mathbf{w}_2 = 0$,

$$\rho_\sigma \frac{\partial \mathbf{w}_2}{\partial t} + \frac{g}{c_\sigma^2} P_2 \hat{\mathbf{z}} = 0,$$

and ϕ_2 is given by

$$P_2 + \rho_\sigma \frac{\partial \phi_2}{\partial t} = -\frac{1}{2}\rho_\sigma \mathbf{v}_1 \cdot \mathbf{v}_1 . \tag{39}$$

With these definitions (34) is satisfied. As in Sec. II E, $\mathbf{w}_2$ can be shown to be smaller than $\nabla \phi_2$, in this case by a single power of $g/\omega_0 c_\sigma$. Consequently, it can be ignored so that one has $\mathbf{v}_2 = \nabla \phi_2$.

Substituting (39) into the second-order equation of continuity (33) and using the first-order Euler equation to simplify, one obtains the second-order wave equation

$$\left(\nabla^2 - \frac{1}{c_\sigma^2}\frac{\partial^2}{\partial t^2}\right)\phi_2 = \frac{1}{c_\sigma^2}\frac{\partial}{\partial t}\mathbf{v}_1 \cdot \mathbf{v}_1 . \tag{40}$$

Note that, since the linear solutions decrease rapidly with distance from the interface, the source term on the right side of (40) does so as well.

The second-order wave equation contains the effects of nonlinearities in the air and the water themselves. The second-order interface conditions contain the nonlinear effects due the fact that the motions of the fluids affect the motion of the interface. To separate the effects of the fluid media from those of the undulating interface one may write the solution of (40) as

$$\phi_2(\mathbf{x}_H, z, t) = \phi_p(\mathbf{x}_H, z, t) + \phi_h(\mathbf{x}_H, z, t), \tag{41}$$

where ϕ_p is a particular solution and ϕ_h is the solution of the homogeneous wave equation required so that the interface conditions (35)–(37) are satisfied.

To determine ϕ_p substitute (27) and (30), into (40). In evaluating the source term the incompressible approximation $\sqrt{|\mathbf{k}|^2 - \omega(\mathbf{k})^2/c_\sigma^2} \approx |\mathbf{k}|$ may be used here since it gives a nonzero result. The source term on the right of (40) becomes

$$\begin{aligned}\frac{1}{c_\sigma^2}\frac{\partial}{\partial t}\mathbf{v}_1 \cdot \mathbf{v}_1 &= \int\int [\mathcal{R}_\sigma^{(+)}(\mathbf{k},\mathbf{q})\hat{\xi}_1(\mathbf{k})\hat{\xi}_1(\mathbf{q}) e^{i((\mathbf{k}+\mathbf{q})\cdot \mathbf{x}_H - (\omega(\mathbf{k})+\omega(\mathbf{q}))t)} \\ &\quad + \mathcal{R}_\sigma^{(-)}(\mathbf{k},\mathbf{q})\hat{\xi}_1(\mathbf{k})\hat{\xi}_1(\mathbf{q})^* e^{i((\mathbf{k}-\mathbf{q})\cdot \mathbf{x}_H - (\omega(\mathbf{k})-\omega(\mathbf{q}))t)}] \\ &\quad \times e^{-(|\mathbf{k}|+|\mathbf{q}|)|z|}\, d^2k\, d^2q + \text{complex conjugate}\end{aligned} \tag{42}$$

with

$$\mathcal{R}_\sigma^{(\pm)}(\mathbf{k},\mathbf{q}) = -\frac{i}{c_\sigma^2}\omega(\mathbf{k})\omega(\mathbf{q})(\omega(\mathbf{k}) \pm \omega(\mathbf{q}))\left(\frac{\mathbf{k}\cdot\mathbf{q}}{|\mathbf{k}||\mathbf{q}|} \mp 1\right).$$

Using (42) the particular solution ϕ_p can be chosen to be

$$\begin{aligned}\phi_p(\mathbf{x}_H, z, t) &= \int\int [\mathcal{Q}_\sigma^{(+)}(\mathbf{k},\mathbf{q})\hat{\xi}_1(\mathbf{k})\hat{\xi}_1(\mathbf{q}) \\ &\quad \times e^{i((\mathbf{k}+\mathbf{q})\cdot \mathbf{x}_H - (\omega(\mathbf{k})+\omega(\mathbf{q}))t) - (|\mathbf{k}|+|\mathbf{q}|)|z|} \\ &\quad + \mathcal{Q}_\sigma^{(-)}(\mathbf{k},\mathbf{q})\hat{\xi}_1(\mathbf{k})\hat{\xi}_1(\mathbf{q})^* \\ &\quad \times e^{i((\mathbf{k}-\mathbf{q})\cdot \mathbf{x}_H - (\omega(\mathbf{k})-\omega(\mathbf{q}))t) - (|\mathbf{k}|+|\mathbf{q}|)|z|}]\, d^2k\, d^2q \\ &\quad + \text{complex conjugate}\end{aligned} \tag{43}$$

with

$$\mathcal{Q}_\sigma^{(\pm)}(\mathbf{k},\mathbf{q}) = \frac{\mathcal{R}_\sigma^{(\pm)}(\mathbf{k},\mathbf{q})}{[(\omega(\mathbf{k}) \pm \omega(\mathbf{q}))^2/c_\sigma^2] + 2(|\mathbf{k}||\mathbf{q}| \mp \mathbf{k}\cdot\mathbf{q})}.$$

The homogeneous solution ϕ_h is then of the form

$$\begin{aligned}\phi_h(\mathbf{x}_H, z, t) &= \int\int [C_\sigma^{(+)}(\mathbf{k},\mathbf{q})\hat{\xi}_1(\mathbf{k})\hat{\xi}_1(\mathbf{q}) \\ &\quad \times e^{i((\mathbf{k}+\mathbf{q})\cdot \mathbf{x}_H + \Omega_+(\mathbf{k},\mathbf{q})|z| - (\omega(\mathbf{k})+\omega(\mathbf{q}))t)} \\ &\quad + C_\sigma^{(-)}(\mathbf{k},\mathbf{q})\hat{\xi}_1(\mathbf{k})\hat{\xi}_1(\mathbf{q})^* \\ &\quad \times e^{i((\mathbf{k}-\mathbf{q})\cdot \mathbf{x}_H + \Omega_-(\mathbf{k},\mathbf{q})|z| - (\omega(\mathbf{k})-\omega(\mathbf{q}))t)}]\, d^2k\, d^2q \\ &\quad + \text{complex conjugate}\end{aligned} \tag{44}$$

with

$$\Omega_\pm(\mathbf{k},\mathbf{q}) = \sqrt{\frac{(\omega(\mathbf{k}) \pm \omega(\mathbf{q}))^2}{c_\sigma^2} - (\mathbf{k} \pm \mathbf{q})^2}$$

and the second-order contribution to the sea state is given by

$$\begin{aligned}\xi_2(x_H, t) &= \int\int [\hat{\xi}^{(+)}(\mathbf{k},\mathbf{q})\hat{\xi}_1(\mathbf{k})\hat{\xi}_1(\mathbf{q}) \\ &\quad \times e^{i((\mathbf{k}+\mathbf{q})\cdot \mathbf{x}_H - (\omega(\mathbf{k})+\omega(\mathbf{q}))t)} + \hat{\xi}^{(-)}(\mathbf{k},\mathbf{q})\hat{\xi}_1(\mathbf{k})\hat{\xi}_1(\mathbf{q})^* \\ &\quad \times e^{i((\mathbf{k}-\mathbf{q})\cdot \mathbf{x}_H - (\omega(\mathbf{k})-\omega(\mathbf{q}))t)}]\, d^2k\, d^2q \\ &\quad + \text{complex conjugate}.\end{aligned} \tag{45}$$

The coefficients $C_\sigma^{(\pm)}(\mathbf{k},\mathbf{q})$ and $\hat{\xi}^{(\pm)}(\mathbf{k},\mathbf{q})$ are determined by the condition that the interface conditions (35)–(37) must be satisfied. The explicit determination of $C_\sigma^{(\pm)}(\mathbf{k},\mathbf{q})$ and $\hat{\xi}^{(\pm)}(\mathbf{k},\mathbf{q})$ will be postponed until the next section where it is shown that, to determine the statistical properties of the microbarom signal, only the coefficients $C_\sigma^{(+)}(\mathbf{k},-\mathbf{k})$ and $\hat{\xi}^{(+)}(\mathbf{k},-\mathbf{k})$ are required. Note that these particular coefficients correspond to the interaction of ocean waves of equal frequency and opposite propagation direction.

The decomposition (41) with the choice (43) for the particular solution separates the second-order velocity potential into a term, the particular solution, which is negligible outside of the source region and a term, the homogeneous solution, which contains the part of the field which radiates into the atmosphere and into the ocean. The particular solution contains the nonlinear effects produced in the air or water in the bulk rather than at the interface. These volume contributions do not directly produce acoustic radiation. The radiation is produced by the requirement that the interface conditions be satisfied. Note that the homogeneous solution must both account for nonlinear contributions to as well as correct for the deviations of the particular solution from the interface conditions. Specifically, the air and the water must accommodate both the nonlinear contributions to the interface motion as well as the nonlinear contributions to the flow in the bulk.

III. THE RECEIVED MICROBAROM SIGNAL

Let $P_\sigma(\mathbf{x}_H,z,t)$ be the acoustic pressure at large distances from the source region, either in the atmosphere, $\sigma=a$ and $z\geq 0$, or in the ocean, $\sigma=w$ and $z\leq 0$. To determine P_σ one must find the outward-propagating solution to the equations for acoustic propagation in the atmosphere/ocean which reduces in the source region to the solution produced in Sec. II F. One may write

$$P_\sigma = P_{\sigma p} + P_{\sigma h},$$

where

(i) in the source region $P_{\sigma p}$ and $P_{\sigma h}$ reduce to the solutions produced in Sec. II F,

$$P_{\sigma p} = -\rho_\sigma \frac{\partial \phi_p}{\partial t} - \frac{1}{2}\rho_\sigma \mathbf{v}_1\cdot\mathbf{v}_1$$

and

$$P_{\sigma h} = -\rho_\sigma \frac{\partial \phi_h}{\partial t},$$

where ϕ_p and ϕ_h are given by (43) and (44);

(ii) away from the source region $P_{\sigma p}$ is negligible, so that $P_\sigma = P_{\sigma h}$, and $P_{\sigma h}$ is taken to satisfy the rigid ground boundary condition at the atmospheric side of the air/water interface and the pressure release boundary condition at the oceanic side

$$0 = \left.\frac{\partial P_{\sigma h}(\mathbf{x}_H,z,t)}{\partial z}\right|_{z=0^+} = P_{\sigma h}(\mathbf{x}_H,z,t)|_{z=-0^+}$$

Let $G_{\mathrm{a}}(\mathbf{x}_H,z,\mathbf{x}'_H,z',t)$ be the Green's function describing the propagation of sound in the atmosphere over a rigid surface and let $G_{\mathrm{w}}(\mathbf{x}_H,z,\mathbf{x}'_H,z',t)$ be the Green's function describing the propagation of sound in the ocean under a pressure release surface. Let S be the part of the $z=0$ plane which is in the source region. Then the Helmholtz-Kirchoff integral theorem gives

$$P_{\sigma h}(\mathbf{x}_H,z,t) = -\rho_\sigma \int_{-\infty}^{\infty}\int_S G_\sigma(\mathbf{x}_H,z,\mathbf{y}_H,0,t-\tau) \times \frac{\partial v_\sigma(\mathbf{y}_H,\tau)}{\partial \tau}\, d^2\mathbf{y}_H\, d\tau, \tag{46}$$

where

$$v_\sigma(\mathbf{x}_H,t) = \left.\frac{\partial \phi_h(\mathbf{x}_H,z,t)}{\partial z}\right|_{(-1)^\sigma z=0^+} \tag{47}$$

is the normal component of the homogeneous part of the fluid velocity at the interface.

A. The statistics of the microbarom signal and the sea state

Let $\langle\cdot\rangle_T$ represent both the sea state ensemble average, $\langle\cdot\rangle_S$, given in (2) as well as, if required, an average, $\langle\cdot\rangle_P$, over fluctuations in the propagation medium. Of interest is the correlation

$$\langle P_\sigma(\mathbf{x}_H,z,0)P_\sigma(\mathbf{x}'_H,z',\tau)^*\rangle_T = \lim_{T\to\infty}\frac{1}{T}\int_0^T P_\sigma(\mathbf{x}_H,z,t)P_\sigma(\mathbf{x}'_H,z',t+\tau)^*\,dt \tag{48}$$

between the signals received at (possibly) spatially separated points (see Fig. 3). The Fourier transform of (48) with respect to τ is known as the cross spectral density; for $\mathbf{x}'_H=\mathbf{x}_H$ it is the power spectrum.[22] Substituting (46) and (47), into (48) one has

$$\begin{aligned}\langle P_\sigma(\mathbf{x}_H,z,0)&P_\sigma(\mathbf{x}'_H,z',\tau)^*\rangle_T\\ &= \rho_\sigma^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_S\int_S \langle G_\sigma(\mathbf{x}_H,z,\mathbf{y}_H,0,-\tau_1)\\ &\quad\times G_\sigma(\mathbf{x}'_H,z',\mathbf{y}'_H,0,\tau-\tau_2)^*\rangle_P\\ &\quad\times\left\langle \frac{\partial v_\sigma(\mathbf{y}_H,\tau_1)}{\partial\tau_1}\frac{\partial v_\sigma(\mathbf{y}'_H,\tau_2)^*}{\partial \tau_2}\right\rangle_S d^2y_H\,d^2y'_H\,d\tau_1\,d\tau_2.\end{aligned} \tag{49}$$

To compute v_σ in (47) one may substitute from (44) for the velocity potential ϕ_h. Note that v_σ is a binomial in the linear sea state amplitudes $\hat{\xi}_1$. As discussed in Sec. II D, to compute the average over the sea state in (49) we assume that $\hat{\xi}_1$ is the Gaussian process $\hat{\xi}_S$ given by (2). Since $\langle\cdot\rangle_S$ is Gaussian[23]

$$\begin{aligned}\langle\hat{\xi}_1(\mathbf{k}_1)&\hat{\xi}_1(\mathbf{q}_1)\hat{\xi}_1(\mathbf{k}_2)^*\hat{\xi}_1(\mathbf{q}_2)^*\rangle_S\\ &=\langle\hat{\xi}_1(\mathbf{k}_1)\hat{\xi}_1(\mathbf{k}_2)^*\rangle_S\langle\hat{\xi}_1(\mathbf{q}_1)\hat{\xi}_1(\mathbf{q}_2)^*\rangle_S\\ &\quad+\langle\hat{\xi}_1(\mathbf{k}_1)\hat{\xi}_1(\mathbf{q}_2)^*\rangle_S\langle\hat{\xi}_1(\mathbf{q}_1)\hat{\xi}_1(\mathbf{k}_2)^*\rangle_S\\ &=\mathcal{F}(\mathbf{k}_1)\mathcal{F}(\mathbf{q}_1)(\delta(\mathbf{k}_1-\mathbf{k}_2)\delta(\mathbf{q}_1-\mathbf{q}_2)+\delta(\mathbf{k}_1-\mathbf{q}_2)\\ &\quad\times\delta(\mathbf{q}_1-\mathbf{k}_2))\end{aligned}$$

so that one has

$$\begin{aligned}\langle v_\sigma(\mathbf{y}_H,\tau_1)v_\sigma(\mathbf{y}'_H,\tau_2)^*\rangle_S &= 2\int\int \mathcal{F}(\mathbf{k})\mathcal{F}(\mathbf{q})\\ &\quad\times|C_\sigma^{(+)}(\mathbf{k},\mathbf{q})|^2|\Omega_+(\mathbf{k},\mathbf{q})|^2\\ &\quad\times e^{i((\mathbf{k}+\mathbf{q})\cdot(\mathbf{y}_H-\mathbf{y}'_H)-(\omega(\mathbf{k})+\omega(\mathbf{q}))(\tau_1-\tau_2))}\,d^2k\,d^2q\\ &\quad+\text{complex conjugate}+\text{time-independent terms}\end{aligned} \tag{50}$$

for the correlation between the normal velocity field v_σ at different positions and times.

Substitute (50) into (49). The integrals over τ_1 and τ_2 are Fourier transforms in time of the G_σ. Let $\hat{G}_\sigma(\mathbf{x}_H,z,\mathbf{x}'_H,z',\nu)$ be these Fourier transforms at angular frequency ν. Since the Green's functions for the acoustic propagation are approximately constant over distances which are small compared to acoustic wavelengths and since the significant sea state wave vectors $\mathbf{K}$ satisfy (6), one has,[24] to leading order in ω_0/k_0c_σ,

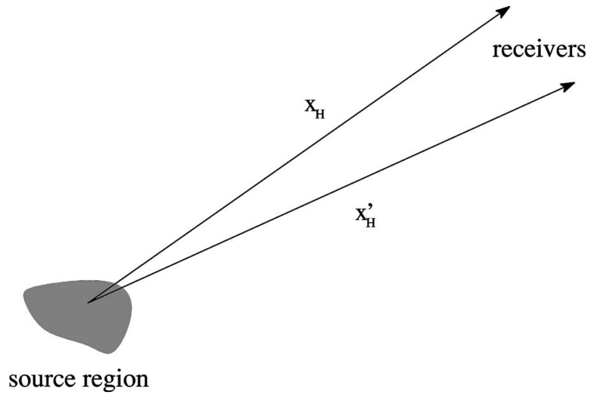

FIG. 3. The geometry. Microbaroms radiated from the source region are detected by distant sensors.

$$\int_S \int_S \hat{G}_\sigma(\mathbf{x}_H, z, \mathbf{y}_H, 0, \nu) \hat{G}_\sigma(\mathbf{x}'_H, z', \mathbf{y}'_H, 0, \nu)^* \times e^{i\mathbf{k}\cdot(\mathbf{y}_H - \mathbf{y}'_H)}\, d^2\mathbf{y}_H\, d^2\mathbf{y}'_H \approx (2\pi)^2 \delta(\mathbf{k}) \int_S \hat{G}_\sigma(\mathbf{x}_H, z, \mathbf{y}_H, 0, \nu) \times \hat{G}_\sigma(\mathbf{x}'_H, z', \mathbf{y}_H, 0, \nu)^*\, d^2\mathbf{y}_H.$$

It follows that

$$\langle P_\sigma(\mathbf{x}_H, z, 0) P_\sigma(\mathbf{x}'_H, z', \tau)^* \rangle_T = \frac{32\rho_\sigma^2 \pi^2}{c_\sigma^2} \int \left(\int_S \langle \hat{G}_\sigma(\mathbf{x}_H, z, \mathbf{y}_H, 0, 2\omega(\mathbf{k})) \times \hat{G}_\sigma(\mathbf{x}'_H, z', \mathbf{y}_H, 0, 2\omega(\mathbf{k}))^* \rangle_P\, d^2\mathbf{y}_H \right) \times \mathcal{F}(\mathbf{k}) \mathcal{F}(-\mathbf{k}) |C_\sigma^{(+)}(\mathbf{k}, -\mathbf{k})|^2 \omega(\mathbf{k})^4 e^{-i2\omega(\mathbf{k})\tau}\, d^2k. \quad (51)$$

Equation (51) was first obtained by Hasselmann for the oceanic microbarom signal. It gives a direct relation between the spectral density function of the sea state and the statistical properties of the received microbarom signals.

The sharp low-frequency cutoff in the wave spectrum means that there is a smallest $|\mathbf{k}|$ and ω, corresponding to a longest wavelength and period in the ocean wave spectrum. At separations, $|\mathbf{y}_H - \mathbf{y}'_H|$, much longer than this longest wavelength or at time delays much greater than this longest period the motions of the sea surface become uncorrelated.[14] Thus, the normal velocity (50) becomes uncorrelated at separations large compared to the longest wavelength and at time delays long compared to the longest period. However, by (6) these separations are small compared to acoustic wavelengths. The physical picture which emerges from the stochastic sea state model is one of intermittent radiation, persisting for times long compared with the longest ocean wave period, from patches of the ocean surface randomly distributed across S, whose diameters are larger than the longest ocean wave wavelengths but smaller than acoustic wavelengths. This picture is consistent with the observations of individual microbarom wave trains reported in Ref. 25. Recall that radiation produced by patches which are small compared to acoustic wavelengths can be represented at large distances as a acoustic monopole fields. Thus the microbarom signal can be thought of as an incoherent superposition of fields produced by monopole sources at the sea surface (see also Ref. 5), precisely as presented by (51).

B. Solving the reduced problem

Taking the dispersion relation $\omega(\mathbf{k})$ to be given by the deep water dispersion relation, (31), all that remains is a determination of the coefficients $C_\sigma^{(\pm)}(\mathbf{k}, \mathbf{q})$ in (44). The task is greatly simplified by the fact that we need only $C_\sigma^{(+)}(\mathbf{k}, -\mathbf{k})$. Hence, only the sum frequency components for sea state wave vectors of equal magnitude and opposite direction are required. Referring to Eqs. (44) and (45), one sees that the restriction to interface waves of equal and opposite wave vectors reduces the acoustic problem to one of plane waves propagating normal to the air/water interface. The determination of the reduced set of coefficients is thus equivalent to the one-dimensional problem of finding the outward propagating velocity potential

$$\phi = C_\sigma^{(+)}(\mathbf{k}, -\mathbf{k}) e^{i[2\omega(\mathbf{k})/c_\sigma]|z| - i2\omega(\mathbf{k})t} \quad (52)$$

and uniformly vibrating surface displacement

$$\xi = \hat{\xi}^{(+)}(\mathbf{k}, -\mathbf{k}) e^{-i2\omega(\mathbf{k})t}, \quad (53)$$

which satisfy the interface conditions.

The interface conditions simplify as well. Substituting (39) and (41) into the pressure condition (35) one obtains

$$\left(\rho_\sigma \frac{\partial \phi_h}{\partial t} + \rho_\sigma g \xi_2 \right) \Bigg|_{z=-0^+}^{0^+} = \left(-\rho_\sigma \frac{\partial \phi_p}{\partial t} - \frac{1}{2}\rho_\sigma \mathbf{v}_1 \cdot \mathbf{v}_1 - \rho_\sigma \xi_1 \frac{\partial v_{1z}}{\partial t} \right) \Bigg|_{z=-0^+}^{0^+}.$$

Referring to (43), the relevant component of ϕ_p, to leading order in $\omega_0/k_0 c_\sigma$, is

$$i\frac{\omega(\mathbf{k})^3}{|\mathbf{k}|^2 c_\sigma^2} e^{-2|\mathbf{k}||z| - i2\omega(\mathbf{k})t}.$$

Substituting the linear form (27) in $\frac{1}{2}\rho_\sigma \mathbf{v}_1 \cdot \mathbf{v}_1 + \rho_\sigma \xi_1 \partial v_{1z}/\partial t$, one finds the relevant component to be

$$2\rho_\sigma \omega(\mathbf{k})^2 e^{i[2\omega(\mathbf{k})/c_\sigma]|z| - i2\omega(\mathbf{k})t}.$$

Thus, recalling that the density of water is much greater than that of air and noting that the contribution from ϕ_p is higher order in $\omega_0/k_0 c_\sigma$, the reduced velocity potential (52) and surface displacement (53) satisfy

$$\left(\rho_\sigma \frac{\partial \phi}{\partial t} + \rho_\sigma g \xi \right) \Bigg|_{z=-0^+}^{0^+} = 2\rho_{\mathrm{w}} \omega(\mathbf{k})^2 e^{-i2\omega(\mathbf{k})t}. \quad (54)$$

Similarly, substituting (41) into the velocity condition (36) one obtains

$$\left.\frac{\partial \phi_h}{\partial z}\right|_{z=-0^+}^{0^+} = \left.\left(-\frac{\partial \phi_p}{\partial z} - \xi_1 \frac{\partial v_{1z}}{\partial z} + \mathbf{v}_1 \cdot \nabla_H \xi_1\right)\right|_{z=-0^+}^{0^+}$$

Using (38) one needs only the relevant component of $-\xi_1 \nabla \cdot \mathbf{v}_1 = (1/\rho_\sigma c_\sigma^2)(\partial P_1/\partial t)$. Referring to (29) one finds this component to be

$$(-1)^\sigma i \frac{\omega(\mathbf{k})^3}{|\mathbf{k}| c_\sigma} e^{i[2\omega(\mathbf{k})/c_\sigma]|z| - i2\omega(\mathbf{k})t}.$$

Adding the contribution from ϕ_p and dropping $1/c_{\mathrm{w}}^2$ relative to $1/c_{\mathrm{a}}^2$ one obtains

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-0^+}^{0^+} = 3i \frac{\omega(\mathbf{k})^3}{|\mathbf{k}| c_{\mathrm{a}}^2} e^{-i2\omega(\mathbf{k})t} \tag{55}$$

and

$$\frac{\partial \xi}{\partial t} = \left.\left(\frac{\partial \phi}{\partial z} - 3i \frac{\omega(\mathbf{k})^3}{|\mathbf{k}| c_\sigma} e^{-i2\omega(\mathbf{k})t}\right)\right|_{z=\pm 0^+}. \tag{56}$$

Substitute (52) and (53), into (54)–(56) to obtain

$$\rho_{\mathrm{a}}[-2i\omega(\mathbf{k}) C_{\mathrm{a}}^{(+)}(\mathbf{k},-\mathbf{k}) + g\hat{\xi}^{(+)}(\mathbf{k},-\mathbf{k})] + \rho_{\mathrm{w}}[2i\omega(\mathbf{k}) C_{\mathrm{w}}^{(+)}(\mathbf{k},-\mathbf{k}) - g\hat{\xi}^{(+)}(\mathbf{k},-\mathbf{k})] = 2\rho_{\mathrm{w}}\omega(\mathbf{k})^2,$$

$$\frac{2}{c_{\mathrm{a}}} C_{\mathrm{a}}^{(+)}(\mathbf{k},-\mathbf{k}) + \frac{2}{c_{\mathrm{w}}} C_{\mathrm{w}}^{(+)}(\mathbf{k},-\mathbf{k}) = 3\frac{\omega(\mathbf{k})^2}{|\mathbf{k}| c_{\mathrm{a}}^2}$$

$$\hat{\xi}^{(+)}(\mathbf{k},-\mathbf{k}) = -\frac{1}{c_{\mathrm{w}}} C_{\mathrm{w}}^{(+)}(\mathbf{k},-\mathbf{k}) + \frac{3\omega(\mathbf{k})^2}{2|\mathbf{k}| c_{\mathrm{w}}^2}.$$

Solving, one finds to leading order

$$C_{\mathrm{a}}^{(+)}(\mathbf{k},-\mathbf{k}) = i\omega(\mathbf{k})\left(\frac{c_{\mathrm{a}}}{c_{\mathrm{w}}} - \frac{3}{2} i \frac{\omega(\mathbf{k})}{|\mathbf{k}| c_{\mathrm{a}}}\right) \tag{57}$$

$$C_{\mathrm{w}}^{(+)}(\mathbf{k},-\mathbf{k}) = -i\omega(\mathbf{k}), \tag{58}$$

and

$$\hat{\xi}(\mathbf{k},-\mathbf{k}) = i\frac{\omega(\mathbf{k})}{c_{\mathbf{w}}}.$$

The atmospheric and oceanic velocity potential amplitudes (57) and (58) are of rather different forms. The oceanic microbarom velocity potential amplitude (58) is equivalent to that obtained by Longuet-Higgens[2] (it is also equivalent to that obtained, incorrectly, by Arendt and Fritts[6] for the atmospheric microbarom amplitude). The imaginary part of the atmospheric microbarom velocity potential amplitude (57) is equivalent to the form predicted by Brekhovskikh *et al.* in Ref. 5 for the normally propagating component of the atmospheric radiation.

The ratio of the velocity potential amplitudes (57) and (58), is given by the term $c_{\mathrm{a}}/c_{\mathrm{w}} - \frac{3}{2} i\omega(\mathbf{k})/|\mathbf{k}| c_{\mathrm{a}}$. This term is plotted in Fig. 4 assuming the deep water dispersion relation (31). Note that for frequencies near 0.1 Hz the velocity potential amplitude in the atmosphere is smaller than in the ocean by about a factor of four. The real and imaginary parts, $c_{\mathrm{a}}/c_{\mathrm{w}}$ and $-\frac{3}{2}\omega(\mathbf{k})/|\mathbf{k}| c_{\mathrm{a}}$, respectively, of the ratio of (57) and (58), are of comparable magnitude. The real part $c_{\mathrm{a}}/c_{\mathrm{w}}$ is the larger of the two, by about a factor of 3, and represents the contribution from sound radiated from the ocean into the atmosphere; that such a term is significant was pointed out in Ref. 5. The imaginary part $-\frac{3}{2}\omega(\mathbf{k})/|\mathbf{k}| c_{\mathrm{a}}$ represents the compression of the air by the ocean waves.

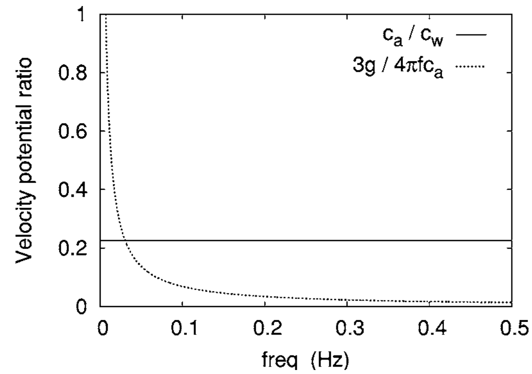

FIG. 4. The real and imaginary parts of the complex conjugate (so that both parts are positive) of the ratio, $c_{\mathrm{a}}/c_{\mathrm{w}} - \frac{3}{2} ig/2\pi f c_{\mathrm{a}}$, of velocity potential amplitude in the atmosphere to that in the ocean for an infinitely deep ocean plotted as a function of frequency f.

Generally speaking, because of the enormous impedance contrast between air and water, the acoustic pressure radiated into the air by sound in the water is reduced by a factor of $\rho_{\mathrm{a}} c_{\mathrm{a}}/\rho_{\mathrm{w}} c_{\mathrm{w}}$. In this case, however, the effective interface pressure source given by the right side of (35) is proportional to the density of the water, ρ_{w}. This is the origin of the factor $c_{\mathrm{a}}/c_{\mathrm{w}}$ in (57) and illustrates the basic physics of the problem. Once the interface is in motion the air and water in the source region must have roughly the same velocity fields [up to phase, see (30)]. However, the changes in pressure required to support changes in velocity are proportional to density. Thus the sound pressure fluctuations in the water associated with the motion of the interface are greater than those in the air by a factor of $\rho_{\mathrm{w}}/\rho_{\mathrm{a}} \approx 1000$.

The compressibility of the air arises in (57) because, as discussed above, the horizontally constant part of the effective interface velocity source given by the right side of (36) is proportional to $\nabla \cdot \mathbf{v}_1$, which is zero in the incompressible approximation. Indeed, to have a horizontally constant velocity source requires the fluid to be compressible.

C. The source spectra

Substituting (57) and (58), in (51), using the deep water dispersion relation (31) to relate the wave vector spectral density to the frequency direction spectral density through $\mathbf{k} = (4\pi^2/g) f^2 (\cos\theta, \sin\theta)$ and

$$\mathcal{F}(\mathbf{k}) = \frac{g^2}{32\pi^4 f^3} F(f,\theta),$$

and choosing as integration variable the frequency of the received acoustic signal one finds

$$\langle P_\sigma(\mathbf{x}_H,z,0)P_\sigma(\mathbf{x}'_H,z',\tau)^*\rangle_T = \int_S\int_0^\infty \mathcal{Q}_\sigma(\mathbf{x}_H,\mathbf{x}'_H,z,z',\mathbf{y}_H,f)\mathcal{D}_\sigma(f)e^{-i2\pi f\tau}\,df\,d^2\mathbf{y}_H, \quad (59)$$

where $\mathcal{Q}_\sigma$ is the cross spectral density,[22]

$$\mathcal{Q}_\sigma(\mathbf{x}_H,\mathbf{x}'_H,z,z',\mathbf{y}_H,f) = \langle \hat{G}_\sigma(\mathbf{x}_H,z,\mathbf{y}_H,0,2\pi f)\hat{G}_\sigma(\mathbf{x}'_H,z',\mathbf{y}_H,0,2\pi f)^*\rangle_P, \quad (60)$$

for propagation from a point source at $\mathbf{y}_H \in S$ to receivers at $(\mathbf{x}_H,z)$ and $(\mathbf{x}'_H,z')$, and $\mathcal{D}_\sigma$ is the source strength spectrum squared

$$\mathcal{D}_\sigma(f) = \int_0^{2\pi} F\left(\frac{f}{2},\theta\right)F\left(\frac{f}{2},\theta+\pi\right)d\theta \cdot \begin{cases} \dfrac{4\rho_a^2 g^2\pi^4 f^3}{c_a^2}\left(\dfrac{9g^2}{4\pi^2 c_a^2 f^2}+\dfrac{c_a^2}{c_w^2}\right) & \text{in the atmosphere,} \quad \sigma=a \\ \dfrac{4\rho_w^2 g^2\pi^4 f^3}{c_w^2} & \text{in the ocean, } \sigma=w. \end{cases} \quad (61)$$

A similar form for the source strength spectrum $\mathcal{D}_w$ for radiation into the ocean was first obtained by Hasselmann in Ref. 3. If $\hat{P}_\sigma$ is the Fourier transform in time of the received acoustic pressure P_σ, then[22] the cross spectral density of the received microbarom signals is given by

$$\langle \hat{P}_\sigma(\mathbf{x}_H,z,2\pi f)\hat{P}_\sigma(\mathbf{x}'_H,z',2\pi f)^*\rangle_T = \int_S \mathcal{Q}_\sigma(\mathbf{x}_H,\mathbf{x}'_H,z,z',\mathbf{y}_H,f)\mathcal{D}_\sigma(f)\,d^2\mathbf{y}_H. \quad (62)$$

The integral over S in (62) is required because, in general, as discussed above, $\mathcal{D}_\sigma$ is not constant over the source region, but varies slowly enough to be considered constant over distances of many ocean wavelengths.

Consider the ratio of the atmospheric to oceanic source strength spectra. Note that this ratio is independent of the sea state. One has

$$\sqrt{\frac{\mathcal{D}_a(f)}{\mathcal{D}_w(f)}} = \frac{\rho_a c_w}{\rho_w c_a}\sqrt{\frac{9g^2}{4\pi^2 c_a^2 f^2}+\frac{c_a^2}{c_w^2}}. \quad (63)$$

The source strength spectra ratio (63) is plotted in Fig. 5. For higher frequencies the ratio is asymptotic to the incompressible value $\rho_a/\rho_w \approx 10^{-3}$. In the frequency range of interest, 0.1 to 0.3 Hz, the ratio is always less than 2×10^{-3}. This explains the observation, pointed out in the Introduction, that atmospheric microbarom levels are generally three orders of magnitude lower than oceanic levels.

The integral,

$$\int_0^{2\pi} F\left(\frac{f}{2},\theta\right)F\left(\frac{f}{2},\theta+\pi\right)d\theta, \quad (64)$$

first obtained by Hasselmann,[3] is the only term in (61) which is not explicit. It is a factor common to both atmospheric and oceanic radiation and is a measure of the density of counter propagating waves at frequency f. Its determination is problematic. The largest component of a wave field consists of traveling waves propagating in the direction of the prevailing wind. Evaluating (64) requires knowing the tails of the angular part of the wave vector distribution. The determination of (64) thus depends on quantities which are difficult to determine, either experimentally or numerically.[7,15,26,27] It is possible that observed atmospheric and oceanic microbarom levels will ultimately be used to constrain ocean wave models.

IV. A SIMPLE MODEL FOR THE ATMOSPHERIC MICROBAROM SIGNAL FROM STORMS OVER THE OPEN OCEAN

In this section an example calculation is presented to illustrate an application of (61) and (62) to the atmospheric microbarom signal. A simple model is considered for the signal received from a distant isolated storm over the deep ocean far from any land masses. It is assumed that the source region is roughly centered at the origin $\mathbf{y}_H=\mathbf{0}$, that the horizontal extent of the source region is small compared with the distance $|\mathbf{x}_H|$ to the receiver, and that the source spectrum is constant over the source region.

To compute the power spectrum,

$$\langle|\hat{P}_\sigma(\mathbf{x}_H,z,2\pi f)|^2\rangle_T = \int_S \mathcal{Q}_\sigma(\mathbf{x}_H,\mathbf{x}_H,z,z,\mathbf{y}_H,f)\mathcal{D}_\sigma(f)\,d^2\mathbf{y}_H,$$

of the received signal both the power spectral density, $\mathcal{Q}_\sigma(\mathbf{x}_H,\mathbf{x}_H,z,z,\mathbf{y}_H,f)$, for acoustic propagation from $(\mathbf{y}_H,0)$ to $(\mathbf{x}_H,z)$ as well as the source spectrum, $\mathcal{D}_\sigma(f)$, are required. Let $|S|$ be the area of the source region interface S. As a consequence of the assumption that the source spectrum is constant one has

$$\langle|\hat{P}_\sigma(\mathbf{x}_H,z,2\pi f)|^2\rangle_T = |S|\mathcal{Q}_\sigma(\mathbf{x}_H,\mathbf{x}_H,z,z,\mathbf{0},f)\mathcal{D}_\sigma(f).$$

The term $\sqrt{\mathcal{Q}_a(\mathbf{x}_H,\mathbf{x}_H,0,0,\mathbf{y}_H,f)}$ will be referred to here as the propagation factor so that to obtain the received microbarom power spectral density one simply takes the product of source strength squared, propagation factor squared, and area $|S|$. Note that -20 times the logarithm of the propagation factor is generally called the transmission loss.[28]

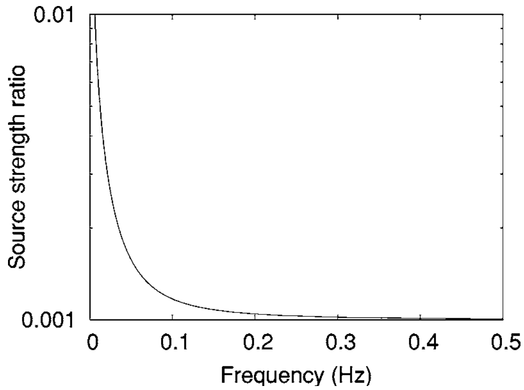

FIG. 5. The ratio of the microbarom source strength spectra for radiation into the atmosphere versus radiation into the ocean.

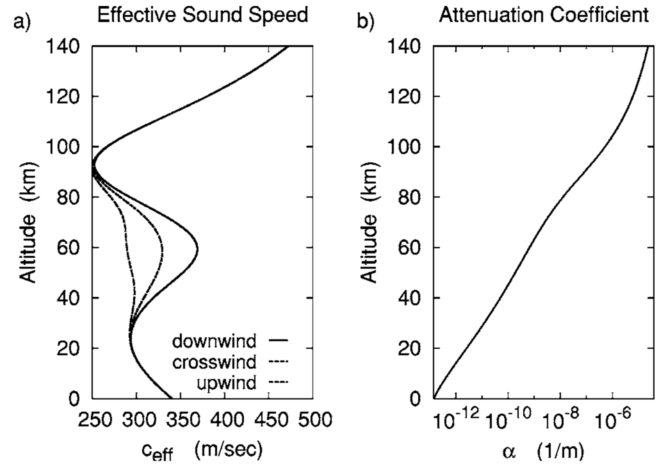

FIG. 6. (a) The model effective sound speed for downwind, crosswind, and upwind propagation. (b) The classical attenuation coefficient, as a function of altitude.

A. Propagation model

To model the propagation a stratified model will be used.[29] In this model the mean thermodynamic properties of the atmosphere, mean temperature, mean pressure, mean density ρ_a, and mean entropy, are assumed to depend only on altitude. It is assumed that the mean winds $\mathbf{v}_0$ have no vertical component and that the horizontal components again depend only on altitude.[30,31] The frequencies of interest here are much larger than the Brunt-Väisälä frequency in the atmosphere so that the effects of buoyancy can be ignored. Thus one may set $g=0$ in the equation for the Green's function. To simplify the model further the effective sound speed approximation, in which the wind velocity in the direction of horizontal propagation,

$$\hat{\mathbf{k}} = \frac{\mathbf{x}_H - \mathbf{y}_H}{|\mathbf{x}_H - \mathbf{y}_H|},$$

is added to the adiabatic sound speed, is used. The effective sound speed approximation is valid for low angle propagation.

Atmospheric attenuation is included by adding an imaginary part, an attenuation coefficient $\alpha(z)$,[32] to the wave number. Here only the classical attenuation coefficient, that due to thermo-viscous effects alone, will be used.

The resulting equation for the Green's function for atmospheric propagation, $\hat{G}_a$, is[30,31,33]

$$\left(\nabla_H^2 + \rho_a \frac{\partial}{\partial z}\frac{1}{\rho_a}\frac{\partial}{\partial z} + \left(\frac{\omega}{c_{\mathrm{eff}}} + i\alpha(z)\right)^2\right) \times \hat{G}_a(\mathbf{x}_H, z, \mathbf{y}_H, z', \omega) = \delta(\mathbf{x}_H - \mathbf{y}_H)\,\delta(z - z')$$

for $z > 0$ with

$$c_{\mathrm{eff}}(z) = c_a(z) + \hat{\mathbf{k}} \cdot \mathbf{v}_0(z)$$

and with rigid ground boundary conditions

$$\left.\frac{\partial P_a(\mathbf{x}_H, z, \omega)}{\partial z}\right|_{z=0} = 0$$

taken at $z=0$. For the purposes of this example the sound speed c_a is taken to be given by the polynomial fit to the "Standard Atmosphere"[34] given in Ref. 35. To simulate downwind versus upwind propagation a simple model for wind speed is used: $\hat{\mathbf{k}} \cdot \mathbf{v}_0(z)$ is assumed to have a Gaussian profile centered at the stratopause (an altitude of 60 km in the model used here) and 17.5 km wide. Explicitly,

$$\hat{\mathbf{k}} \cdot \mathbf{v}_0(z) = \pm 40 \cdot \exp\left[-\left(\frac{z - 6.0 \times 10^4}{1.75 \times 10^4}\right)^2\right] \mathrm{m/s}.$$

The effective sound speeds that result for downwind, upwind, and crosswind propagation, as well as the classical attenuation coefficient, are plotted in Fig. 6.

To solve for $\hat{G}_a$ one may expand in vertical normal modes.[28,36] At long ranges from the source ($\mathbf{x}_H$ and $\mathbf{y}_H$ widely separated) one has

$$\hat{G}_a(\mathbf{x}_H, z, \mathbf{y}_H, z', \omega) = \frac{1}{\rho_a(z')}\sqrt{\frac{i}{8\pi|\mathbf{x}_H - \mathbf{y}_H|}}\sum_j \frac{e^{ik_j|\mathbf{x}_H - \mathbf{y}_H|}}{\sqrt{k_j}}\psi_j(z)\psi_j(z') \qquad (65)$$

with

$$\left(\rho_a \frac{\partial}{\partial z}\frac{1}{\rho_a}\frac{\partial}{\partial z} + \left(\frac{\omega}{c_{\mathrm{eff}}} + i\alpha(z)\right)^2 - k_j^2\right)\psi_j(z) = 0$$

and $\psi_j'(0)=0$. The modes satisfy the bi-orthogonality (no complex conjugation) condition

$$\int_0^\infty \psi_j(z)\psi_n(z)\frac{1}{\rho_a(z)}\,dz = \delta_{jn}.$$

The condition that this integral converges for $j=n$ uniquely determines the mode numbers.

Rather than the Green's function, the cross spectral density $\mathcal{Q}$ from (60) is required. For a stationary atmosphere the cross spectral density is the product of two Green's functions. In this case the propagation factor is simply the magnitude $|G_a(\mathbf{x}_H, 0, \mathbf{y}_H, 0, 2\pi f)|$ of the Green's function. For a fluctuating atmosphere a simple model for the average over the propagation medium $\langle\cdot\rangle_P$ is obtained by assuming that the different modes have statistically independent, uniformly distributed phases. Under such assumptions the square of the propagation factor for surface to surface propagation is given by the "incoherent modal sum"[28]

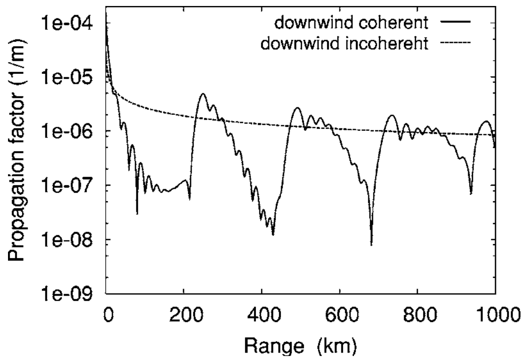

FIG. 7. The propagation factor for downwind propagation. Both the coherent modal sums, representing propagation in a stationary atmosphere, and the incoherent modal sum, representing the mean for propagation in a fluctuating atmosphere, are shown.

$$\mathcal{Q}_a(\mathbf{x}_H,\mathbf{x}_H,0,0,\mathbf{0},f) = \frac{1}{8\pi\rho_a(0)^2|\mathbf{x}_H|} \cdot \sum_j \frac{e^{-2\,\mathrm{Im}\,k_j|\mathbf{x}_H|}}{|k_j|}|\psi_j(0)^4|. \tag{66}$$

The mode functions ψ_j and wave numbers k_j have been determined using the finite difference methods described in Ref. (28). The modal attenuation coefficients, $\mathrm{Im}\,k_j$, are determined perturbatively: the modes are determined for the lossless case $\alpha=0$ and then first-order perturbation theory[36] is used to approximate the imaginary parts of the modal wave numbers. Both upwind and crosswind the surface-to-surface propagation paths must pass through the thermosphere (the region above 120 km). In these cases there is a severe reduction in predicted levels as a consequence of the increased attenuation in the thermosphere. While this is consistent with observations[1,9,11,12,37] the numerical values for the propagation as predicted by such linear models should not be taken literally since, in the thermosphere, the density ρ_a decreases dramatically so that the accuracy of the linear approximation to the atmospheric response is doubtful.[38] For downwind propagation, however, the sound gets trapped in the duct formed by the stratosphere (below 60 km) so that received levels are not greatly influenced by the thermosphere. Thus, for downwind propagation a linear propagation model is reasonable. In Fig. 7 the propagation factor for the downwind model is plotted as a function of range $|\mathbf{x}_H|$ at frequency $f=0.2$ Hz.

B. Sea state model

Now consider the sea state angular integral (64). It is common to write $F(f,\theta)$ as[15]

$$F(f,\theta) = \bar{F}(f)a(f,\theta)$$

where $\bar{F}(f)$ is the frequency spectrum given by (4). For purposes of illustration it will be assumed here that the frequency spectrum $\bar{F}(f)$ can be taken to be given by the JONSWAP[15,16] model of highly excited seas, that both $\bar{F}(f)$ and $a(f,\theta)$ are independent of position in the source region, and that $a(f,\theta)=a(\theta)$ is independent of frequency. Then one has

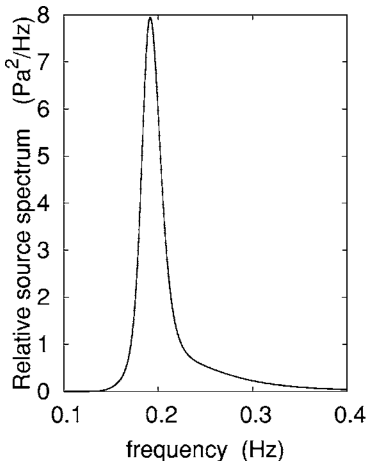

FIG. 8. The relative source spectrum for atmospheric microbaroms, $\mathcal{D}_a(f)/b$.

$$\int_0^{2\pi} F\left(\frac{f}{2},\theta\right)F\left(\frac{f}{2},\theta+\pi\right)\,d\theta = \bar{F}\left(\frac{f}{2}\right)^2\int_0^{2\pi} a(\theta)a(\theta+\pi)\,d\theta.$$

If $\int_0^{2\pi}a(\theta)a(\theta+\pi)d\theta=b$, then

$$\mathcal{D}_a(f) = 4b\bar{F}\left(\frac{f}{2}\right)^2\frac{\rho_a^2 g^2\pi^4 f^3}{c_a^2}\left(\frac{9g^2}{4\pi^2c_a^2f^2}+\frac{c_a^2}{c_w^2}\right).$$

In Fig. 8 the atmospheric microbarom source strength squared divided by b, $\mathcal{D}_a(f)/b$, is plotted for a significant wave height of 10 m and peak frequency of 0.95 Hz. Referring to Figs. 8 and 7, one finds that to produce a microbarom signal with amplitude of 0.1 Pa/$\sqrt{\mathrm{Hz}}$ at the peak frequency 1000 km from the source region would require

$$\left(10^{-12}\frac{1}{\mathrm{m}^2}\right)\left(8\frac{\mathrm{Pa}^2}{\mathrm{Hz}}\right)b|S| = 0.01\frac{\mathrm{Pa}^2}{\mathrm{Hz}}.$$

This gives $b|S|\approx 10^9$ m^2. Assuming the source region to be a disk of radius 200 km gives $b\approx 0.01$. For comparison, note that assuming the sea to be isotropic gives $b=2\pi$. It should be emphasized, however, that the value for b obtained here is illustrative only. The forms used for both the propagation factor and the source strength are based on simplifying assumptions: for the propagation factor these are the use of the standard model for the atmospheric temperature and the exponential profile for the atmospheric wind; for the source strength these are the use of the JONSWAP model for the frequency spectrum and the assumption that the sea state statistics are constant over the source region.

V. DISCUSSION OF THE RESULTS

A complete solution of the problem of the radiation of infrasound by ocean waves, the so-called microbarom radiation, has been presented. A direct connection is made between the statistical properties of the microbarom signal and the stochastic models commonly used to describe ocean wave systems. Since the acoustic wavelengths are so much longer than the correlation lengths for the ocean waves the received microbarom signal can be described as an incoherent superposition of fields produced by monopole sources at the sea surface. The problem of finding the source strength of the radiation has been reduced to oceanography: the determination of the density of counter-propagating waves of equal frequency (standing waves) on the ocean surface, as given by the integral (64), is required.

The source strength spectral density function for the radiation of atmospheric microbaroms has been derived and compared to that for oceanic microbaroms. The primary radiation is into the ocean; the source strength for radiation into the atmosphere is three orders of magnitude smaller than that for radiation into the ocean. This is in accord with observations: typical peak atmospheric microbarom levels are measured in the tenths of Pascals while typical peak oceanic microbarom levels can reach 100 Pascals. The reason for this difference lies in the fact that sea water is three orders of magnitude denser than air. The velocity of the air/water interface is fixed by the ocean waves. Consequently, the air and water have similar velocity fields near the interface. However, the variations in pressure required to maintain variations in velocity are proportional to the density of the fluid. Thus the pressure fluctuations in the ocean are three orders of magnitude larger than those in the air. Indeed, the ratio of the atmospheric to the oceanic microbarom source strengths is shown to be approximately equal to the ratio of the atmospheric to the oceanic densities.

The full atmospheric microbarom signal is shown to be the sum of two terms. The first term, responsible for about 80% of the radiation, represents sound radiated from pressure fluctuations produced in the ocean by the motion of the interface plus a term, responsible for the remaining 20% of the radiation, due to the compression of the air by the ocean waves. The single unknown factor, the standing wave density (64), is common to both the atmospheric and oceanic source strength spectra.

ACKNOWLEDGMENTS

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