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An analysis of the scattering of high-frequency electromagnetic radiation from rough surfaces with application to pulse radar operating in backscatter mode

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Abstract. The scattering of high-frequency (HF) electromagnetic radiation from slightly rough, good conducting surfaces is presented. The analysis is based on a decomposition of the relevant space using generalized functions. The fundamental analysis incorporates a general source and involves all scattering orders for the normal component of the field. Subsequently, derivation of the scattered electric field (to third order in scatter) using a pulsed dipole source is effected. The first 2 orders are used to deduce an estimate of radar cross sections of bounded regions or targets when operation is carried out in the backscatter mode. Conditions of small height and small slope are imposed. Application is made to the determination of the first-order cross section of a perfectly conducting sphere (within the limits of the imposed contraints) and of an exponential boss. The results are shown to be consistent with Rayleigh scattering theory.

1. Introduction

Since the presentations of such fundamental works as those of *Sommerfeld* [1909] and *Norton*, [1936, 1937], a vast quantity of literature has appeared on topics pertinent to the scattering of electromagnetic (em) radiation as it impinges a surface separating homogeneous media. A few of the techniques which are either relevant to the treatment at hand or to its possible applications are briefly mentioned here.

Of the several approaches to the problem of the scattering of wave energy from rough surfaces, one of the earliest was that of perturbation, initiated by Lord Rayleigh in 1896 [*Strutt*, 1945] in the context of acoustics and implemented by *Rice*, [1951] in relation to electromagnetics. Following Rice, many investigators, including *Wait* [1971], *Barrick* [1971a, 1971b, 1972a, 1972b], *Mitzner* [1964], *Rosich and Wait* [1977], and *Rodriguez and Kim* [1992], have adapted, refined, and applied this method to a variety of scattering problems. The basic requirements in the application of the perturbation method are that the surface profile variations be small compared to the radio wavelength and the surface slopes be much smaller than unity. It transpires that similar restrictions in the ensuing procedures lead to simplifications of the analysis, while providing results which are especially useful for the many applications involving radio transmission in the high-frequency (HF) band (3–30 MHz).

The so-called full-wave scattering solutions introduced by *Bahar* [1972] have been refined over the last two and one half decades [see, e.g., *Bahar*, 1980, 1991; *Collin*, 1992]. The results of *Bahar et al.* [1995] are in agreement with those of *Rice* [1951] when the product of the root-mean-square surface height and the free space wavenumber along with the surface slopes are on the same order of smallness.

Walsh [1980] proposed an alternate technique for the study of rough surface propagation and scatter.

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The method is based on a decomposition of surface characteristics and em field components in terms of Heaviside functions dictated by the various regions of the total scattering space. The application and interpretation of Maxwell's are carried out in the sense of generalized functions. As noted by *Walsh and Donnelly* [1987a], the technique contains a minimum of sophisticated analysis and has the following properties: (1) The analysis proceeds directly from Maxwell's equations; (2) the source field is arbitrary; (3) the equations directly relate the scattered electric fields to the known source fields without the use of intermediate Hertz potentials; and (4) the boundary conditions are generated naturally from the initial formulation of the problem. The basic approach has been to develop the analyses and applications to the problem of propagation and scattering for mixed paths with discontinuities [*Walsh et al.*, 1986; *Walsh and Donnelly*, 1987a], as well as to surface propagation and scatter for periodic surfaces [*Walsh and Srivastava*, 1987a].

In this paper, the results given by *Walsh and Donnelly* [1987b] are first used to develop new, general expressions for the normal and tangential electric fields for a surface with arbitrary profile. Second, with a view to applying the results to the operation of a pulsed HF vertical dipole source in a marine environment, the initial equations, which appear in operator form and are of great generality, are vastly simplified by imposing the conditions of (1) a good conducting surface and (2) small surface height variations as compared to the radio wavelength. The resulting integral equation, involving only the normal electric field component, is analyzed to first order in slope. Third, the scattering surface is considered to be representable as a Fourier series, and a general series solution is derived. Fourth, the condition of backscatter is imposed, and the field equations are developed for a pulsed radar. Finally, using the results for the first and second order, the backscatter cross section for a small deterministic target is deduced and is shown to be consistent with Rayleigh scattering.

The results presented here are directly applicable to developing high-frequency ground wave radar cross sections of the ocean surface, and this has motivated the work. The reader will no doubt see other applications at various levels of sophistication, depending on the chosen starting point in the general analysis. In the case of the ocean surface, preliminary considerations for the bistatic case are made by *Walsh et al.* [1996] and *Gill and Walsh* [1997]. It is intended that rigorous derivation and validation of these cross sections, rooted in the theory presented here, should constitute other publications.

2. Formulation of the Electric Field Equations for Rough Surfaces

The general two-body em scattering problem addressed by *Walsh and Donnelly* [1987b] is significantly simplified for the case of an infinite rough surface which is the interface between free space and a region of known em parameters. The entire space satisfies the definition

$$h_R(x,y,z) = 1 - h[z - \xi(x,y)], \tag{1}$$

where the h are the usual Heaviside functions and $\xi(x,y)$ is the two-dimensional rough surface which is assumed to be bounded for all x and y and forms the boundary of the lower half-space. This is depicted in Figure 1. The permittivity, permeability, and conductivity are ϵ_0, μ_0, and zero, respectively, above the surface. The corresponding values below the surface are ϵ_1, μ_0, and σ_0. Using the Heaviside notation while suppressing the argument of h_R, the expressions given by *Walsh and Donnelly* [1987b] for the electric field, **E**, reduce to

$$h_R\mathbf{E} = -\left\{\nabla\cdot\left[\mathbf{E}^{-}\nabla h_R\right] + (\nabla\mathbf{E})^{-}\cdot\nabla h_R\right\} \overset{xyz}{*} G_1, \tag{2}$$

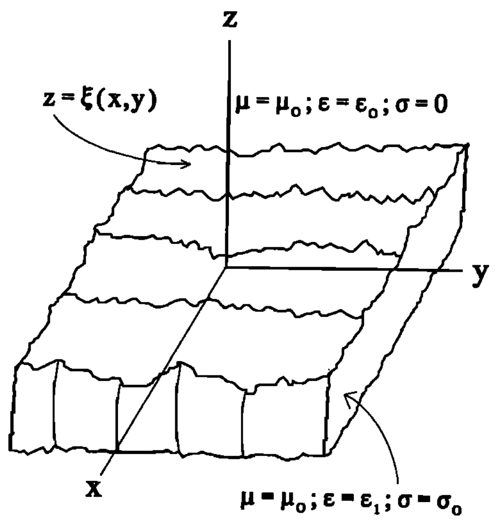

Figure 1. Typical rough surface with parameters as discussed in the text.

$$(1-h_R)\mathbf{E} = \mathbf{E}_s + \left\{\nabla\cdot\left[\mathbf{E}^+\nabla h_R\right] + (\nabla\mathbf{E})^+\cdot\nabla h_R\right\} \overset{xyz}{*} G_0, \quad (3)$$

with boundary condition

$$\left[(\nabla\mathbf{E})^+ - (\nabla\mathbf{E})^-\right]\cdot\nabla h_R + \nabla\cdot\left[\left(\mathbf{E}^+ - \mathbf{E}^-\right)\nabla h_R\right] = \frac{n_0^2-1}{n_0^2}\nabla\left(\mathbf{E}^+\cdot\nabla h_R\right). \quad (4)$$

The plus and minus superscripts on the various expressions indicate values of these variables in the limit as the surface is approached from above and below, respectively. The xyz star represents a three-dimensional spatial convolution with the appropriate Green's functions, G_0 and G_1, which are themselves defined as

$$G_0 = \frac{e^{-jkr}}{4\pi r}, \quad G_1 = \frac{e^{-jkn_0 r}}{4\pi r}. \quad (5)$$

Here, $r = \sqrt{x^2+y^2+z^2}$, k is the wavenumber of the radiation, and n_0 is the index of refraction of the lower medium. The field $\mathbf{E}_s$ is assumed to be derived from a source entirely within the vacuum half-space. Equations (2) and (3) require that

$$(1-h_R)\left\{\left[\nabla\cdot\left(\mathbf{E}^-\nabla h_R\right) + (\nabla\mathbf{E})^-\cdot\nabla h_R\right] \overset{xyz}{*} G_1\right\} = 0, \quad (6)$$

$$h_R\left\{\mathbf{E}_s + \left[\nabla\cdot\left(\mathbf{E}^+\nabla h_R\right) + (\nabla\mathbf{E})^+\cdot\nabla h_R\right] \overset{xyz}{*} G_0\right\} = 0. \quad (7)$$

In the notation of (2)–(4) and (6)–(7) the divergences ($\nabla\cdot$) are with respect to ∇h_R, with $\mathbf{E}^+$ and $\mathbf{E}^-$ being treated formally as scalars [see *Walsh and Donnelly*, 1987b].

From (1) it is easily shown that

$$\nabla h_R(x,y,z) = -\mathbf{n}\delta\left(z-\xi(x,y)\right), \quad (8)$$

where δ is the Dirac delta function and the normal, $\mathbf{n}$, to the surface $z=\xi(x,y)$ is given by

$$\mathbf{n} = \hat{\mathbf{z}} - \nabla\xi(x,y).$$

On substituting (8) into (2)–(4), the latter become

$$[1-h(z-\xi)]\,\mathbf{E} = \left\{\nabla\cdot\left[\mathbf{E}^-\mathbf{n}\delta(z-\xi)\right] + (\nabla\mathbf{E})^-\cdot\mathbf{n}\delta(z-\xi)\right\} \overset{xyz}{*} G_1, \quad (9)$$

$$h(z-\xi)\mathbf{E} = \mathbf{E}_s - \left\{\nabla\cdot\left[\mathbf{E}^+\mathbf{n}\delta(z-\xi)\right] + (\nabla\mathbf{E})^+\cdot\mathbf{n}\delta(z-\xi)\right\} \overset{xyz}{*} G_0, \quad (10)$$

$$\left[(\nabla E)^+ - (\nabla E)^-\right]\cdot\mathbf{n}\delta(z-\xi) + \nabla\cdot\left[\left(\mathbf{E}^+ - \mathbf{E}^-\right)\cdot\mathbf{n}\delta(z-\xi)\right] = \frac{n_0^2-1}{n_0^2}\nabla\left[\mathbf{E}^+\cdot\mathbf{n}\delta(z-\xi)\right], \quad (11)$$

with the requirements in (6) and (7) being recast as

$$h(z-\xi)\left(\left\{\nabla\cdot\left[\mathbf{E}^-\mathbf{n}\delta(z-\xi)\right] + (\nabla\mathbf{E})^-\cdot\mathbf{n}\delta(z-\xi)\right\} \overset{xyz}{*} G_1\right) = 0, \quad (12)$$

$$[1-h(z-\xi)]\left\{\mathbf{E}_s - \left[\nabla\cdot\left[\mathbf{E}^+\mathbf{n}\delta(z-\xi)\right] + (\nabla\mathbf{E})^+\cdot\mathbf{n}\delta(z-\xi)\right] \overset{xyz}{*} G_0\right\} = 0. \quad (13)$$

The coordinate arguments have been suppressed in the various functions of (9)–(13). In particular, referring to (9) and (11), it is noted that

$$\mathbf{E}^-\mathbf{n}\delta(z-\xi) = \mathbf{E}^-(x,y)\mathbf{n}(x,y)\delta[z-\xi(x,y)],$$

with the divergence in those equations being with respect to $\mathbf{n}$ while $\mathbf{E}^-$ is treated formally as a scalar. Consequently, it is easily confirmed that

$$\nabla\cdot\left[\mathbf{E}^-\delta(z-\xi)\mathbf{n}\right] = |\mathbf{n}|^2\,\mathbf{E}^-\delta'(z-\xi) - \nabla_{xy}\cdot\left(\mathbf{E}^-\nabla\xi\right)\delta(z-\xi). \quad (14)$$

Here, $|\mathbf{n}|$ is the magnitude of $\mathbf{n}$, δ' denotes the derivative of δ, $\nabla_{xy}\cdot$ is the planar divergence which, on the basis of the preceding discussion, is with respect to surface gradient, and $\mathbf{E}^-$ is being treated formally as a scalar. In a similar fashion,

$$\nabla\cdot\left[\mathbf{E}^+\delta(z-\xi)\mathbf{n}\right] = |\mathbf{n}|^2\,\mathbf{E}^+\delta'(z-\xi) - \nabla_{xy}\cdot\left(\mathbf{E}^+\nabla\xi\right)\delta(z-\xi). \quad (15)$$

On the basis of (14) and (15) and defining the vector fields

$$\mathbf{R}^-(x,y) = \nabla_{xy}\cdot\left(\mathbf{E}^-\nabla\xi\right) - (\nabla\mathbf{E})^-\cdot\mathbf{n}, \quad (16)$$

$$\mathbf{R}^+(x,y) = \nabla_{xy}\cdot\left(\mathbf{E}^+\nabla\xi\right) - (\nabla\mathbf{E})^+\cdot\mathbf{n}, \quad (17)$$

(9)–(11) may be written as

$$[1-h(z-\xi)]\,\mathbf{E} = \left\{|\mathbf{n}|^2\,\mathbf{E}^-\delta'(z-\xi) - \mathbf{R}^-\delta(z-\xi)\right\} \overset{xyz}{*} G_1 \quad (18)$$

$$h(z-\xi)\mathbf{E} = \mathbf{E}_s - \left\{|\mathbf{n}|^2\,\mathbf{E}^+\delta'(z-\xi) - \mathbf{R}^+\delta(z-\xi)\right\} \overset{xyz}{*} G_0\,, \quad (19)$$

$$|\mathbf{n}|^2\left[\mathbf{E}^+ - \mathbf{E}^-\right]\delta'(z-\xi) - \left[\mathbf{R}^+ - \mathbf{R}^-\right]\delta(z-\xi) = \frac{n_0^2-1}{n_o^2}\boldsymbol{\nabla}\left[|\mathbf{n}|\,E_n^+\delta(z-\xi)\right] \quad (20)$$

with $E_n^+ = \hat{\mathbf{n}}\cdot\mathbf{E}^+$, where $\hat{\mathbf{n}}$ is the unit normal, $\mathbf{n}/|\mathbf{n}|$. Similarly, (12) and (13) become

$$h(z-\xi)\left\{\left[|\mathbf{n}|^2\,\mathbf{E}^-\delta'(z-\xi) - \mathbf{R}^-\delta(z-\xi)\right] \overset{xyz}{*} G_1\right\} = 0\,, \quad (21)$$

$$[1-h(z-\xi)]\left\{\mathbf{E}_s - \left[|\mathbf{n}|^2\,\mathbf{E}^+\delta'(z-\xi) - \mathbf{R}^+\delta(z-\xi)\right] \overset{xyz}{*} G_0\right\} = 0\,. \quad (22)$$

It may be further deduced that

$$\boldsymbol{\nabla}\left[|\mathbf{n}|\,E_n^+\delta(z-\xi)\right] = |\mathbf{n}|^2\,\mathbf{E}_n^+\delta'(z-\xi) + \boldsymbol{\nabla}_{xy}\left(|\mathbf{n}|\,E_n^+\right)\delta(z-\xi)\,, \quad (23)$$

so that the boundary condition dictated by (20) reduces to

$$|\mathbf{n}|^2\left[\mathbf{E}^+ - \mathbf{E}^-\right]\delta'(z-\xi) - \left[\mathbf{R}^+ - \mathbf{R}^-\right]\delta(z-\xi) = \frac{n_0^2-1}{n_o^2}\left[|\mathbf{n}|^2\,\mathbf{E}_n^+\delta'(z-\xi) + \boldsymbol{\nabla}_{xy}\left(|\mathbf{n}|\,E_n^+\right)\delta(z-\xi)\right]. \quad (24)$$

The last equation is satisfied if

$$\mathbf{E}^- = \mathbf{E}^+ - \frac{n_0^2-1}{n_0^2}\mathbf{E}_n^+ = \mathbf{E}_t^+ + \frac{1}{n_0^2}\mathbf{E}_n^+, \quad (25)$$

$$\mathbf{R}^- = \mathbf{R}^+ + \frac{n_0^2-1}{n_0^2}\boldsymbol{\nabla}_{xy}\left(|\mathbf{n}|\,E_n^+\right), \quad (26)$$

and $\mathbf{E}_t^+$ is to be interpreted as the tangential component of $\mathbf{E}^+$. On the basis of the foregoing, the original pair, (3) and (4), has been reduced to the relatively simple set

$$[1-h]\,\mathbf{E} = \left\{|\mathbf{n}|^2\,\mathbf{E}^-\delta'(z-\xi) - \mathbf{R}^-\delta(z-\xi)\right\} \overset{xyz}{*} G_1, \quad (27)$$

$$h\mathbf{E} = \mathbf{E}_s - \left\{|\mathbf{n}|^2\,\mathbf{E}^+\delta'(z-\xi) - \mathbf{R}^+\delta(z-\xi)\right\} \overset{xyz}{*} G_0, \quad (28)$$

which require (21) and (22) and are subject to the boundary condition imposed by (25) and (26). For compactness, the arguments of the Heaviside functions in (27) and (28) are suppressed. At this juncture it is now obvious that if (21) and (22) can be solved for the surface quantities subject to (25) and (26), the complete electric field, $\mathbf{E}$, in all regions may be determined from (27) and (28).

3. Fourier Analysis and Solution Technique for Good Conducting Surfaces

If (27) is Fourier transformed in a plane $z = z^+ > \xi(x,y)$ for all (x,y), it may be determined without undue difficulty that the result is

$$0 = \mathcal{F}_{xy}\left[|\mathbf{n}|^2\,\mathbf{E}^- e^{\xi u_1}\right] + \frac{1}{u_1}\mathcal{F}_{xy}\left[\mathbf{R}^- e^{\xi u_1}\right], \quad (29)$$

$$u_1 = \sqrt{K^2 - n_0^2k^2},$$

$$K^2 = K_x^2 + K_y^2\,.$$

Here, $\mathcal{F}_{xy}[\ ]$ indicates the x–y spatial Fourier transform, and K_x and K_y are the transform variables with respect to x and y. Thus K is interpreted as a surface wavenumber. A similar transformation in a plane $z = z^- < 0 < \xi(x,y)$ for all (x,y) gives for (26)

$$2\mathcal{F}_{xy}\left[\mathbf{E}_s^{z^-}\right] = \mathcal{F}_{xy}\left[|\mathbf{n}|^2\,\mathbf{E}^+ e^{(z^- -\xi)u}\right] - \frac{1}{u}\mathcal{F}_{xy}\left[\mathbf{R}^+ e^{(z^- -\xi)u}\right], \quad (30)$$

$$u = \sqrt{K^2 - k^2}\,.$$

Therefore, as an alternative to solving (21) and (22) subject to (25) and (26), (29) and (30) may be solved subject to the same boundary condition.

As noted, for example, by *Walsh and Srivastava* [1987a], for a good conducting surface it is permissible to write

$$u_1 = \sqrt{K^2 - n_0^2k^2} \sim jkn_0\,.$$

Applying this approximation to (29) yields

$$\mathbf{R}^- = -jkn_0 |\mathbf{n}|^2 \mathbf{E}^-,$$

from which it may be readily deduced that (25) and (26) give

$$\mathbf{R}^+ = -\frac{n_0^2-1}{n_0^2}\nabla_{xy}\left(|\mathbf{n}| E_n^+\right) -jkn_0 |\mathbf{n}|^2 \mathbf{E}_t^+ - jk\Delta |\mathbf{n}|^2 \mathbf{E}_n^+, \quad (31)$$

where $\Delta = 1/n_0$. For the good conductor the problem has therefore reduced to that of solving (30) subject to (31). If, as is true for the class of surfaces being considered,

$$\frac{n_0^2-1}{n_0^2} \sim 1,$$

using (31) in (30) produces

$$\begin{aligned} 2u\mathcal{F}_{xy}\left[\mathbf{E}_s^{z^-}\right] &= (u+jk\Delta)\,\mathcal{F}_{xy}\left[|\mathbf{n}|^2 \mathbf{E}_n^+ e^{(z^- - \xi)u}\right] \\ &+ (u+jkn_0)\,\mathcal{F}_{xy}\left[|\mathbf{n}|^2 \mathbf{E}_t^+ e^{(z^- - \xi)u}\right] \\ &+ \mathcal{F}_{xy}\left[\nabla_{xy}\left(|\mathbf{n}| E_n^+\right) e^{(z^- - \xi)u}\right]. \end{aligned} \quad (32)$$

This result may be written as

$$\begin{aligned} \frac{2u\mathcal{F}_{xy}\left[\mathbf{E}_s^{z^-}\right]}{u+jk\Delta} &= \mathcal{F}_{xy}\left[|\mathbf{n}|^2 \mathbf{E}_n^+ e^{(z^- - \xi)u}\right] \\ &+ \mathcal{F}_{xy}\left[|\mathbf{n}|^2 \mathbf{E}_t^+ e^{(z^- - \xi)u}\right] \\ &+ \frac{1}{u+jk\Delta}\left\{\mathcal{F}_{xy}\left[\nabla_{xy}\left(|\mathbf{n}| E_n^+\right) e^{(z^- - \xi)u}\right]\right. \\ &\left. + jk(n_0-\Delta)\,\mathcal{F}_{xy}\left[|\mathbf{n}|^2 \mathbf{E}_t^+ e^{(z^- - \xi)u}\right]\right\} \end{aligned} \quad (33)$$

or as

$$\begin{aligned} \frac{2u\mathcal{F}_{xy}\left[\mathbf{E}_s^{z^-}\right]}{u+jkn_0} &= \mathcal{F}_{xy}\left[|\mathbf{n}|^2 \mathbf{E}_n^+ e^{(z^- - \xi)u}\right] \\ &+ \mathcal{F}_{xy}\left[|\mathbf{n}|^2 \mathbf{E}_t^+ e^{(z^- - \xi)u}\right] \\ &+ \frac{1}{u+jkn_0}\left\{\mathcal{F}_{xy}\left[\nabla_{xy}\left(|\mathbf{n}| E_n^+\right) e^{(z^- - \xi)u}\right]\right. \\ &\left. - jk(n_0-\Delta)\,\mathcal{F}_{xy}\left[|\mathbf{n}|^2 \mathbf{E}_n^+ e^{(z^- - \xi)u}\right]\right\}. \end{aligned} \quad (34)$$

To facilitate the analysis, a linear operator $\mathcal{L}$ may be defined as

$$\mathcal{L}[E(x,y)] = \mathcal{F}_{xy}\left\{|\mathbf{n}|^2 \mathbf{E} e^{(z^- - \xi)u}\right\}$$

or, explicitly,

$$\begin{aligned} \mathcal{L}[\mathbf{E}] &= \int\!\!\int |\mathbf{n}|^2 (x,y)\mathbf{E}(x,y) \\ &\cdot \exp\left\{\left[z^- - \xi(x,y)\right]u\right\} \\ &\cdot \exp\left(-jk_x x - jk_y y\right) dxdy\,. \end{aligned} \quad (35)$$

Further, a normal projection operator $\mathcal{N}$ and a tangential propagation operator $\mathcal{T}$ may be defined as

$$\mathcal{N}(\mathbf{E}) = \hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\mathbf{E} = \mathbf{E}_n, \quad (36)$$

$$\mathcal{T}(\mathbf{E}) = \mathbf{E} - \mathcal{N}(\mathbf{E}) = \mathbf{E}_t\,. \quad (37)$$

Then, assuming a proper inverse, $\mathcal{L}^{-1}$, exists for $\mathcal{L}$, (33) and (34) take the form

$$\begin{aligned} \mathcal{N}\mathcal{L}^{-1}\left[\frac{2u\mathcal{F}_{xy}\left[\mathbf{E}_s^{z^-}\right]}{u+jk\Delta}\right] &= \mathbf{E}_n^+ + \mathcal{N}\mathcal{L}^{-1} \\ &\cdot\left\{\frac{1}{u+jk\Delta}\left[\mathcal{L}\left(\frac{\nabla_{xy}\left(|\mathbf{n}| E_n^+\right)}{|\mathbf{n}|^2}\right.\right.\right. \\ &\left.\left.\left. + jk(n_0-\Delta)\,\mathbf{E}_t^+\right)\right]\right\}, \end{aligned} \quad (38)$$

$$\begin{aligned} \mathcal{T}\mathcal{L}^{-1}\left[\frac{2u\mathcal{F}_{xy}\left[\mathbf{E}_s^{z^-}\right]}{u+jkn_0}\right] &= \mathbf{E}_t^+ + \mathcal{T}\mathcal{L}^{-1} \\ &\cdot\left\{\frac{1}{u+jkn_0}\left[\mathcal{L}\left(\frac{\nabla_{xy}\left(|\mathbf{n}| E_n^+\right)}{|\mathbf{n}|^2}\right.\right.\right. \\ &\left.\left.\left. - jk(n_0-\Delta)\,\mathbf{E}_n^+\right)\right]\right\}. \end{aligned} \quad (39)$$

Again, invoking the good conductor assumption, so that $u+jkn_0 \sim jkn_0$ and $n_0 \gg \Delta = 1/n_0$, the problem of generating solutions to (38) and (39) is considerably simplified. In this case, (39) may easily be shown to reduce to

$$\begin{aligned} jkn_0\mathbf{E}_t^+ &= \mathcal{T}\left[\mathcal{L}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)\right]\right. \\ &\left. - \frac{\nabla_{xy}\left(|\mathbf{n}| E_n^+\right)}{|\mathbf{n}|^2}\right]. \end{aligned} \quad (40)$$

Substitution of this result into (38) gives, after minimal rearrangement of terms,

$$\mathcal{N}\mathcal{L}^{-1}\left[\frac{2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)-\mathcal{L}\mathcal{T}\mathcal{L}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)\right]}{u+jk\Delta}\right]$$
$$=\mathbf{E}_n^+ +\mathcal{N}\mathcal{L}^{-1}\left[\frac{\mathcal{L}\mathcal{N}\left(\frac{\nabla_{xy}(|\mathbf{n}|E_n^+)}{|\mathbf{n}|^2}\right)}{u+jk\Delta}\right],$$

which, on observing that

$$2u\mathcal{F}_{xy}\left[\mathbf{E}_s^{z^-}\right]-\mathcal{L}\mathcal{T}\mathcal{L}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)\right]$$
$$=\mathcal{L}\left\{\mathcal{L}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)\right]\right.$$
$$\left.-\mathcal{T}\mathcal{L}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)\right]\right\}$$
$$=\mathcal{L}\mathcal{N}\mathcal{L}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)\right],$$

may be written as

$$\mathcal{N}\mathcal{L}^{-1}\left[\frac{\mathcal{L}\mathcal{N}\mathcal{L}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)\right]}{u+jk\Delta}\right]$$
$$=\mathbf{E}_n^+ +\mathcal{N}\mathcal{L}^{-1}\left[\frac{\mathcal{L}\mathcal{N}\left(\frac{\nabla_{xy}(|\mathbf{n}|E_n^+)}{|\mathbf{n}|^2}\right)}{u+jk\Delta}\right]. \tag{41}$$

It should be noted that if (41) can be solved for $\mathbf{E}_n^+$, $\mathbf{R}^+$ can be determined from (31) via (40). Then, the electric field above the surface may be found from (28).

4. Small-Height Analysis

The result for a good conducting surface in (41) may be further simplified by imposing a so-called small-height analysis while requiring $z^-(<0) \ll \xi(x,y)$ for all (x,y). This constraint dictates that $k\xi \ll 1$ and permits the operator in (35) to be written as

$$\mathcal{L}(\mathbf{E}) \sim e^{z^- u}\mathcal{F}_{xy}\left[|\mathbf{n}|^2\mathbf{E}\right]. \tag{42}$$

Its inverse is therefore

$$\mathcal{L}^{-1}(\ ) \sim \frac{1}{|\mathbf{n}|^2}\mathcal{F}_{xy}^{-1}\left[e^{-z^- u}(\ )\right]. \tag{43}$$

Here $\mathcal{F}_{xy}^{-1}[\ \ ]$ denotes the inverse two-dimensional spatial Fourier transform [see *Walsh*, 1984]

$$\mathcal{F}_{xy}^{-1}\left[\frac{1}{u+jk\Delta}\right] \sim F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}, \tag{44}$$

where ρ is the planar distance variable given by $\rho=\sqrt{x^2+y^2}$ and $F(\rho)$ is the familiar Sommerfeld attenuation function incorporating the surface impedance Δ. Then, applying (42), (43), and (44) to (41), while recalling the definition of $\mathcal{N}$ in (36) yields

$$\mathbf{E}_{0_n}^+ + \frac{\hat{\mathbf{n}}\hat{\mathbf{n}}}{|\mathbf{n}|^2}\cdot\left\{\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\nabla_{xy}\left(|\mathbf{n}|\,E_{0_n}^+\right) \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\}$$
$$=\frac{\hat{\mathbf{n}}\hat{\mathbf{n}}}{|\mathbf{n}|^2}\cdot\left\{\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\mathcal{F}_{xy}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)e^{-z^- u}\right]\right.$$
$$\left.\overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\}, \tag{45}$$

where $E_{0_n}^+$ has replaced E_n^+ to indicate that the small-height approximation is being invoked. Noting that the unit normal, $\hat{\mathbf{n}}$, may be written as

$$\hat{\mathbf{n}}=\frac{\mathbf{n}}{|\mathbf{n}|}=\frac{\hat{\mathbf{z}}-\nabla\xi}{|\mathbf{n}|}$$

and taking $\hat{\mathbf{n}}\cdot$ on each side of (45) yields, after minimal manipulation, the scalar equation

$$E_{0_n}^+ - \frac{1}{|\mathbf{n}|^3}\cdot\left\{\frac{\nabla\xi}{|\mathbf{n}|^2}\cdot\nabla_{xy}\left(|\mathbf{n}|\,E_{0_n}^+\right)\overset{xy}{*}F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\}$$
$$-\frac{\nabla\xi}{|\mathbf{n}|^3}\cdot\left\{\frac{\nabla\xi\nabla\xi}{|\mathbf{n}|^2}\cdot\nabla_{xy}\left(|\mathbf{n}|\,E_{0_n}^+\right)\overset{xy}{*}F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\}$$
$$=\frac{\mathbf{n}}{|\mathbf{n}|^3}\cdot\left\{\frac{\mathbf{n}\mathbf{n}}{|\mathbf{n}|^2}\cdot\mathcal{F}_{xy}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)e^{-z^- u}\right]\right.$$
$$\left.\overset{xy}{*}F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\}. \tag{46}$$

This may be written more compactly, in operator form, as

$$E_{0_n}^+ - \mathcal{T}_1\left(E_{0_n}^+\right)-\mathcal{T}_2\left(E_{0_n}^+\right)=E^s, \tag{47}$$

$$\mathcal{T}_1(\ )=\frac{1}{|\mathbf{n}|^3}\cdot\left\{\frac{\nabla\xi}{|\mathbf{n}|^2}\cdot\nabla_{xy}\left[|\mathbf{n}|(\ )\right]\overset{xy}{*}F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\},$$
$$\mathcal{T}_2(\ )=\frac{\nabla\xi}{|\mathbf{n}|^3}\cdot\left\{\frac{\nabla\xi\nabla\xi}{|\mathbf{n}|^2}\cdot\nabla_{xy}\left[|\mathbf{n}|(\ )\right]\overset{xy}{*}F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\},$$
$$E^s=\frac{\mathbf{n}}{|\mathbf{n}|^3}\cdot\left\{\frac{\mathbf{n}\mathbf{n}}{|\mathbf{n}|^2}\cdot\mathcal{F}_{xy}^{-1}\left[2u\mathcal{F}_{xy}\left(\mathbf{E}_s^{z^-}\right)e^{-z^- u}\right]\right.$$
$$\left.\overset{xy}{*}F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\}.$$

It should be noted that (46) or (47) is valid for arbitrary slope and is in the proper form for a Neumann series or iterative solution. However, in the section 5 where a vertical dipole source is specified, we have for simplicity imposed the condition of small slope.

5. Field Equations Incorporating a Vertical Dipole Source

It is now assumed that the source field is the far field of a vertical dipole at the (x, y) origin and elevated slightly above the surface; i.e., the location is the point $(x, y, z) \equiv (0, 0, 0^+)$. The far field is then the well-known result

$$\mathbf{E}_s = C_0 G_0 \hat{\mathbf{z}} \,, \tag{48}$$

where G_0 is defined in (5) and

$$C_0 = \frac{I \Delta \ell k^2}{j\omega\epsilon_0} \,,$$

where the source current I on the dipole of length $\Delta\ell$ has a radian frequency ω. Fourier transformation in a plane $z = z^- < 0$ thus gives

$$\mathcal{F}_{xy}\left[\mathbf{E}_s^{z-}\right] = \frac{C_0 e^{z^- u}}{2u}\hat{\mathbf{z}} \,,$$

which immediately implies

$$\mathcal{F}_{xy}^{-1}\left\{2u\mathcal{F}_{xy}\left[\mathbf{E}_s^{z-}\right]e^{-z^- u}\right\} = C_0\delta(x)\delta(y)\hat{\mathbf{z}} \,,$$

where $\delta(x)\delta(y)$ is the two-dimensional Dirac delta function. Substitution of this inverse transform expression into the form of E^s defined following (47) results in

$$\begin{aligned} E^s &= \frac{\mathbf{n}}{|\mathbf{n}|^3}\cdot\left\{\frac{\mathbf{n}\mathbf{n}}{|\mathbf{n}|^2}\cdot C_0\delta(x)\delta(y)\hat{\mathbf{z}} \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\} \\ &= \frac{C_0}{|\mathbf{n}|^3}F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} \,. \end{aligned} \tag{49}$$

Here, it has been assumed that at the (x, y) origin $\mathbf{n} = \hat{\mathbf{z}}$.

If it is now assumed that $|\nabla\xi| \ll 1$, then clearly

$$|\mathbf{n}|^2 = 1 + |\nabla\xi|^2 \sim 1 \,.$$

If it is further agreed to neglect powers of slope greater than 1 in a single scatter, then the term of (47) which contains the $\mathcal{T}_2$ operator is eliminated and that equation, for vertical dipole excitation, becomes

$$\begin{aligned} E_{0_n}^+ - \left\{\nabla\xi\cdot\nabla_{xy}\left(E_{0_n}^+\right) \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\} \\ = C_0 F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} \,. \end{aligned} \tag{50}$$

A Neumann series solution of (50) may be given as

$$\begin{aligned} E_{0_n}^+ &= E^s + \mathcal{T}_1(E^s) + \mathcal{T}_1^2(E^s) + \cdots \\ &= \left(E_{0_n}^+\right)_0 + \left(E_{0_n}^+\right)_1 + \left(E_{0_n}^+\right)_2 + \cdots \,, \end{aligned} \tag{51}$$

where it may be noted that the zeroth-order term, $\left(E_{0_n}^+\right)_0$, is simply the expression for propagation over a smooth plane surface with surface impedance Δ. The form of the first-order solution in (51) may now be written down directly as

$$\begin{aligned} \left(E_{0_n}^+\right)_1 = \mathcal{T}_1(E^s) &= \nabla\xi\cdot\nabla_{xy}\left[C_0 F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right] \\ &\quad \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} \,. \end{aligned} \tag{52}$$

Then, given that $\nabla_{xy} = \hat{\boldsymbol{\rho}}\partial/\partial\rho + \hat{\boldsymbol{\theta}}(1/\rho)(\partial/\partial\theta)$, in an asymptotic sense,

$$\nabla_{xy}\left[C_0 F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right] \sim -jkC_0 F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\hat{\boldsymbol{\rho}},$$

so that

$$\begin{aligned} \left(E_{0_n}^+\right)_1 &\sim -jkC_0\left[\hat{\boldsymbol{\rho}}\cdot\nabla_{xy}(\xi)F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right. \\ &\quad \left.\overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right] \,. \end{aligned} \tag{53}$$

Using this formulation, the next 2 orders of scatter may be written as

$$\begin{aligned} \left(E_{0_n}^+\right)_2 &= \mathcal{T}_1^2(E^s) = \mathcal{T}_1\left\{\mathcal{T}_1\left[C_0 F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right]\right\} \\ &= C_0\mathcal{T}_1\left\{\nabla\xi\cdot\nabla_{xy}\left[F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right] \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\} \\ &\sim -jkC_0\left\{\nabla\xi\cdot\nabla_{xy}\left[\hat{\boldsymbol{\rho}}\cdot\nabla\xi F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right.\right. \\ &\quad \left.\left.\overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right] \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\}, \end{aligned} \tag{54}$$

$$\begin{aligned}\left(E_{0_n}^{+}\right)_3 &= \mathcal{T}_1^3\left(E^s\right)\\ &\sim -jkC_0\left(\nabla\xi\cdot\nabla_{xy}\left\{\nabla\xi\cdot\nabla_{xy}\right.\right.\\ &\cdot\left[\hat{\rho}\cdot\nabla\xi F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} * F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right]\\ &\left.\left.\overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\}\overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right). \qquad (55)\end{aligned}$$

6. Scattering From Surfaces Representable as Fourier Series

In this section, the preceding analysis is applied to rough surfaces which may be represented as Fourier series. Each scattering order is examined separately, and results are given for the backscatter condition.

6.1. First-Order Case

It is straightforward to show that (53) for first-order scattering may be written as an asymptotic integral in the form

$$\begin{aligned}\left(E_{0_n}^{+}\right)_1 &\sim -jkC_0\frac{1}{(2\pi)^2}\int_{x_1}\int_{y_1}\hat{\rho}_1\cdot\nabla_{x_1y_1}[\xi(x_1,y_1)]\\ &\cdot F(\rho_1)\,F(\rho_2)\,\frac{e^{-jk(\rho_1+\rho_2)}}{\rho_1\rho_2}dx_1\,dy_1 . \qquad (56)\end{aligned}$$

The various quantities in (56) are illustrated in Figure 2. It should be recalled that the source is at the origin and the scatter occurs at point (x_1, y_1). Of course, the scattered radiation travels in all directions over the scattering surface, but it is "observed" at position (x, y). The distances ρ, ρ_1, ρ_2 are from the source to the reception point, from the source to the scatter point, and from the scatter point to the reception point, respectively. The integration limits are over the entire x–y plane.

It is now required that the surface, $\xi(x, y)$, be characterized in some fashion, the choice here taking the form of a two-dimensional Fourier series. It shall be assumed that the fundamental surface wavenumber N is the same in both the x and y directions. The appropriate series is

$$\xi(x,y) = \sum_{m,n} P_{mn} e^{jN(mx+ny)} . \qquad (57)$$

The indices m, n span the set of integers, and P_{mn} is the Fourier coefficient corresponding to the wavenumber components Nm and Nn. We may therefore express the mnth surface wave vector component as

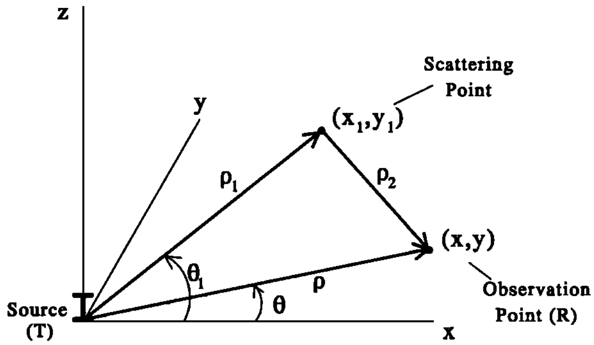

Figure 2. Geometry of the first-order scatter.

$$\mathbf{K}_{mn} = Nm\hat{\mathbf{x}} + Nn\hat{\mathbf{y}} ,$$

$\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ being the usual unit vectors. Since a general planar displacement vector $\boldsymbol{\rho}$ on the surface may be written as

$$\boldsymbol{\rho} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} ,$$

(57) is, equivalently,

$$\begin{aligned}\xi(x,y) &= \sum_{m,n} P_{\mathbf{K}_{mn}} e^{j\boldsymbol{\rho}\cdot\mathbf{K}_{mn}}\\ &= \sum_{m,n} P_{\mathbf{K}_{mn}} \exp\left[j\rho K_{mn}\cos\left(\theta_{mn}-\theta\right)\right] . \qquad (58)\end{aligned}$$

Here θ_{mn} is the direction of $\mathbf{K}_{mn}$ and θ is the direction of $\boldsymbol{\rho}$ (i.e., $\theta = \tan^{-1}(x/y)$). Applying (58) to the scattering point (x_1, y_1) of Figure 2, it may be easily seen that on writing ∇_{xy} in polar coordinates,

$$\begin{aligned}\hat{\rho}_1\cdot\nabla_{x_1y_1}[\xi(x_1,y_1)] &= j\sum_{m,n} P_{\mathbf{K}_{mn}} K_{mn}\cos\left(\theta_{mn}-\theta_1\right)\\ &\cdot\exp\left[j\rho K_{mn}\cos\left(\theta_{mn}-\theta_1\right)\right] .\end{aligned}$$

Using this result in (56) produces

$$\begin{aligned}\left(E_{0_n}^{+}\right)_1 &\approx \frac{kC_0}{(2\pi)^2}\sum_{m,n} P_{\mathbf{K}_{mn}} K_{mn}\\ &\cdot\int_{y_1}\int_{x_1}\cos\left(\theta_{mn}-\theta_1\right)\frac{F(\rho_1)\,F(\rho_2)}{\rho_1\rho_2}\\ &\cdot\exp\left\{j\rho_1\left[K_{mn}\cos\left(\theta_{mn}-\theta_1\right)-k\right]\right\}\\ &\cdot e^{-jk\rho_2}dx_1dy_1 \qquad (59)\end{aligned}$$

for the first-order field. This equation represents the integral form of the field observed at a general point (x, y) or (ρ, θ) when a single scatter occurs at a point (x_1, y_1) far from the source and the surface profile is not a function of time.

Equation (59) may be simplified for the condition of backscatter. Referring to Figure 2, for backscatter, $\rho = 0$, giving $\rho_1 = \rho_2$. Therefore, in polar coordinates, (59) becomes

$$(E_{0_n}^+)_1 \approx \frac{kC_0}{(2\pi)^2} \sum_{m,n} P_{\mathbf{K}_{mn}} K_{mn} \int_{\rho_1} \frac{F^2(\rho_1)}{\rho_1} \cdot e^{-j2k\rho_1} \int_0^{2\pi} g(\theta_1) \cos(\theta_{mn} - \theta_1) \cdot \exp[j\rho_1 K_{mn} \cos(\theta_{mn} - \theta_1)]\, d\theta_1 d\rho_1. \quad (60)$$

In this equation a receiving antenna "directivity" function $g(\theta_1)$ has been included. At this point, g will be taken as a measure of the ability of the receiving antenna to discriminate in direction, in the sense of electric field. It is necessary to introduce it, at this point, as the simplest way to model the receiving antenna's directivity. For an omnidirectional system, $g(\theta) = 1$. The function g may be appropriately scaled later to agree with convention.

The θ_1 integral in (60) is available in closed form if $g(\theta) = 1$. In any event a stationary phase integration may be performed [see, e.g., *Ishimaru*, 1991, appendix to chapter 11, section C]. It requires the usual assumptions that $\rho_1 K_{mn} \gg 1$ and $g(\theta_1)\cos(\theta_{mn} - \theta_1)$ is slowly varying. Consider

$$I(K_{mn}, \theta_{mn}) = \int_0^{2\pi} g(\theta_1) \cos(\theta_{mn} - \theta_1) \cdot \exp[j\rho_1 K_{mn} \cos(\theta_{mn} - \theta_1)]\, d\theta_1. \quad (61)$$

Applying the stationary phase method requires the solution of

$$\frac{d}{d\theta_1}[\cos(\theta_{mn} - \theta_1)] = 0,$$

which gives

$$\theta_1 = \theta_{mn}\,, \quad \theta_{mn} + \pi$$

as stationary points. Further, the value of $d^2[\cos(\theta_{mn} - \theta_1)]/d\theta_1^2$ is required at the stationary values of θ_1. Clearly,

$$\frac{d^2[\cos(\theta_{mn} - \theta_1)]}{d\theta_1^2} = \begin{cases} -1 & \theta_1 = \theta_{mn}, \\ 1 & \theta_1 = \theta_{mn} + \pi. \end{cases}$$

According to the theory, (61) may be approximated as

$$I(K_{mn}, \theta_{mn}) \sim \sqrt{\frac{2\pi}{K_{mn}\rho_1}} \left\{ g(\theta_{mn})\, e^{j\rho_1 K_{mn}} e^{-j(\pi/4)} - g(\theta_{mn} + \pi)\, e^{-j\rho_1 K_{mn}} e^{j(\pi/4)} \right\}.$$

By using this result, (60) may be written as

$$(E_{0_n}^+)_1 \sim \frac{kC_0}{(2\pi)^{3/2}} \sum_{m,n} P_{\mathbf{K}_{mn}} \sqrt{K_{mn}} \int_{\rho_1} \frac{F^2(\rho_1)}{\rho_1^{3/2}} \cdot \left\{ g(\theta_{mn})\, e^{-j(\pi/4)} e^{j\rho_1(K_{mn} - 2k)} - g(\theta_{mn} + \pi)\, e^{j(\pi/4)} e^{-j\rho_1(K_{mn} + 2k)} \right\} d\rho_1. \quad (62)$$

Equation (62) represents the first-order normal backscatter field from the rough surface when the source is a continuously excited vertical dipole. It is well known that for the given excitation, and surface constraints, it is the normal component which predominates. It is modified in section 7 to determine the backscattered first-order field when the excitation is a time-pulsed dipole.

6.2. Second-Order Case

The second-order field component dictated by (54) is represented geometrically in Figure 3. The first scatter at (x_1, y_1) may occur anywhere on the rough surface, and the result is a scatter to any other surface point (x_2, y_2). Subsequently, a portion of this doubly scattered radiation may be received, for the moment, at some arbitrary position (x, y). Not surprisingly, this adds considerable tedium to the ensuing analysis as compared to the first-order case. The significant portions of the procedure are outlined here, and details of the technique are appropriately referenced.

Designating the inner convolution of (54) by I_1, say, and referring to Figure 3 for various position and distance parameters, we write

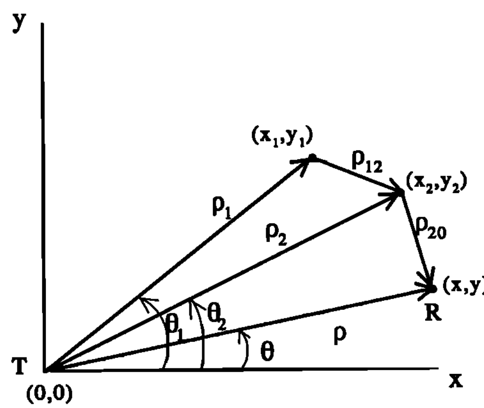

Figure 3. Geometry of the second-order scatter.

$$\hat{\rho}\cdot\nabla\xi F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} = \frac{1}{(2\pi)^2} \cdot\int_{x_1}\int_{y_1}[\hat{\rho}\cdot\nabla\xi]\,(x_1,y_1)F(\rho_1)F(\rho_{12}) \cdot\frac{e^{-jk(\rho_1+\rho_{12})}}{\rho_1\rho_{12}}dx_1dy_1 = I_1. \quad (63)$$

For the surface being considered (see (58)),

$$(\hat{\rho}\cdot\nabla\xi)_{x_1,y_1} = j\sum_{m,n}P_{\mathbf{K}_{mn}}K_{mn}\cos(\theta_{mn}-\theta_1) \cdot\exp[j\rho_1K_{mn}\cos(\theta_{mn}-\theta_1)],$$

giving, for (63),

$$I_1 = \frac{j}{(2\pi)^2}\sum_{m,n}P_{\mathbf{K}_{mn}}K_{mn}\int_{y_1}\int_{x_1}\cos(\theta_{mn}-\theta_1) \cdot\exp\{j\rho_1[K_{mn}\cos(\theta_{mn}-\theta_1)-k]-jk\rho_{12}\} \cdot\frac{F(\rho_1)F(\rho_{12})}{\rho_1\rho_{12}}dx_1dy_1\,. \quad (64)$$

This result may be reduced via a two-dimensional stationary phase approach [see, e.g., *Bleistein and Handelsman*, 1975; *Friedman*, 1969]. In order to initiate this process, a transformation of coordinates may be effected as follows: (1) rotation of the axes by θ_2, (2) a shift of the origin to a position halfway along ρ_2, and (3) conversion to elliptic coordinates. It is easy to show that the following result:

$$\begin{aligned}\rho_1 &= \sqrt{x_1^2+y_1^2} = \frac{\rho_2}{2}(\cosh\mu+\cos\delta),\\ \rho_{12} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \frac{\rho_2}{2}(\cosh\mu-\cos\delta),\\ \theta_1 &= \tan^{-1}\left(\frac{y_1}{x_1}\right)\\ &=\tan^{-1}\left[\frac{(1+\cosh\mu\cos\delta)\sin\theta_2+\sinh\mu\sin\delta\cos\theta_2}{(1+\cosh\mu\cos\delta)\cos\theta_2-\sinh\mu\sin\delta\sin\theta_2}\right].\end{aligned} \quad (65)$$

Here, μ and δ are the usual cylindrical elliptic coordinates [e.g., *Stratton*, 1941]. Using (65), (64) becomes

$$I_1 = \frac{j}{(2\pi)^2}\sum_{m,n}P_{\mathbf{K}_{mn}}K_{mn}\int_0^{2\pi}\int_0^{\infty}\cos(\theta_{mn}-\theta_1) \cdot F(\rho_1)F(\rho_{12})\exp(j(\rho_2/2)\{K_{mn}[(1+\cosh\mu\cos\delta) \cdot\cos(\theta_{mn}-\theta_2)+\sinh\mu\sin\delta\sin(\theta_{mn}-\theta_2)] -2k\cosh\mu\})\,d\mu d\delta\,. \quad (66)$$

It is understood that ρ_1, ρ_{12}, and θ_1 are functions of μ and δ as required in (65).

Equation (66) may now be simplified by a two-dimensional stationary phase analysis. The stationary points may be shown to be

$$\mu = 0\ ,\ \delta = 0, \quad (67a)$$

$$\mu = 0\ ,\ \delta = \pm\pi \quad (67b)$$

(only one of the pair is distinct),

$$\tan\delta = \frac{\sqrt{K_{mn}^2-4k^2\cos^2(\theta_{mn}-\theta_2)}}{2k\cos(\theta_{mn}-\theta_2)},$$

$$\tanh\mu = \frac{\sqrt{K_{mn}^2-4k^2\cos^2(\theta_{mn}-\theta_2)}}{2k\sin(\theta_{mn}-\theta_2)}. \quad (67c)$$

In (67c) the restriction $2k|\cos(\theta_{mn}-\theta_2)| < K_{mn} < 2k$ obviously applies. Furthermore, $\theta_{mn}\neq\theta_2$, $\theta_2\neq\theta_{mn}\pm\pi/2$ in the points designated by (67c), but these values are covered by the first two stationary points. These points may be seen to represent the following physical situations:

1. From the first two equations in (65) and Figure 3, $(\mu,\delta)\equiv(0,0)$ means that $\rho_1=\rho_2$ and $\rho_{12}=0$. It is seen that this indicates a double scatter at (x_1,y_1) (i.e., $(x_2,y_2)\equiv(x_1,y_1)$). For reasons that will become evident when this analysis is applied to a pulse radar (section 7), it is customary to refer to this phenomenon as patch scatter.

2. For the $(0,\pi)$ stationary point the first two equations of (65) reduce to $\rho_1=0$ and $\rho_{12}=\rho_2$. This time, Figure 3 indicates that point (x_1,y_1) has shifted to the transmitting sight T and the second scatter occurs remotely from T at (x_2,y_2). Given that $\theta_{12}=\tan^{-1}[(y_2-y_1)/(x_2-x_1)]$, it is easy to show that $\theta_{12}\to\theta_2$ uniquely as $(\mu,\delta)\to(0,\pi)$. Thus the $(0,\pi)$ represents a first scatter near the transmitter and a second on a patch of ocean which is "viewed" from R.

3. The restrictions on the third stationary "point" (which is really a set of points) stated following (65) make it distinct from the previous two points. Thus the scatter for the third "point" must occur elsewhere than at the transmitter or at the remote patch whereon (x_1,y_1) or (x_2,y_2) resides. In monostatic operation this case has been referred to as "off-patch" scatter [*Srivastava*, 1984].

In this work, only the patch scatter condition, i.e. the $(0,0)$ stationary point, is addressed. Such may be shown to be the important contribution when the source and receiver are "narrow beam" in nature and neither is totally surrounded by the highly conducting scattering medium. Deviations from these conditions are addressed by *Gill* [1999]. For the patch scatter condition the integral in (66), by direct application of the two-dimensional stationary phase procedure, may be shown to be well approximated by

$$\begin{aligned} I_1 \sim\ & jF(\rho_2)\frac{e^{-jk\rho_2}}{2\pi\rho_2}\sum_{m,n} P_{\mathbf{K}_{mn}} \\ & \cdot\frac{K_{mn}\cos(\theta_{mn}-\theta_2)}{\sqrt{K_{mn}^2-2kK_{mn}\cos(\theta_{mn}-\theta_2)}} \\ & \exp\left[j\rho_2 K_{mn}\cos(\theta_{mn}-\theta_2)\right] . \end{aligned} \tag{68}$$

That is, I_1 contains this asymptotic patch scatter term for large $\rho_2 K_{mn}$. On the basis of this discussion it may therefore be seen that an asymptotical term of (52) is given by

$$\begin{aligned} (E_{0_n}^+)_2 \sim\ & -jkC_0\left\{\nabla\xi\cdot\nabla_{xy}\left[jF(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right.\right. \\ & \cdot\sum_{m,n} P_{\mathbf{K}_{mn}}\exp\left[j\rho K_{mn}\cos(\theta_{mn}-\theta)\right] \\ & \left.\cdot\frac{K_{mn}\cos(\theta_{mn}-\theta)}{\sqrt{K_{mn}^2-2kK_{mn}\cos(\theta_{mn}-\theta)}}\right] \\ & \left.\overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\} . \end{aligned} \tag{69}$$

A geometrical representation of (69) is given in Figure 4, and the similarity with Figure 2 for first-order scatter is evident. Again, using the polar representation of ∇_{xy} and dropping terms of $1/\rho^2$, (69) becomes

$$\begin{aligned} (E_{0_n}^+)_2 \sim\ & jkC_0\left\{\left(F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right.\right. \\ & \cdot\sum_{m,n} P_{\mathbf{K}_{mn}}\frac{K_{mn}\cos(\theta_{mn}-\theta)}{\sqrt{K_{mn}^2-2kK_{mn}\cos(\theta_{mn}-\theta)}}\nabla\xi \\ & \cdot\left\{\left[K_{mn}\cos(\theta_{mn}-\theta)-k\right]\hat{\rho}+K_{mn}\sin(\theta_{mn}-\theta)\hat{\theta}\right\} \\ & \left.\left.\cdot\ \exp\left[j\rho K_{mn}\cos(\theta_{mn}-\theta)\right]\right)\overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\} . \end{aligned} \tag{70}$$

For the type of surface assumed,

$$\xi(\rho,\theta)=\sum_{pq} P_{\mathbf{K}_{pq}}\exp\left[j\rho K_{pq}\cos(\theta_{pq}-\theta)\right] .$$

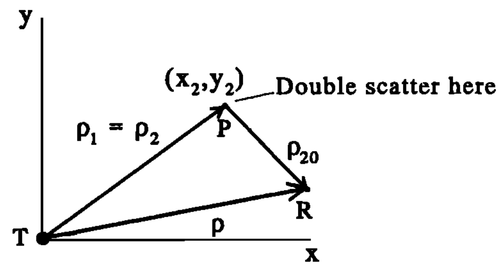

Figure 4. Geometry of the second-order "patch scatter."

Then, it may be verified that the dot product in (70) may be written as

$$\begin{aligned} \nabla\xi\cdot & \left\{\left[K_{mn}\cos(\theta_{mn}-\theta)-k\right]\hat{\rho}\right. \\ & \left.+K_{mn}\sin(\theta_{mn}-\theta)\hat{\theta}\right\} \\ & = j\sum_{p,q} P_{\mathbf{K}_{pq}}\left[\mathbf{K}_{mn}\cdot\mathbf{K}_{pq}-k\hat{\rho}\cdot\mathbf{K}_{pq}\right] \\ & \cdot\exp\left[j\rho K_{pq}\cos(\theta_{pq}-\theta)\right] , \end{aligned}$$

so that (70) becomes

$$\begin{aligned} (E_{0_n}^+)_2 \sim & -kC_0\left(\left\{F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\sum_{m,n}\sum_{p,q} P_{\mathbf{K}_{mn}}P_{\mathbf{K}_{pq}}\right.\right. \\ & \cdot\exp\left[j\rho K_{rs}\cos(\theta_{rs}-\theta)\right] \\ & \left.\left.\cdot\frac{(\mathbf{K}_{mn}\cdot\hat{\rho})\left[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k\hat{\rho})\right]}{\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k\hat{\rho})}}\right\}\overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right) . \end{aligned} \tag{71}$$

Here, $\mathbf{K}_{rs}=\mathbf{K}_{mn}+\mathbf{K}_{pq}$. In integral form, referring to Figure 4, and using the notation there, (71) is

$$\begin{aligned} (E_{0_n}^+)_2 \sim\ & \frac{-kC_0}{(2\pi)^2}\sum_{m,n}\sum_{p,q} P_{\mathbf{K}_{mn}}P_{\mathbf{K}_{pq}} \\ & \cdot\int_{x_2}\int_{y_2}\frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_2)\left[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k\hat{\rho}_2)\right]}{\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k\hat{\rho}_2)}}F(\rho_2)F(\rho_{20}) \\ & \cdot\frac{\exp\left\{j\rho_2\left[K_{rs}\cos(\theta_{rs}-\theta_2)-k\right]-jk\rho_{20}\right\}}{\rho_2\rho_{20}}dx_2\,dy_2 . \end{aligned} \tag{72}$$

Referring to Figure 4, it is evident that for the special case of backscatter, $\rho=0$ and $\rho_{20}=\rho_2$. Equation (72) in this instance becomes

$$
\begin{aligned}
(E_{0_n}^{+})_2 \sim\ & \frac{-kC_0}{(2\pi)^2} \sum_{m,n}\sum_{p,q} P_{\mathbf{K}_{mn}} P_{\mathbf{K}_{pq}} \int_{\rho_2} \frac{F^2(\rho_2)}{\rho_2} \\
& \cdot e^{-j2k\rho_2} \int_{-\pi}^{\pi} g(\theta_2) \frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_2)\,[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k\hat{\rho}_2)]}{\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k\hat{\rho}_2)}} \\
& \cdot \exp\left[j\rho_2 K_{rs}\cos\left(\theta_{rs}-\theta_2\right)\right] d\theta_2\, d\rho_2. \qquad (73)
\end{aligned}
$$

As for the first-order case, a factor $g(\theta_2)$ has been included to account for receiver antenna directivity, and the equation has been written using polar coordinates. By direct comparison with first-order backscatter and assuming a stationary phase integration with respect to θ_2 in (73), there results

$$
\begin{aligned}
(E_{0_n}^{+})_2 \sim\ & \frac{-kC_0}{(2\pi)^{3/2}} \sum_{m,n}\sum_{p,q} P_{\mathbf{K}_{mn}} P_{\mathbf{K}_{pq}} \int_{\rho_2} \frac{F^2(\rho_2)}{\rho_2^{3/2}} \\
& \cdot \Bigg\{ e^{-j(\pi/4)} g(\theta_{rs}) e^{j\rho_2(K_{rs}-2k)} \\
& \cdot \frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_{rs})\,[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k\hat{\rho}_{rs})]}{\sqrt{K_{rs}}\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k\hat{\rho}_{rs})}} \\
& + e^{j(\pi/4)} g(\theta_{rs}+\pi) e^{-j\rho_2(K_{rs}+2k)} \\
& \cdot \frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_{rs})\,[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}+k\hat{\rho}_{rs})]}{\sqrt{K_{rs}}\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}+2k\hat{\rho}_{rs})}} \Bigg\}\, d\rho_2. \qquad (74)
\end{aligned}
$$

In (74) the following definitions hold:

$$\mathbf{K}_{rs} = \mathbf{K}_{mn} + \mathbf{K}_{pq},$$

$$\hat{\rho}_{rs} = \hat{\mathbf{x}}\cos\theta_{rs} + \hat{\mathbf{y}}\sin\theta_{rs}\ .$$

It may also be noted that

$$\hat{\rho}_{rs}(\theta_{rs}+\pi) = -\hat{\rho}_{rs}\ .$$

6.3. Third-Order Case

The triple convolution governing the third-order scatter as it appears in (55) is interpreted graphically in Figure 5. We now seek a result for the situation in which all three interactions occur near each other on the scattering surface, a phenomenon which shall be referred to as third-order patch scatter. It follows directly from the second-order result that the stationary phase contribution for the point $\rho_{12} \to 0$, $\rho_1 \to \rho_2$ gives

$$
\begin{aligned}
& \nabla\xi\cdot\nabla_{xy}\left\{\hat{\rho}\cdot\nabla\xi F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\} \\
& \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} \sim -\frac{j}{(2\pi)^2}\sum_{m,n}\sum_{p,q} P_{\mathbf{K}_{mn}} P_{\mathbf{K}_{pq}} \\
& \cdot \int_{x_2}\int_{y_2} \frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_2)\,[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k\hat{\rho}_2)]}{\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k\hat{\rho}_2)}} F(\rho_2)F(\rho_{23}) \\
& \cdot \frac{\exp\left\{j\rho_3\left[K_{rs}\cos\left(\theta_{rs}-\theta_2\right)-k\right]-jk\rho_{23}\right\}}{\rho_2\rho_{23}} dx_2\, dy_2. \qquad (75)
\end{aligned}
$$

Here, for clarity, the variables have been labeled as in Figure 5. Again, after the fashion of the second order, a further two-dimensional stationary phase integration may be carried out. With a view to the patch scatter condition the contribution to the right-hand side (RHS) of (75) from the point $\rho_{23} \to 0$, $\rho_2 \to \rho_3$ may be correspondingly written as

$$
\begin{aligned}
\mathrm{RHS}_{(75)} \sim\ & -jF(\rho_3)\frac{e^{-jk\rho_3}}{2\pi\rho_3}\sum_{m,n}\sum_{p,q} P_{\mathbf{K}_{mn}} P_{\mathbf{K}_{pq}} \\
& \cdot \frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_3)\,[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k\hat{\rho}_3)]}{\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k\hat{\rho}_3)}\sqrt{\mathbf{K}_{rs}\cdot(\mathbf{K}_{rs}-2k\hat{\rho}_3)}} \\
& \cdot \exp\left[j\rho_3 K_{rs}\cos\left(\theta_{rs}-\theta_3\right)\right], \qquad (76)
\end{aligned}
$$

where, as before, $\mathbf{K}_{rs} = \mathbf{K}_{mn} + \mathbf{K}_{pq}$.

Carrying out the final convolution in (55), again following the second-order analysis, leads to the expression

$$
\begin{aligned}
& \nabla\xi\cdot\nabla_{xy}\left\{\nabla\xi\cdot\nabla_{xy}\left[\hat{\rho}\cdot\nabla\xi F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right]\right. \\
& \left.\overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho}\right\} \overset{xy}{*} F(\rho)\frac{e^{-jk\rho}}{2\pi\rho} \\
& \sim jF(\rho_3)\frac{e^{-jk\rho_3}}{2\pi\rho_3}\sum_{m,n}\sum_{p,q}\sum_{w,v} P_{\mathbf{K}_{mn}} P_{\mathbf{K}_{pq}} P_{\mathbf{K}_{wv}} \\
& \cdot \exp\left[j\rho_3 K_{ab}\cos\left(\theta_{ab}-\theta_3\right)\right] \\
& \cdot \frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_3)\,[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k\hat{\rho}_3)]\,[\mathbf{K}_{wv}\cdot(\mathbf{K}_{rs}-k\hat{\rho}_3)]}{\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k\hat{\rho}_3)}\sqrt{\mathbf{K}_{rs}\cdot(\mathbf{K}_{rs}-2k\hat{\rho}_3)}}, \qquad (77)
\end{aligned}
$$

where the third-order surface wave vector $\mathbf{K}_{ab}$ is given by

$$\mathbf{K}_{ab} = \mathbf{K}_{mn} + \mathbf{K}_{pq} + \mathbf{K}_{wv}\ .$$

The backscatter condition, with reference to Figure 5, is clearly $\rho = 0$, $\rho_{30} = \rho_3$. Therefore, using polar coordinates as for the first 2 orders and introducing an antenna directivity $g(\theta_3)$, (77) becomes

$$(E_{0_n}^+)_3 \sim \frac{kC_0}{(2\pi)^2} \sum_{m,n}\sum_{p,q}\sum_{w,v} P_{\mathbf{K}_{mn}} P_{\mathbf{K}_{pq}} P_{\mathbf{K}_{wv}}$$
$$\cdot \int_{\rho_3} \frac{F^2(\rho_3)}{\rho_3} e^{-j2k\rho_3} \int_{-\pi}^{\pi} g(\theta_3)$$
$$\cdot \frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_3)\,[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k\hat{\rho}_3)]\,[\mathbf{K}_{wv}\cdot(\mathbf{K}_{rs}-k\hat{\rho}_3)]}{\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k\hat{\rho}_3)}\sqrt{\mathbf{K}_{rs}\cdot(\mathbf{K}_{rs}-2k\hat{\rho}_3)}}$$
$$\cdot \exp\left[j\rho_3 K_{ab}\cos(\theta_{ab}-\theta_3)\right] d\theta_3 d\rho_3. \tag{78}$$

This time, a stationary phase integration with respect to θ_3 leads to

$$(E_{0_n}^+)_3 \sim \frac{kC_0}{(2\pi)^{3/2}} \sum_{m,n}\sum_{p,q}\sum_{w,v} P_{\mathbf{K}_{mn}} P_{\mathbf{K}_{pq}} P_{\mathbf{K}_{wv}}$$
$$\cdot \int_{\rho_3} \frac{F^2(\rho_3)}{\rho_3^{3/2}} \Big\{ e^{-j(\pi/4)} g(\theta_{ab})$$
$$\cdot \frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_{ab})\,[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k\hat{\rho}_{ab})]\,[\mathbf{K}_{wv}\cdot(\mathbf{K}_{rs}-k\hat{\rho}_{ab})]}{\sqrt{K_{ab}}\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k\hat{\rho}_{ab})}\sqrt{\mathbf{K}_{rs}\cdot(\mathbf{K}_{rs}-2k\hat{\rho}_{ab})}}$$
$$\cdot e^{j\rho_3(K_{ab}-2k)} + e^{j(\pi/4)} g(\theta_{ab}+\pi)$$
$$\cdot \frac{(\mathbf{K}_{mn}\cdot\hat{\rho}_{ab})\,[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}+k\hat{\rho}_{ab})]\,[\mathbf{K}_{wv}\cdot(\mathbf{K}_{rs}+k\hat{\rho}_{ab})]}{\sqrt{K_{ab}}\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}+2k\hat{\rho}_{ab})}\sqrt{\mathbf{K}_{rs}\cdot(\mathbf{K}_{rs}+2k\hat{\rho}_{ab})}}$$
$$\cdot e^{-j\rho_3(K_{ab}+2k)} \Big\} d\rho_3. \tag{79}$$

Collecting the patch scatter results of (62), (74), and (79), the backscattered field to third order, given a dipole source whose excitation for the moment is general, is

$$(E_{0_n}^+) \sim \frac{kC_0}{(2\pi)^{3/2}} e^{-j(\pi/4)}$$
$$\cdot \sum_{\mathbf{K}_1} \frac{g(\phi_1)}{\sqrt{K_1}} \int_{\rho} \frac{F^2(\rho)}{\rho^{3/2}} e^{j\rho(K_1-2k)} d\rho$$
$$\cdot \Big\{ P(\mathbf{K}_1)(\mathbf{K}_1\cdot\hat{\rho}_1) - \sum_{\mathbf{K}_2} P(\mathbf{K}_2)P(\mathbf{K}_1-\mathbf{K}_2)$$
$$\cdot \frac{(\mathbf{K}_2\cdot\hat{\rho}_1)\,[(\mathbf{K}_1-\mathbf{K}_2)\cdot(\mathbf{K}_2-k\hat{\rho}_1)]}{\sqrt{\mathbf{K}_2\cdot(\mathbf{K}_2-2k\hat{\rho}_1)}}$$
$$+ \sum_{\mathbf{K}_3}\sum_{\mathbf{K}_2} P(\mathbf{K}_3)P(\mathbf{K}_2-\mathbf{K}_3)P(\mathbf{K}_1-\mathbf{K}_2)$$
$$\cdot \frac{(\mathbf{K}_3\cdot\hat{\rho}_1)\,[(\mathbf{K}_2-\mathbf{K}_3)\cdot(\mathbf{K}_3-k\hat{\rho}_1)]}{\sqrt{\mathbf{K}_3\cdot(\mathbf{K}_3-2k\hat{\rho}_1)}}$$
$$\cdot \frac{[(\mathbf{K}_1-\mathbf{K}_2)\cdot(\mathbf{K}_2-k\hat{\rho}_1)]}{\sqrt{\mathbf{K}_2\cdot(\mathbf{K}_2-2k\hat{\rho}_1)}} \Big\}. \tag{80}$$

Higher orders of scatter may be written down immediately by observing the pattern developed in the second- and third-order analyses. For example, for fourth order the term in the braces of (80) becomes

$$-\sum_{\mathbf{K}_4}\sum_{\mathbf{K}_3}\sum_{\mathbf{K}_2} P(\mathbf{K}_4)P(\mathbf{K}_3-\mathbf{K}_4)$$
$$\cdot P(\mathbf{K}_2-\mathbf{K}_3)P(\mathbf{K}_1-\mathbf{K}_2)$$
$$\cdot \frac{(\mathbf{K}_4\cdot\hat{\rho}_1)\,[(\mathbf{K}_3-\mathbf{K}_4)\cdot(\mathbf{K}_4-k\hat{\rho}_1)]}{\sqrt{\mathbf{K}_4\cdot(\mathbf{K}_4-2k\hat{\rho}_1)}}$$
$$\cdot \frac{[(\mathbf{K}_2\mathbf{K}_3)\cdot(\mathbf{K}_3-k\hat{\rho}_1)]\,[(\mathbf{K}_1-\mathbf{K}_2)\cdot(\mathbf{K}_2-k\hat{\rho}_1)]}{\sqrt{\mathbf{K}_3\cdot(\mathbf{K}_3-2k\hat{\rho}_1)}\sqrt{\mathbf{K}_2\cdot(\mathbf{K}_2-2k\hat{\rho}_1)}}. \tag{81}$$

To aid in compactly writing the results in the latter two equations, the letter subscripts on the K have been changed to numbers, with the meaning of the latter being obvious when compared to (62), (74), and (79) and the wave vector definitions which are associated with them.

7. Application to Pulse Radar

In this section, the backscattered fields derived in section 5 are modified by imposition of a particular source excitation, namely that of a pulsed dipole. The aim of this procedure is to develop suitable expressions which may be used to model a variety of backscatter cross sections for pulse radar applications. Since the higher-order derivatives follow directly from the first-order analysis, the latter is treated in some detail here. It should be mentioned that for pulse excitation the integral (for example, equation (46)) are of the Volterra type. Hence a Neumann series solution will converge [*Walsh and Srivastava*, 1987b].

Recalling that kC_0 is a function of the transformed time variable ω, (62) may be inverse Fourier transformed to yield

$$(E_{0_n}^+)_1(t) \sim \frac{1}{(2\pi)^{3/2}} \Big\{ \mathcal{F}_t^{-1}(kC_0) \overset{t}{*} \sum_{m,n} P_{\mathbf{K}_{mn}} \sqrt{K_{mn}}$$
$$\cdot \int_{\rho_1} \frac{F^2(\rho_1,\omega_0)}{\rho_1^{3/2}} \delta\left[t - 2\left(\frac{\rho_1}{c}\right)\right] \left[e^{-j(\pi/4)} g(\theta_{mn})\right.$$
$$\left. e^{j\rho_1 K_{mn}} - e^{j(\pi/4)} g(\theta_{mn}+\pi) e^{-j\rho_1 K_{mn}}\right] d\rho_1 \Big\}. \tag{82}$$

Here t star denotes time convolution, and $\mathcal{F}_t^{-1}(\ )$ is the inverse temporal Fourier transform. It is assumed that

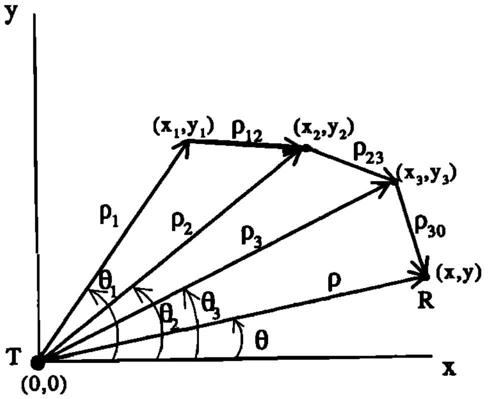

Figure 5. The geometry of the third-order scatter.

$$\mathcal{F}_t^{-1}\left[F^2(\rho_1,\omega)e^{-j2k\rho_1}\right] \sim F^2(\rho_1,\omega_0)\delta\left(t-\frac{2\rho_1}{c}\right),$$

where ω_0 is the dominant or representative frequency of excitation. This approximation is common in high-frequency applications and is discussed in detail, for example, by *Srivastava* [1984]. The ρ_1 integration in (82) is facilitated by the Dirac delta function $\delta[t-(2\rho_1/c)]$. Setting $x=(2\rho_1/c)-t$ and $d\rho_1=(c/2)dx$, while noting $x=0$ when $\rho_1=ct/2$, permits (82) to be written as

$$\begin{aligned}(E_{0_n}^{+})_1(t) \sim\ & \frac{1}{(2\pi)^{3/2}} \\ & \cdot\left\{\mathcal{F}_t^{-1}(kC_0) \overset{t}{*} \sum_{m,n} P_{\mathbf{K}_{mn}}\sqrt{K_{mn}}\left(\frac{c}{2}\right)\right. \\ & \cdot\frac{F^2(ct/2,\omega_0)}{(ct/2)^{3/2}}\left[g(\theta_{mn})e^{-j(\pi/4)}e^{j(ct/2)K_{mn}}\right. \\ & \left.\left.-g(\theta_{mn}+\pi)e^{j(\pi/4)}e^{-j(ct/2)K_{mn}}\right]\right\}. \qquad (83)\end{aligned}$$

It should be remembered that the earlier stationary phase integration which preceded the result in (82) requires that $\rho_1 K_{mn}=K_{mn}(ct/2)\gg 1$.

Before performing the time convolution in (83), the function $[kC_0](t)$ must be specified. In (48), C_0 was given as

$$C_0=\frac{I\Delta\ell k^2}{j\omega\epsilon_0}.$$

Emphasizing that the current I is, strictly, a function of ω, it is easily seen that

$$kC_0=-j\frac{\eta_0\Delta\ell}{c^2}\omega^2 I(\omega),$$

where $\eta_0=\sqrt{\mu_0/\epsilon_0}$ and $c=\sqrt{1/(\mu_0\epsilon_0)}$ is the (vacuum) speed of light. Therefore

$$\mathcal{F}_t^{-1}(kC_0)=j\frac{\eta_0\Delta\ell}{c^2}\frac{\partial^2}{\partial t^2}[i(t)], \qquad (84)$$

where $i(t)$ is the time domain dipole current.

Equations (83) and (84) together are useful in modeling a wide variety of transmitted signals. In the present discussion the particular case of a pulse radar is to be examined so that the antenna current may be modeled as

$$i(t)=I_0e^{j\omega_0 t}\left[h(t)-h(t-\tau_0)\right]. \qquad (85)$$

Here, $h(\ )$ is the usual Heaviside function, I_0 is the current magnitude, and τ_0 is the pulse duration. This current is obviously complex, but it may be physically realized by in-phase and quadrature components. Also, real trigonometric sines and cosines are given by linear combinations of (85) at positive and negative ω_0. Thus a variety of excitations may be derived from (85), and the basic structure of the ensuing analysis would be unaltered since the linear response equation ensures valid application of superposition. If it is agreed to ignore the leading and trailing edge terms, then from (85)

$$\frac{\partial^2 i(t)}{\partial t^2}=-\omega_0^2I_0e^{j\omega_0 t}\left[h(t)-h(t-\tau_0)\right].$$

In this case, (84) becomes

$$\mathcal{F}_t^{-1}(kC_0)\sim -j\eta_0\Delta\ell I_0k_0^2e^{j\omega_0 t}\left[h(t)-h(t-\tau_0)\right], \qquad (86)$$

with $k_0=\omega_0/c$. Consequently, (83) may be cast as

$$\begin{aligned}(E_{0_n}^{+})_1(t) \sim\ & -j\frac{\eta_0\Delta\ell I_0k_0^2}{(2\pi)^{3/2}}\sum_{m,n}P_{\mathbf{K}_{mn}}\sqrt{K_{mn}}\left(\frac{c}{2}\right) \\ & \cdot\left(\left\{e^{j\omega_0 t}\left[h(t)-h(t-\tau_0)\right]\right\}\right. \\ & \overset{t}{*}\left\{\frac{F^2(ct/2,\omega_0)}{(ct/2)^{3/2}}\left[g(\theta_{mn})e^{-j(\pi/4)}e^{j(ct/2)K_{mn}}\right.\right. \\ & \left.\left.\left.-g(\theta_{mn}+\pi)e^{j(\pi/4)}e^{-j(ct/2)K_{mn}}\right]\right\}\right). \qquad (87)\end{aligned}$$

Focusing on the convolution inside the braces, that factor may be written as

$$\begin{aligned}
&\int_{t'} e^{j\omega_0(t-t')}\left[h(t-t')-h(t-t'-\tau_0)\right]\\
&\cdot\frac{F^2(ct'/2,\omega_0)}{(ct'/2)^{3/2}}\Big\{e^{-j(\pi/4)}g(\theta_{mn})e^{j(ct'/2)K_{mn}}\\
&-e^{j(\pi/4)}g(\theta_{mn}+\pi)e^{-j(ct'/2)K_{mn}}\Big\}\,dt'\\
&=e^{j\omega_0 t}\int_{t-\tau_0}^{t}\frac{F^2(ct'/2,\omega_0)}{(ct'/2)^{3/2}}\\
&\cdot\Big(e^{-j(\pi/4)}g(\theta_{mn})\exp\{j\left[(cK_{mn}/2)-\omega_0\right]t'\}\\
&-e^{j(\pi/4)}g(\theta_{mn}+\pi)\\
&\cdot\exp\{-j\left[(cK_{mn}/2)+\omega_0\right]t'\}\Big)\,dt'.
\end{aligned}\tag{88}$$

For the intended application of this analysis, i.e., pulse radar operation, it may be reasonably assumed that $ct/2 \gg 1$ and $c\tau_0/2 \ll ct/2$. It follows that for t' in the range $t-\tau_0 < t' < t$, $ct'/2 \gg 1$. Denoting

$$\rho_0=\frac{ct/2+c(t-\tau_0)/2}{2}=\frac{c\,(t-\tau_0/2)}{2}$$

and using the fact that the Sommerfeld attenuation function may be assumed to be slowly varying over the integration range, the right-hand side (RHS) of (88) may be approximated as

$$\begin{aligned}
&\mathrm{RHS}_{(88)}\sim e^{j\omega_0 t}\frac{F^2(\rho_0,\omega_0)}{\rho_0^{3/2}}\Bigg(\Big\{g(\theta_{mn})e^{-j(\pi/4)}\\
&\cdot\int_{t-\tau_0}^{t}\exp\left[j\,(ct'/2)\,(K_{mn}-2k_0)\right]dt'\Big\}\\
&-\Big\{g(\theta_{mn}+\pi)e^{j(\pi/4)}\\
&\cdot\int_{t-\tau_0}^{t}\exp\left[-j\,(ct'/2)\,(K_{mn}+2k_0)\right]dt'\Big\}\Bigg)\\
&=\frac{2}{c}\Delta\rho e^{j\omega_0 t}\frac{F^2(\rho_0,\omega_0)}{\rho_0^{3/2}}\Bigg(\Big\{g(\theta_{mn})e^{-j(\pi/4)}\\
&\cdot e^{j\rho_0(K_{mn}-2k_0)}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{mn}-2k_0)\right]\Big\}\\
&-\Big\{g(\theta_{mn}+\pi)e^{j(\pi/4)}e^{-j\rho_0(K_{mn}+2k_0)}\\
&\cdot\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{mn}+2k_0)\right]\Big\}\Bigg).
\end{aligned}\tag{89}$$

In addition to the k_0 and ρ_0 definitions above,

$$\Delta\rho=c\tau_0/2,$$

$$\mathrm{SA}[\ ]=\frac{\sin(\ )}{(\ )}.$$

By using (89) in (87) there results

$$\begin{aligned}
\left(E_{0_n}^{+}\right)_1(t)\ &\sim\ -j\eta_0\Delta\ell\Delta\rho I_0 k_0^2\frac{F^2(\rho_0,\omega_0)}{(2\pi\rho_0)^{3/2}}\\
&\cdot\exp\{j\omega_0\left[t-2\,(\rho_0/c)\right]\}\sum_{m,n}P_{\mathbf{K}_{mn}}\sqrt{K_{mn}}\\
&\cdot\Bigg(\Big\{g(\theta_{mn})e^{-j(\pi/4)}e^{j\rho_0K_{mn}}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{mn}-2k_0)\right]\Big\}\\
&-\Big\{g(\theta_{mn}+\pi)e^{j(\pi/4)}e^{-j\rho_0K_{mn}}\\
&\cdot\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{mn}+2k_0)\right]\Big\}\Bigg).
\end{aligned}\tag{90}$$

In (90) it should be noted that

$$t-2\rho_0/c=\tau_0/2=\Delta\rho/c\,.$$

Further, since the original distance variable, ρ_1, in (62) is given by $\rho_1=ct'/2$, the requirement that

$$t-\tau_0<t'<t$$

implies

$$c(t-\tau_0)/2<\rho_1<ct/2$$

or

$$\rho_0-\Delta\rho/2<\rho_1<\rho_0+\Delta\rho/2\,.$$

That is, $\Delta\rho=c\tau_0/2$ is the potential range resolution (commonly referred to as "patch width") for the pulsed signal.

With a view to pulse Doppler radar applications a pulse to pulse time variation may be introduced into (90). One means of doing this is by introducing a time variation into the Fourier surface coefficients according to the equation

$$P_{\mathbf{K}_{mn}}=\sum_{\ell}P_{\mathbf{K}_{mn},\omega_\ell}e^{j\ell Wt},$$

i.e., by redefining the surface as

$$\xi(x,y,t)=\sum_{mn\ell}P_{\mathbf{K}_{mn},\omega_\ell}\exp\left[jN(mx+ny)+j\ell Wt\right],$$

with W being the fundamental temporal frequency of the surface and $\omega_\ell=\ell W$. Using this surface definition in (90), that expression becomes

$$\begin{aligned}(E_{0_n}^{+})_1(t,t_0) \quad &\sim \quad -j\eta_0 \Delta\ell \Delta\rho I_0 k_0^2 \frac{F^2(\rho_0,\omega_0)}{(2\pi\rho_0)^{3/2}} \\ &\cdot \exp\left\{j\omega_0\left(t-2\left(\frac{\rho_0}{c}\right)\right]\right\}\sum_{mn\ell} P_{\mathbf{K}_{mn},\omega_\ell} e^{j\ell Wt}\sqrt{K_{mn}} \\ &\cdot\left(\left\{g(\theta_{mn})e^{-j(\pi/4)}e^{j\rho_0 K_{mn}}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{mn}-2k_0)\right]\right\}\right. \\ &-\left\{g(\theta_{mn}+\pi)e^{j(\pi/4)}e^{-j\rho_0 K_{mn}}\right. \\ &\left.\left.\cdot\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{mn}-2k_0)\right]\right\}\right). \qquad (91)\end{aligned}$$

Here, t_0 is to be understood as the time of observation of the electric field after the beginning of the pulse. The variable t refers to the variation in the field for successive pulses; that is, the "experiment" is repeated, and the surface variation between pulses is accounted for by the dependency on t. It is, of course, important that the rate of variation of the surface be much smaller than the time necessary to make a single observation.

Equation (74), for the second-order backscatter, may be compared directly with its first-order counterpart in (62) and then the first-order pulse radar result in (90). The corresponding second-order return for the pulse radar model may thus be written down immediately as

$$\begin{aligned}(E_{0_n}^{+})_2(t) \quad &\sim \quad j\eta_0 \Delta\ell \Delta\rho I_0 k_0^2 \frac{F^2(\rho_0,\omega_0)}{(2\pi\rho_0)^{3/2}} \\ &\cdot \exp\left\{j\omega_0\left[t-2\left(\frac{\rho_0}{c}\right)\right]\right\}\sum_{m,n}\sum_{p,q} P_{\mathbf{K}_{mn}}P_{\mathbf{K}_{pq}} \\ &\cdot\left\{\left[g(\theta_{rs})e^{-j(\pi/4)}e^{j\rho_0 K_{rs}}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{rs}-2k_0)\right]\right.\right. \\ &\left.\cdot\frac{(\mathbf{K}_{mn}\cdot\hat{\boldsymbol{\rho}}_{rs})\left[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k_0\hat{\boldsymbol{\rho}}_{rs})\right]}{\sqrt{K_{rs}}\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k_0\hat{\boldsymbol{\rho}}_{rs})}}\right] \\ &+\left[g(\theta_{rs}+\pi)e^{j(\pi/4)}e^{-j\rho_0 K_{rs}}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{rs}+2k_0)\right]\right. \\ &\left.\left.\frac{(\mathbf{K}_{mn}\cdot\hat{\boldsymbol{\rho}}_{rs})\left[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}+k_0\hat{\boldsymbol{\rho}}_{rs})\right]}{\sqrt{K_{rs}}\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}+2k_0\hat{\boldsymbol{\rho}}_{rs})}}\right]\right\}. \qquad (92)\end{aligned}$$

The definitions of τ_0, ρ_0, $\Delta\rho$, and k_0 hold as before. Also, pulse to pulse surface variation over time may be accounted for, as before, by introducing time dependency into the surface coefficients, $P_{\mathbf{K}_{mn}}$ and $P_{\mathbf{K}_{pq}}$.

Following from the first- and second-order procedures, the third-order result may be written analogously as

$$\begin{aligned}(E_{0_n}^{+})_3(t) \quad &\sim \quad -j\eta_0 \Delta\ell \Delta\rho I_0 k_0^2 \frac{F^2(\rho_0,\omega_0)}{(2\pi\rho_0)^{3/2}} \\ &\cdot \exp\left\{j\omega_0\left[t-2\left(\frac{\rho_0}{c}\right)\right]\right\}\sum_{m,n}\sum_{p,q}\sum_{w,v} P_{\mathbf{K}_{mn}}P_{\mathbf{K}_{pq}}P_{\mathbf{K}_{wv}} \\ &\cdot\left(\left\{g(\theta_{ab})e^{-j(\pi/4)}e^{j\rho_0 K_{ab}}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{ab}-2k_0)\right]\right.\right. \\ &\cdot\frac{(\mathbf{K}_{mn}\cdot\hat{\boldsymbol{\rho}}_{ab})\left[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k_0\hat{\boldsymbol{\rho}}_{ab})\right]}{\sqrt{K_{ab}}\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k_0\hat{\boldsymbol{\rho}}_{ab})}} \\ &\left.\cdot\frac{\left[\mathbf{K}_{wv}\cdot(\mathbf{K}_{rs}-k_0\hat{\boldsymbol{\rho}}_{ab})\right]}{\sqrt{\mathbf{K}_{rs}\cdot(\mathbf{K}_{rs}-2k_0\hat{\boldsymbol{\rho}}_{ab})}}\right\} \\ &+\left\{g(\theta_{ab}+\pi)e^{j(\pi/4)}e^{-j\rho_0 K_{ab}}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{ab}+2k_0)\right]\right. \\ &\cdot\frac{(\mathbf{K}_{mn}\cdot\hat{\boldsymbol{\rho}}_{ab})\left[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}+k_0\hat{\boldsymbol{\rho}}_{ab})\right]}{\sqrt{K_{ab}}\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}+2k_0\hat{\boldsymbol{\rho}}_{ab})}} \\ &\left.\left.\cdot\frac{\left[\mathbf{K}_{wv}\cdot(\mathbf{K}_{rs}+k_0\hat{\boldsymbol{\rho}}_{ab})\right]}{\sqrt{\mathbf{K}_{rs}\cdot(\mathbf{K}_{rs}+2k_0\hat{\boldsymbol{\rho}}_{ab})}}\right\}\right) \qquad (93)\end{aligned}$$

In addition to the previous wavenumber definitions in (93), $\mathbf{K}_{ab} = \mathbf{K}_{mn} + \mathbf{K}_{pq} + \mathbf{K}_{wv}$. As usual, the surface time dependency may be introduced by appropriately modifying $P_{\mathbf{K}_{mn}}$, $P_{\mathbf{K}_{pq}}$, and $P_{\mathbf{K}_{wv}}$. Thus, to third order, the pulse radar results may be summarized as

$$\begin{aligned}(E_{0_n}^{+})(t) \quad &\sim \quad -j\eta_0 \Delta\ell \Delta\rho I_0 k_0^2 \frac{F^2(\rho_0,\omega_0)}{(2\pi\rho_0)^{3/2}} \\ &\cdot \exp\left\{j\omega_0\left(t-2\left(\frac{\rho_0}{c}\right)\right]\right\}e^{-j(\pi/4)}\sum_{\mathbf{K}_1} e^{j\rho_0 K_1}\frac{g(\phi_1)}{\sqrt{K_1}} \\ &\cdot\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_1-2k_0)\right]\left\{P_{\mathbf{K}_1}(\mathbf{K}_1\cdot\hat{\boldsymbol{\rho}}_1)\right. \\ &-\sum_{\mathbf{K}_2} P_{\mathbf{K}_2}P_{\mathbf{K}_1-\mathbf{K}_2}\frac{(\mathbf{K}_2\cdot\hat{\boldsymbol{\rho}}_1)\left[(\mathbf{K}_1-\mathbf{K}_2)\cdot(\mathbf{K}_2-k_0\hat{\boldsymbol{\rho}}_1)\right]}{\sqrt{\mathbf{K}_2\cdot(\mathbf{K}-2k_0\hat{\boldsymbol{\rho}}_1)}} \\ &+\sum_{\mathbf{K}_3}\sum_{\mathbf{K}_2} P_{\mathbf{K}_3}P_{\mathbf{K}_2-\mathbf{K}_3}P_{\mathbf{K}_1-\mathbf{K}_2} \\ &\cdot\frac{(\mathbf{K}_3\cdot\hat{\boldsymbol{\rho}}_1)\left[(\mathbf{K}_2-\mathbf{K}_3)\cdot(\mathbf{K}_3-k_0\hat{\boldsymbol{\rho}}_1)\right]}{\sqrt{\mathbf{K}_3\cdot(\mathbf{K}_3-2k_0\hat{\boldsymbol{\rho}}_1)}} \\ &\left.\cdot\frac{\left[(\mathbf{K}_1-\mathbf{K}_2)\cdot(\mathbf{K}_2-k_0\hat{\boldsymbol{\rho}}_1)\right]}{\sqrt{\mathbf{K}_2\cdot(\mathbf{K}_2-2k_0\hat{\boldsymbol{\rho}}_1)}}\right\}, \qquad (94)\end{aligned}$$

where, as mentioned in section 6.3, the K have been subscripted numerically. This equation may be extended to any order. Finally, it is noteworthy that only the sampling function with the minus sign has been retained. The term containing the other sampling function has been legitimately discarded because of its relatively insignificant contribution.

8. Application to Estimates of Target Radar Cross Sections

With a view to estimating the radar cross section of discrete targets it is next considered that the scattering surface may be described, with reference to Figure 6, as

$$\xi(x,y) = \begin{cases} z & x, y \in R, \\ 0 & \text{otherwise}, \end{cases}$$

where R is a small bounded region in the (x, y) plane. In order to apply the results of the preceding analysis the spatial Fourier transform

$$\begin{aligned} \mathcal{F}_{xy}\left[\xi(x,y)\right] &= \Xi(K_x, K_y) \\ &= \int_y \int_x \xi(x,y) e^{-jK_x x - jK_y y} dx dy \end{aligned} \quad (95)$$

is required. It is also useful to introduce a change of variables in (95) as

$$x' = x - x_1 \ , \ y' = y - y_1,$$

$(x_1, y_1) \in R$ as illustrated in Figure 6. Then, (95) becomes

$$\Xi(K_x, K_y) = e^{-j\mathbf{K}\cdot\boldsymbol{\rho}_1} \Xi'(K_x, K_y), \quad (96)$$

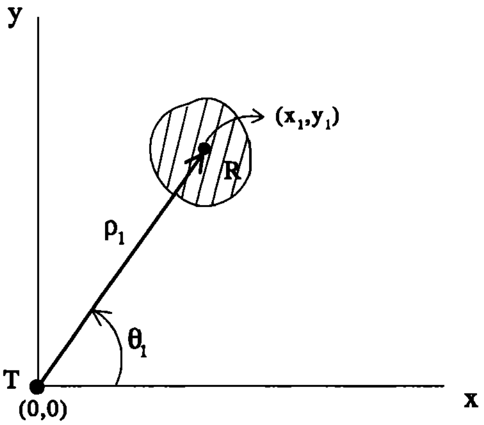

Figure 6. A discrete scattering region in the x–y plane.

$$\begin{aligned} \Xi'(K_x, K_y) &= \Xi'(\mathbf{K}) \\ &= \int_{y'} \int_{x'} \xi(x_1 + x', y_1 + y') e^{-j\mathbf{K}\cdot\boldsymbol{\rho}'} dx' dy' \\ &= \mathcal{F}_{x'y'}\left[\xi(x_1 + x', y_1 + y')\right] \ . \end{aligned}$$

8.1. First-Order, Pulse Radar

Retaining only the first term of (90) as mentioned in section 7, the first-order field expression becomes, in this present context,

$$\begin{aligned} \left(E_{0_n}^+\right)_1(t) \sim\ & -j\eta_0 \Delta\ell \Delta\rho I_0 k_0^2 \frac{F^2(\rho_0, \omega_0)}{(2\pi\rho_0)^{3/2}} \\ & \cdot \exp\left\{j\omega_0\left[t - 2\left(\frac{\rho_0}{c}\right)\right]\right\} e^{-j(\pi/4)} \\ & \cdot \frac{1}{(2\pi)^2} \int_{\mathbf{K}} e^{-j\mathbf{K}\cdot\boldsymbol{\rho}_1} \sqrt{K}\, \Xi'(\mathbf{K}) g(\theta_{\mathbf{K}}) \\ & \cdot e^{j\rho_0 K} \mathrm{SA}\left[\frac{\Delta\rho}{2}(K - 2k_0)\right] d\mathbf{K}. \end{aligned} \quad (97)$$

Here, $\mathbf{K}_{mn}$ is identified as $\mathbf{K}$, and the summation has been converted to an integral in view of the surface transform in (96), which logically suggests identifying the Fourier surface coefficients $P_{\mathbf{K}_{mn}}$ as

$$e^{-j\mathbf{K}\cdot\boldsymbol{\rho}_1} \Xi'(\mathbf{K}) \frac{d\mathbf{K}}{(2\pi)^2} \ .$$

It should be recalled that in (97)

$$k_0 = \omega/c \ , \ \Delta\rho = c\tau_0/2 \ , \ \rho_0 = (t - \tau_0/2)/2 \ .$$

Next, it is noted that $\left[\Delta\rho/(2\pi)^2\right] \int (\ ) d\mathbf{K}$ of (97) may be expressed in polar coordinates, $(K, \theta_{\mathbf{K}})$, as

$$\begin{aligned} \frac{\Delta\rho}{(2\pi)^2} \int (\ ) d\mathbf{K} =\ & \frac{\Delta\rho}{(2\pi)^2} \int_0^\infty \Bigg\{ K^{3/2} e^{j\rho_0 K} \\ & \cdot \mathrm{SA}\left[\frac{\Delta\rho}{2}(K - 2k_0)\right] \int_{-\pi}^{\pi} g(\theta_{\mathbf{K}})\, \Xi'(K, \theta_{\mathbf{K}}) \\ & \cdot \exp\left[-j\rho_1 K \cos(\theta_{\mathbf{K}} - \theta_1)\right] d\theta_{\mathbf{K}} \Bigg\} dK. \end{aligned} \quad (98)$$

Assuming $\rho_1 K \gg 1$, the $\theta_{\mathbf{K}}$ integral in (98) may be estimated by the method of stationary phase. It is easily verified that (98) may be approximated as

$$\begin{aligned} \frac{\Delta\rho}{(2\pi)^2} \int (\ ) d\mathbf{K} \sim\ & \frac{\Delta\rho}{2\pi (2\pi\rho_1)^{1/2}} e^{j(\pi/4)} g(\theta_1) \\ & \cdot \int_0^\infty K\, \Xi'(K, \theta_1) e^{jK(\rho_0 - \rho_1)} \\ & \cdot \mathrm{SA}\left[\frac{\Delta\rho}{2}(K - 2k_0)\right] dK. \end{aligned} \quad (99)$$

Further, if the pulse is long,

$$\Delta\rho \mathrm{SA}\left[\frac{\Delta\rho}{2}(K-2k_0)\right] \longrightarrow 2\pi\delta(K-2k_0),$$

and assuming $\rho_0 = \rho_1$,

$$\frac{\Delta\rho}{(2\pi)^2}\int(\ )d\mathbf{K} \sim \frac{2k_0}{(2\pi\rho_1)^{1/2}}e^{j(\pi/4)}g(\theta_1)\Xi'(2k_0,\theta_1). \tag{100}$$

Finally, using (100) in (97), the expression for the backscattered first-order field as received from the region R becomes

$$\begin{aligned}\left(E_{0_n}^{+}\right)_1(t) &\sim -2j\eta_0\Delta\ell I_0 k_0^3\frac{F^2(\rho_1,\omega_0)}{(2\pi\rho_1)^2}\\ &\cdot\exp\left\{j\omega_0\left[t-2\left(\frac{\rho_1}{c}\right)\right]\right\}g(\theta_1)\Xi'(2k_0,\theta_1).\end{aligned} \tag{101}$$

8.2. Second-Order, Pulse Radar

A result in keeping with (101) is now sought for the second-order field. Again, retention of only the $\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{rs}-2k_0)\right]$ gives from (92)

$$\begin{aligned}\left(E_{0_n}^{+}\right)_2(t) &\sim j\eta_0\Delta\ell\Delta\rho I_0 k_0^2\frac{F^2(\rho_0,\omega_0)}{(2\pi\rho_0)^{3/2}}\\ &\cdot\exp\left\{j\omega_0\left[t-2\left(\frac{\rho_0}{c}\right)\right]\right\}e^{-j(\pi/4)}\sum_{m,n}\sum_{p,q}P_{\mathbf{K}_{mn}}P_{\mathbf{K}_{pq}}\\ &\cdot g(\theta_{rs})\frac{(\mathbf{K}_{mn}\cdot\hat{\boldsymbol{\rho}}_{rs})\left[\mathbf{K}_{pq}\cdot(\mathbf{K}_{mn}-k_0\hat{\boldsymbol{\rho}}_{rs})\right]}{\sqrt{K_{rs}}\sqrt{\mathbf{K}_{mn}\cdot(\mathbf{K}_{mn}-2k_0\hat{\boldsymbol{\rho}}_{rs})}}\\ &\cdot e^{j\rho_0 K_{rs}}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_{rs}-2k_0)\right].\end{aligned} \tag{102}$$

In a manner analogous to that which led to (97), application of (102) to the problem at hand suggests the following identifications:

$$\begin{aligned}&\mathbf{K}_{mn}\rightarrow\mathbf{K}_1,\\ &P_{\mathbf{K}_{mn}}\rightarrow e^{-j\mathbf{K}_1\cdot\boldsymbol{\rho}_1}\Xi'(\mathbf{K}_1)\tfrac{d\mathbf{K}_1}{(2\pi)^2},\\ &\mathbf{K}_{pq}\rightarrow\mathbf{K}_2,\\ &P_{\mathbf{K}_{pq}}\rightarrow e^{-j\mathbf{K}_2\cdot\boldsymbol{\rho}_1}\Xi'(\mathbf{K}_2)\tfrac{d\mathbf{K}_2}{(2\pi)^2},\\ &\mathbf{K}_{rs}=\mathbf{K}_{mn}+\mathbf{K}_{pq}\rightarrow\mathbf{K}_1+\mathbf{K}_2=\mathbf{K}_s,\end{aligned}$$

for example. With these changes in labeling, (102) becomes

$$\begin{aligned}\left(E_{0_n}^{+}\right)_2(t) &\sim j\eta_0\Delta\ell\Delta\rho I_0 k_0^2\frac{F^2(\rho_0,\omega_0)}{(2\pi\rho_0)^{3/2}}\\ &\cdot\exp\left\{j\omega_0\left[t-2\left(\frac{\rho_0}{c}\right)\right]\right\}e^{-j(\pi/4)}\frac{1}{(2\pi)^4}\\ &\cdot\int_{\mathbf{K}_2}\int_{\mathbf{K}_1}\Xi'(\mathbf{K}_1)\Xi'(\mathbf{K}_2)g(\theta_s)\\ &\cdot\frac{(\mathbf{K}_1\cdot\hat{\boldsymbol{\rho}}_s)\left[\mathbf{K}_2\cdot(\mathbf{K}_1-k_0\hat{\boldsymbol{\rho}}_s)\right]}{\sqrt{K_s}\sqrt{\mathbf{K}_1\cdot(\mathbf{K}_1-2k_0\hat{\boldsymbol{\rho}}_s)}}e^{-j\mathbf{K}_s\cdot\boldsymbol{\rho}_1}\\ &\cdot e^{j\rho_0 K_s}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_s-2k_0)\right]d\mathbf{K}_1 d\mathbf{K}_2.\end{aligned} \tag{103}$$

Changing the integration variable from $\mathbf{K}_2$ to $\mathbf{K}_s$ and using polar coordinates $(K_s,\theta_{\mathbf{K}_s})$, a portion of (103) may be written as

$$\begin{aligned}&\frac{\Delta\rho}{(2\pi)^4}\int_{\mathbf{K}_2}\int_{\mathbf{K}_1}\Xi'(\mathbf{K}_1)\Xi'(\mathbf{K}_2)g(\theta_s)\\ &\cdot\frac{(\mathbf{K}_1\cdot\hat{\boldsymbol{\rho}}_s)\left[\mathbf{K}_2\cdot(\mathbf{K}_1-k_0\hat{\boldsymbol{\rho}}_s)\right]}{\sqrt{K_s}\sqrt{\mathbf{K}_1\cdot(\mathbf{K}_1-2k_0\hat{\boldsymbol{\rho}}_s)}}e^{-j\mathbf{K}_s\cdot\boldsymbol{\rho}_1}e^{j\rho_0 K_s}\\ &\cdot\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_s-2k_0)\right]d\mathbf{K}_1 d\mathbf{K}_2\\ &=\frac{\Delta\rho}{(2\pi)^4}\int_{\mathbf{K}_1}\int_{K_s}\sqrt{K_s}e^{j\rho_0 K_s}\,\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_s-2k_0)\right]\\ &\Xi'(\mathbf{K}_1)\int_{-\pi}^{\pi}g(\theta_s)\Xi'(K_s\hat{\boldsymbol{\rho}}_s-\mathbf{K}_1)\\ &\cdot\frac{(\mathbf{K}_1\cdot\hat{\boldsymbol{\rho}}_s)\left[(K_s\hat{\boldsymbol{\rho}}_s-\mathbf{K}_1)\cdot(\mathbf{K}_1-k_0\hat{\boldsymbol{\rho}}_s)\right]}{\sqrt{\mathbf{K}_1\cdot(\mathbf{K}_1-2k_0\hat{\boldsymbol{\rho}}_s)}}\\ &\exp\left[-jK_s\rho_1\cos(\theta_1-\theta_{\mathbf{K}_s})\right]d\theta_{\mathbf{K}_s}dK_s d\mathbf{K}_1.\end{aligned} \tag{104}$$

If, as similar to the first-order result, a stationary phase integration may be implemented with respect to $\theta_{\mathbf{K}_s}$, then the RHS of (104) becomes

$$\begin{aligned}\mathrm{RHS}_{(104)} &= \frac{\Delta\rho}{(2\pi)^3}\frac{e^{j(\pi/4)}}{\sqrt{2\pi\rho_1}}g(\theta_1)\int_{\mathbf{K}_1}\int_{K_s}\Xi'(\mathbf{K}_1)\\ &\cdot\Xi'(K_s\hat{\boldsymbol{\rho}}_s-\mathbf{K}_1)e^{jK_s(\rho_0-\rho_1)}\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_s-2k_0)\right]\\ &\cdot\frac{(\mathbf{K}_1\cdot\hat{\boldsymbol{\rho}}_1)\left[(K_s\hat{\boldsymbol{\rho}}_1-\mathbf{K}_1)\cdot(\mathbf{K}_1-k_0\hat{\boldsymbol{\rho}}_1)\right]}{\sqrt{\mathbf{K}_1\cdot(\mathbf{K}_1-2k_0\hat{\boldsymbol{\rho}}_1)}}dK_s d\mathbf{K}_1.\end{aligned} \tag{105}$$

Again, invoking the long-pulse assumption so that

$$\Delta\rho\mathrm{SA}\left[\frac{\Delta\rho}{2}(K_s-2k_0)\right]\longrightarrow 2\pi\delta(K_s-2k_0),$$

(105) may be written as

$$\begin{aligned}\mathrm{RHS}_{(104)} &= \frac{e^{j(\pi/4)}}{(2\pi)^2\sqrt{2\pi\rho_1}}g(\theta_1)\\ &\cdot\int_{\mathbf{K}_1}\Xi'(\mathbf{K}_1)\Xi'(2k_0\hat{\boldsymbol{\rho}}_1-\mathbf{K}_1)\\ &\cdot\frac{(\mathbf{K}_1\cdot\hat{\boldsymbol{\rho}}_1)\left[(2k_0\hat{\boldsymbol{\rho}}_1-\mathbf{K}_1)\cdot(\mathbf{K}_1-k_0\hat{\boldsymbol{\rho}}_1)\right]}{\sqrt{\mathbf{K}_1\cdot(\mathbf{K}_1-2k_0\hat{\boldsymbol{\rho}}_1)}}d\mathbf{K}_1.\end{aligned} \tag{106}$$

Then, changing the $\mathbf{K}_1$ integration variable to $\mathbf{K}$ as defined by $\mathbf{K} = 2k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K}_1$, the integral portion on the RHS of (106) may be written as

$$\int_{\mathbf{K}_1} [\ ]\, d\mathbf{K}_1 = \int_{\mathbf{K}} \Xi'(2k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K})\Xi'(\mathbf{K}) \cdot \frac{[(2k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K})\cdot\hat{\boldsymbol{\rho}}_1]\,[\mathbf{K}\cdot(k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K})]}{\sqrt{\mathbf{K}\cdot(\mathbf{K} - 2k_0\hat{\boldsymbol{\rho}}_1)}} d\mathbf{K}\,. \quad (107)$$

Furthermore, since both the $\mathbf{K}_1$ and $\mathbf{K}$ integrals are over all wavenumber space, the $\mathbf{K}$ integration in (107) may be written as

$$\int_{\mathbf{K}} [\ ]\, d\mathbf{K} = \frac{1}{2}\int_{\mathbf{K}} \frac{\Xi'(2k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K})\Xi'(\mathbf{K})}{\sqrt{\mathbf{K}\cdot(\mathbf{K} - 2k_0\hat{\boldsymbol{\rho}}_1)}} \cdot \{(\mathbf{K}\cdot\hat{\boldsymbol{\rho}}_1)\,[(2k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K})\cdot(\mathbf{K} - k_0\hat{\boldsymbol{\rho}}_1)] - [(2k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K})\cdot\hat{\boldsymbol{\rho}}_1]\,[\mathbf{K}\cdot(\mathbf{K} - k_0\hat{\boldsymbol{\rho}}_1)]\}\, d\mathbf{K}\,. \quad (108)$$

It is not difficult to show that the term in the braces of the integrand of (108) may be simplified to

$$\{\cdots\} = -2k_0K^2\sin^2(\theta_{\mathbf{K}} - \theta_1) = -2k_0\left|\mathbf{K}\times\hat{\boldsymbol{\rho}}_1\right|^2\ ,$$

giving

$$\frac{1}{2}\int_{\mathbf{K}} [\ ]\, d\mathbf{K} = -k_0\int_{\mathbf{K}} \Xi'(2k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K})\Xi'(\mathbf{K}) \cdot \frac{\left|\mathbf{K}\times\hat{\boldsymbol{\rho}}_1\right|^2}{\sqrt{\mathbf{K}\cdot(\mathbf{K} - 2k_0\hat{\boldsymbol{\rho}}_1)}} d\mathbf{K}\,. \quad (109)$$

A further simplification is possible via the variable change

$$\mathbf{K} = \mathbf{K}' + k_0\hat{\boldsymbol{\rho}}_1,$$

so that

$$\mathbf{K}\cdot(\mathbf{K} - 2k_0\hat{\boldsymbol{\rho}}_1) = K'^2 - k_0^2,$$
$$\left|\mathbf{K}\times\hat{\boldsymbol{\rho}}_1\right|^2 = \left|\mathbf{K}'\times\hat{\boldsymbol{\rho}}_1\right|^2\ .$$

Hence, using these relationships and the fact that $\mathbf{K}$ and $\mathbf{K}'$ are over all wavenumber space, expression (109) may be cast as

$$-k_0\int_{\mathbf{K}} \Xi'(k_0\hat{\boldsymbol{\rho}}_1 + \mathbf{K})\Xi'(k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K})\frac{\left|\mathbf{K}\times\hat{\boldsymbol{\rho}}_1\right|^2}{\sqrt{K^2 - k_0^2}} d\mathbf{K}\,. \quad (110)$$

Substituting (110) into (106) for the $\mathbf{K}_1$ integral, and further substituting this result into (107), gives for the second-order backscattered field of (103) received from region R of Figure 6

$$\left(E_{0_n}^{+}\right)_2(t) \sim -2j\eta_0\Delta\ell I_0 k_0^3\frac{F^2(\rho_1,\omega_0)}{(2\pi\rho_1)^2} \cdot \exp\left\{j\omega_0\left[t - 2\left(\frac{\rho_1}{c}\right)\right]\right\} g(\theta_1)\frac{1}{2(2\pi)^2} \cdot \left\{\int_{\mathbf{K}} \Xi'(k_o\hat{\boldsymbol{\rho}}_1 + \mathbf{K})\Xi'(k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K})\frac{\left|\mathbf{K}\times\hat{\boldsymbol{\rho}}_1\right|^2}{\sqrt{K^2 - k_0^2}} d\mathbf{K}\right\}. \quad (111)$$

Using (51), (101), and (111), the backscatter field to second order for the case being considered is

$$E_{0_n}^{+}(t) \sim -2j\eta_0\Delta\ell I_0 k_0^3\frac{F^2(\rho_1,\omega_0)}{(2\pi\rho_1)^2} \cdot \exp\left\{j\omega_0\left[t - 2\left(\frac{\rho_1}{c}\right)\right]\right\} g(\theta_1)\left\{\Xi'(2k_o,\theta_1) + \frac{1}{2(2\pi)^2}\int_{\mathbf{K}} \Xi'(k_o\hat{\boldsymbol{\rho}}_1 + \mathbf{K})\Xi'(k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K}) \cdot \frac{\left|\mathbf{K}\times\hat{\boldsymbol{\rho}}_1\right|^2}{\sqrt{K^2 - k_0^2}} d\mathbf{K}\right\}. \quad (112)$$

It is worth noting that by definition,

$$\Xi'(2k_o,\theta_1) = \int_y\int_x \xi(x,y)e^{-j2k_0x\cos\theta_1} \cdot e^{-j2k_0y\sin\theta_1} dx dy$$

or, in polar coordinates (ρ,ϕ),

$$\Xi'(2k_o,\theta_1) = \int_\rho\int_\phi \xi(\rho,\phi)\rho \cdot \exp\left[-j2k_0\rho\cos(\phi - \theta_1)\right] d\phi d\rho.$$

This may be interpreted as a Bragg-type scattering, the reaction being with spatial wavenumbers which are $2k_0$ in magnitude in the look direction. On the other hand, if the exponent in the integral is expanded in ascending powers of ρ, it may be seen that the leading term is the volume bounded by $z = \xi(x,y)$ and $z = 0$.

The integral term in (112) may be interpreted as a "corner reflector" effect. We note that $(k_0\hat{\boldsymbol{\rho}}_1 + \mathbf{K})\cdot(k_0\hat{\boldsymbol{\rho}}_1 - \mathbf{K}) = k_0^2 - K^2$. The integrand is unbounded when $K = k_0$. At this value the vectors are perpendicular. Furthermore, the cross product term is maximum when $\mathbf{K}$ is perpendicular to the look direction $\hat{\boldsymbol{\rho}}_1$. It may then be seen that $\hat{\boldsymbol{\rho}}_1$ bisects the right angle between $k_0\hat{\boldsymbol{\rho}}_1 + k_0\hat{K}$ and $k_0\hat{\boldsymbol{\rho}}_1 - k_0\hat{K}$.

8.3. Radar Cross Sections

Beginning with (112), a general form for the radar cross section of a small bounded region, R of Figure 6, may be developed. From (112) the backscattered electric field is estimated by the expression

$$\begin{aligned} E_{0_n}^{+}(t) \quad &\sim \quad -j\eta_0 \Delta\ell I_0 k_0 \frac{F^2(\rho_1, \omega_0)}{(2\pi\rho_1)^2} \\ &\cdot \exp\left\{ j\omega_0 \left[t - 2\left(\frac{\rho_1}{c}\right)\right]\right\} \left(2k_0^2 \mathcal{P}\right), \quad (113) \end{aligned}$$

with $\mathcal{P}$ being defined as

$$\begin{aligned} \mathcal{P} \quad &= \quad \Xi'(2k_o, \theta_1) + \frac{1}{2(2\pi)^2} \int_{\mathbf{K}} \Xi'(k_o\hat{\rho}_1 + \mathbf{K}) \\ &\cdot \Xi'(k_0\hat{\rho}_1 - \mathbf{K}) \frac{|\mathbf{K} \times \hat{\rho}_1|^2}{\sqrt{K^2 - k_0^2}} d\mathbf{K}. \quad (114) \end{aligned}$$

Since the transmitting antenna is assumed to be a vertical dipole, the vertical electric field E_z in the (x, y) plane is given by

$$E_z(t) = -j\eta_0 I_0 \Delta\ell k_0 \frac{e^{-jk_0\rho_1}}{4\pi\rho_1}. \quad (115)$$

The free space gain, G_t, of the transmitting antenna in the horizontal plane may be defined as

$$G_t = \frac{2\pi \left|E_z\right|^2}{\eta_0 P_t} \rho_1^2, \quad (116)$$

where P_t is the transmitted power. Then, using E_z from (115), it follows that

$$G_t = \frac{2\pi}{\eta_0 P_t} \rho_1^2 \left|\frac{\eta_0 I_0 \Delta\ell k_0}{4\pi\rho_1}\right|^2 = \frac{\eta_0 k_0^2}{8\pi P_t} \left|I_0 \Delta\ell\right|^2 . \quad (117)$$

The power, P_r, received by an antenna placed at the origin is estimated by

$$P_r = \frac{A_r}{2\eta_0} \left|E_{0_n}^{+}\right|^2, \quad (118)$$

where $E_{0_n}^{+}$ appears in (113) and the effective free space area of the receiving antenna whose free space gain is G_r is defined by

$$A_r = \frac{\lambda_0^2 G_r}{4\pi}, \quad (119)$$

with λ_0 being the radiation wavelength. Hence, (118), (119), and (113) together give for the received power

$$P_r = \frac{G_r \lambda_0^2}{8\pi\eta_0} \left|\eta_0 I_0 \Delta\ell k_0\right|^2 \frac{\left|F(\rho_1, \omega_0)\right|^4}{\left(2\pi\rho_1\right)^4} \left(2k_0^2\right)^2 \left|\mathcal{P}\right|^2 . \quad (120)$$

From (117) and (120) it is easily seen that (120) may be expressed as

$$P_r = \frac{\lambda_0^2 G_r G_t P_t \left|F(\rho_1, \omega_0)\right|^4}{\left(4\pi\right)^3 \rho_1^4} \left[\frac{16k_0^4}{\pi} \left|\mathcal{P}\right|^2\right] . \quad (121)$$

Comparing (121) with the standard radar range equation yields for the radar cross section σ

$$\sigma = \left[\frac{16k_0^4}{\pi}\right] \left|\mathcal{P}\right|^2, \quad (122)$$

where $\mathcal{P}$ is defined in (114). It may be observed that since $\sigma \propto k_0^4$ and the leading term in $\mathcal{P}$ is the surface protrusion volume, the cross section in (122) is consistent with Rayleigh theory [see, e.g., *Ishimaru*, 1991]. However, the result may also be interpreted in terms of Bragg scatter.

8.3.1. Cross section of a perfectly conducting sphere. In application of (122) we first consider a perfectly conducting hemisphere of radius a embedded in the highly conducting surface. See Figure 6 and consider that R is a circle forming the lower surface of the hemisphere. Of course, it should be recalled that in addition to the small-height approximation the magnitude of the surface gradient must be much less than unity (i.e., $|\nabla\xi| \ll 1$). Consequently, while the small-height approximation may be met by stipulating a appropriately, it is clearly not possible to guarantee $|\nabla\xi| \ll 1$ everywhere on the sphere. Thus it should be expected that the spherical cross section using this analysis will be some fraction of its actual value.

Using the transformation following (95) and introducing polar coordinates gives

$$\Xi'(\mathbf{K}) = \int_0^a \int_0^{2\pi} \rho' \left[a^2 - \rho'^2\right]^{1/2} e^{-j\mathbf{K}\cdot\boldsymbol{\rho}'} d\theta' d\rho' \quad (123)$$

for the expression immediately following (96). Since $0 \le \theta' \le 2\pi$ and, for a particular $\mathbf{K}$, θ_K is fixed,

$$\Xi'(\mathbf{K}) = \int_0^a \int_0^{2\pi} \rho' \left[a^2 - \rho'^2\right]^{\frac{1}{2}} e^{-jK\rho'\cos\theta'} d\theta' d\rho'. \quad (124)$$

It is readily determined that the θ' integral in is given by

$$I_{\theta'} = \int_0^{2\pi} e^{-jK\rho'\cos\theta'} d\theta' = 2\pi J_0(K\rho'), \quad (125)$$

where J_0 is the usual zeroth-order Bessel function. Substituting (125) into (124) and evaluating the ρ'

integral produces

$$\Xi'(\mathbf{K}) = \frac{2\sqrt{2}\pi a^{3/2}}{K^{3/2}} \Gamma(3/2)\, J_{3/2}(aK), \tag{126}$$

with Γ and J being gamma and Bessel functions, respectively. Any suitable handbook of mathematical functions yields

$$\Gamma(3/2) = \sqrt{\pi}/2,$$

$$J_{3/2}(aK) = \sqrt{\frac{2}{\pi a K}} \left(\frac{\sin aK}{aK} - \cos aK \right) .$$

Then,

$$\Xi'(\mathbf{K}) = \frac{2\pi a}{K^2} \left(\frac{\sin aK}{aK} - \cos aK \right) . \tag{127}$$

Expanding the bracketed term, while noting that the stationary phase value of K is $2k_0$, yields

$$\Xi'(\mathbf{K}) = \frac{2\pi a^3}{3} + O[k_0 a]^4, \tag{128}$$

with $O[k_0a]^4$ signifying terms of order $(k_0a)^4$. Retaining only the first terms of (114) and (128), there results

$$\mathcal{P} = \frac{2\pi a^3}{3} ,$$

the volume of the hemisphere, so that from (122), the first-order cross section of the hemisphere embedded in the highly conducting surface becomes

$$\sigma = (64\pi/9)\, k_0^4 a^6 . \tag{129}$$

Adjusting this by a factor of 1/4 to obtain the equivalent free space expression for the spherical cross section, we obtain

$$\sigma_{fs} = (16\pi/9)\, k_0^4 a^6 . \tag{130}$$

The accepted scattering cross section of the conducting sphere in free space, for small $k_0 a$, appears in the literature [*Ruck et al.*, 1970] as

$$\sigma_{fs} = 9\pi k_0^4 a^6 . \tag{131}$$

Thus it is observed that the $k_0^4 a^6$ factor appearing in (130) is of the correct form. However, as expected from the small-slope approximation, the multiplier $16\pi/9$ is only 20% of that appearing in (131).

8.3.2. Cross section of a perfectly conducting exponential boss. As a final application of (122), we consider that there is a surface protuberance above the region R of Figure 6 which may be characterized by

$$\xi(\rho') = h_e e^{-\rho'^2/(2a^2)} \quad 0 \le \rho' \le \infty, \tag{132}$$

that is, an exponential boss of maximum elevation h_e. The constraint on the positive parameter, a, as dictated by the small-slope condition may be guaranteed by requiring that the maximum slope be much less than unity (i.e., $|\nabla\xi|_{\max} \ll 1$). This leads to the fact that $a \gg h_e e^{-1/2}$.

Analogously to (124), (95) now becomes

$$\Xi'(\mathbf{K}) = h_e \int_0^\infty \int_0^{2\pi} \rho' e^{-\rho'^2/(2a^2)} e^{-jK\rho' \cos\theta'} d\theta' d\rho'. \tag{133}$$

Evaluating the θ' integral as in (125) then leads to

$$\Xi'(\mathbf{K}) = 2\pi h_e \int_0^\infty \rho' e^{-\rho'^2/(2a^2)} J_0(K\rho') d\rho' . \tag{134}$$

This integral evaluates exactly to

$$\Xi'(\mathbf{K}) = 2\pi h_e a^2 e^{-K^2a^2/2} . \tag{135}$$

Applying the same constraints as used in (128)–(129), the first-order cross section of the Gaussian boss may be written as

$$\sigma = 64\pi k_0^4 h_e^2 a^4 e^{-(2k_0 a)^2} . \tag{136}$$

We note that since a must have SI dimensions of meters, σ has dimensions of m^2 as required.

9. Concluding Remarks

A general technique for electromagnetic scattering, founded in the theory presented by *Walsh and Donnelly* [1987b], is applied to rough surfaces which are representable as a Fourier series (or transform). The approach is initially similar to that of *Walsh and Srivastava* [1987a], but a significant simplification of notation leads to electric field expressions which may be readily applied to a variety of scattering problems.

Field equations are first developed without specification of a particular source. However, the surface impedance boundary condition as well as the "good conductor" and "small-height" assumptions are imposed. These approximations lead to an expression for the component of the field normal to the scattering surface. Subsequent imposition of a vertical dipole source leads to a successive approximation

(Neumann series) solution for the electric field to first order in surface slope. Each term of the series, starting with the direct field (or zero-order) term, may be simply interpreted as successively higher orders of scatter. For a static surface the field equations are developed to third order, and the extension to higher orders is shown to be obvious.

With a view to radar applications the general field expressions are reduced to the case of backscatter, and a pulsed dipole source is introduced. The result is a set of electric field equations which, in conjunction with the radar range equation, are used to determine an expression (to second order) for the HF radar backscatter cross section of a finite target. Initially, the target geometry has been kept general so that it may be verified that the theory leads to the condition of Rayleigh scattering. This was subsequently shown specifically to first order for the case of a conducting sphere and an exponential boss.

The theory presented here has been used elsewhere [*Walsh et al.*, 1990] to generate monostatic cross sections of time-varying rough surfaces with particular emphasis on the ocean. Subsequent application [*Howell and Walsh*, 1993a, 1993b; *Gill and Walsh*, 1992; *Gill et al.*, 1996] has proven very encouraging.

Extension of the analysis presented here to the case of bistatic reception of the scattered field has been analyzed by *Gill and Walsh*, this issue. Additionally, cross sections derived by E. Gill and J. Walsh from this basic work have been recently submitted to the open literature (High-frequency bistatic cross sections of the ocean surface, submitted to *Radio Science*, 2000).

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