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Energy transport in a nonlinear and inhomogeneous random gravity wave field†

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Certain tertiary resonant interactions of gravity waves which have been found previously can be obtained more easily by using a simple extension of Whitham's formalism. The contribution of these interactions to the total energy transfer in an inhomogeneous random field of gravity waves is calculated. It is found to be small for open-ocean waves, but to be of some importance for shallow-water waves, where topography or mean shear currents may produce strong inhomogeneities. The nonlinear splitting of the group velocity is found to be unimportant in wave fields with sufficiently smooth spectra.

1. Introduction

The generation, propagation and dissipation of surface waves in the ocean have been objects of intense study for a long time. From theoretical considerations it turned out that the evolution of wave fields is governed by an equation similar to Boltzmann's transport equation in statistical mechanics:

$$\frac{\partial A}{\partial t}+\dot{\mathbf{x}}.\frac{\partial A}{\partial \mathbf{x}}+\dot{\mathbf{k}}.\frac{\partial A}{\partial \mathbf{k}}=S \tag{1}$$

(Hasselmann 1968, 1970). Here $A(\mathbf{k},\mathbf{x},t)$ denotes the action density, equal to the spectral energy density divided by the intrinsic frequency, $\dot{\mathbf{x}}=\partial\Omega/\partial\mathbf{k}$, the group velocity, and $\dot{\mathbf{k}}=-\partial\Omega/\partial\mathbf{x}$, the rate of change of the wavenumber due to refraction. The left-hand side of (1) describes, besides horizontal transport and refraction of energy, the interaction with inhomogeneous or time-varying mean currents and depths, studied originally by Longuet-Higgins & Stewart (1961, 1962), who expressed the energy transfer in terms of the radiation stress associated with the waves.

The source function $S(\mathbf{k},\mathbf{x},t)$ is the rate of change of action density and is due to several physical processes, e.g. wave generation caused by the mechanisms investigated by Miles (1957) and Phillips (1957), dissipation due to turbulent bottom friction (Hasselmann & Collins 1968) or white-capping (Longuet-Higgins & Turner 1974; Hasselmann 1973), and energy transfer due to resonant interactions (Phillips 1960; Hasselmann 1962). This last mechanism has been found to be of major importance during the stage of wave generation.

† Shortened version of the author's Ph.D. thesis, hereafter cited as I.

While the left-hand side of (1) is linear in A, the source function is not; for example, the resonant interactions produce a term cubic in A. It is the purpose of this paper to look in detail at the approximations which lead to (1), and to derive a more accurate version which contains nonlinear corrections on the left-hand side. It is found that certain tertiary interactions are of importance; these were discussed by Longuet-Higgins & Phillips (1962) and are usually neglected since they do not contribute to the total energy transfer. However, this result is valid only for homogeneous wave fields, and in the following the effect of these interactions on the propagation of an inhomogeneous wave field is considered.

A powerful mathematical technique for such problems has been developed by Whitham (1965*a*, 1967). It had to be generalized slightly since in its original form it is suitable for dealing with single wave trains rather than a random wave field. The procedure outlined below is applicable to all those wave problems where the interactions are *very* weak, i.e. where resonant three-wave processes are prevented by the structure of the dispersion relation. It is therefore different from the method used by Simmons (1969) to calculate capillary–gravity wave interactions.

In addition, it is found that the nonlinear splitting of the group velocity mentioned by Whitham (1965*b*) does not occur in a random wave field with a sufficiently smooth spectrum.

In the following some of the nonlinear coefficients which are defined by cumbersome algebraic expressions are given explicitly in appendix B; further details can be found in I.

2. Analysis

We start with a form of the equations of motion for surface waves which can be deduced using a variational principle (Luke 1967):

$$\delta \int L \, d\mathbf{x} \, dt = 0, \tag{2}$$

with
$$L = \int_{-h}^{Z+\zeta} \left\{ \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial \mathbf{x}}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right] - \frac{\partial \Phi}{\partial t} - \mathbf{U}.\frac{\partial \Phi}{\partial \mathbf{x}} + gz \right\} dz.$$

Here $\Phi(\mathbf{x}, z, t)$ and $\zeta(\mathbf{x}, t)$ denote the velocity potential and surface elevation of the wave field, which is superimposed on a mean current of velocity $\mathbf{U}(\mathbf{x}, t)$ and elevation $Z(\mathbf{x}, t)$; $h(\mathbf{x})$ is the bottom profile and $\mathbf{x}$ and z are the horizontal and vertical co-ordinates, respectively. Mean quantities, which are allowed to be slowly varying functions of $\mathbf{x}$ and t, describe the state of the fluid in the *absence* of the wave field.

For the following perturbation expansion, whose details are given in I, we introduce two small parameters: ϵ and η, where ϵ is a measure of nonlinearity, i.e. a typical wave steepness, and η is a measure of inhomogeneity or unsteadiness, i.e. the ratio of a typical wavelength or wave period to the scale of the mean motion. The wave field is then expanded in a way similar to that of Whitham

(1967), except that we do not restrict ourselves to a single wave train but consider a superposition of a great number, say N, of wave components:

$$\zeta(\mathbf{x},t) = a(\mathbf{x},t)+\sum_n a_n(\mathbf{x},t)\exp(is_n)+\sum_{m,n} a_{mn}(\mathbf{x},t)\exp[i(s_m+s_n)]+O(\epsilon^3), \quad (3a)$$

$$\Phi(\mathbf{x},z,t) = b(\mathbf{x},z,t)+\sum_n b_n(\mathbf{x},z,t)\exp(is_n)+\sum_{m,n} b_{mn}(\mathbf{x},z,t)\exp[i(s_m+s_n)]+O(\epsilon^3). \quad (3b)$$

So that ζ and Φ are real, the sums in (3) are extended to negative indices with the conventions $s_{-n} = -s_n$, $a_{-n} = a_n^*$, etc.

The frequencies and wavenumbers are related to the phases $s_n(\mathbf{x},t)$ by

$$\omega_n = -\partial s_n/\partial t, \quad \mathbf{k}_n = \partial s_n/\partial \mathbf{x}. \quad (4)$$

The *linear* amplitudes a_n and b_n are $O(\epsilon)$; the nonlinear amplitudes a_{mn} and b_{mn} and the wave-induced changes a and b in the mean field are $O(\epsilon^2)$. All these quantities are assumed to be slowly varying functions of $\mathbf{x}$ and t, i.e.

$$(\partial/\partial t, \partial/\partial \mathbf{x}) = O(\eta)\,(\omega_n, \mathbf{k}_n).$$

For $N = 1$, (3) and the following calculations of this section are similar to those of Whitham (1967).

The vertical structure of the coefficients b, b_n and b_{mn} can be obtained by inserting (3 b) in the potential equation

$$\Delta\Phi = 0, \quad (5)$$

which follows from (2) by considering the Φ variation. This procedure is not quite straightforward since the variational principle (2) should make the use of the equations of motion unnecessary. The complication is due to the fact that the upper limit of the integral in (2) depends on the phases $s_n(\mathbf{x},t)$; a detailed discussion of this problem has been given by Bisshop & Wilson (1968). One obtains

$$\left.\begin{aligned} b(\mathbf{x},z,t) &= A(\mathbf{x},t),\\ b_n(\mathbf{x},z,t) &= i\frac{\cosh k_n(z+h)}{\cosh k_n H}A_n(\mathbf{x},t),\\ b_{mn}(\mathbf{x},z,t) &= i\frac{\cosh|\mathbf{k}_n+\mathbf{k}_m|(z+h)}{\cosh|\mathbf{k}_n+\mathbf{k}_m|H}A_{mn}(\mathbf{x},t),\end{aligned}\right\} \quad (6)$$

with A, A_n and A_{mn} still to be determined. The (arbitrary) factors $\cosh k_n H$ and $\cosh|\mathbf{k}_n+\mathbf{k}_m|H$, where $H(\mathbf{x},t) = h(\mathbf{x})+Z(\mathbf{x},t)$ is the mean water depth, are introduced for convenience.

With (3b), (4) and (6) the integration in (2) can be carried out analytically and the Lagrangian can be expressed as

$$L = L(a, \boldsymbol{\beta}, \gamma, a_n, A_n, a_{mn}, A_{mn}, s_n, \omega_n, \mathbf{k}_n), \quad (7)$$

where $\boldsymbol{\beta} = -\partial A/\partial \mathbf{x}$ and $\gamma = \partial A/\partial t$. The explicit form of (7) is given in I.

The *averaged* Lagrangian

$$\mathscr{L} = \bar{L} = \frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} L\,ds_1\ldots ds_N \quad (8)$$

is now calculated in a simple way. After Taylor expansion of L in powers of the perturbation parameter ϵ, the phases appear in L only in expressions of the form $\exp[i(s_m+s_n+\ldots)]$. It is easily verified that

$$\overline{\exp(is_n)} = 0, \quad \overline{\exp(i[s_m+s_n])} = \delta_{m,-n}, \tag{9a, b}$$

$$\overline{\exp(i[s_l+s_m+s_n])} = 0, \tag{9c}$$

$$\overline{\exp(i[s_j+s_l+s_m+s_n])} = (\delta_{j,-l}\delta_{m,-n}+\delta_{j,-n}\delta_{l,-m}+\delta_{j,-m}\delta_{l,-n}) \times [1-\tfrac{1}{2}(\delta_{jl}\delta_{mn}+\delta_{jm}\delta_{ln}+\delta_{jn}\delta_{lm})], \tag{9d}$$

and one finally obtains the averaged Lagrangian†

$$\begin{aligned}\mathscr{L} = {} & (\mathbf{U}.\boldsymbol{\beta}-\gamma)(H+a)+\tfrac{1}{2}ga^2+\tfrac{1}{2}H\beta^2 \\ & + \sum_{n>0}\{A_n A_n^*[k_n \tanh k_n H + ak_n^2(1+\tanh^2 k_n H)] + ga_n a_n^* \\ & - (A_n a_n^* + A_n^* a_n)[\omega_n - (\mathbf{U}+\boldsymbol{\beta}).\mathbf{k}_n](1+ak_n \tanh k_n H)\} \\ & + \tfrac{1}{2}\sum_{m,n>0} V_{mn} + O(\epsilon^6).\end{aligned} \tag{10}$$

The interaction term $V_{mn} = V(\mathbf{k}_m, \mathbf{k}_n, \omega_m, \omega_n, a_m, a_n, A_m, A_n, a_{mn}, A_{mn})$ is given explicitly in I.

It should be noted that the Lagrangian contains terms $O(\epsilon^4)$ whereas in (3) the $O(\epsilon^3)$ terms have already been neglected. It is easy to show, however, that these terms do not contribute to the *averaged* Lagrangian.

The governing equations are now obtained by applying the modified variational principle

$$\delta \int \mathscr{L}\, d\mathbf{x}\, dt = 0. \tag{11}$$

First we consider the variations with respect to a and A, the wave-induced changes in elevation and velocity potential:

$$\frac{\partial \mathscr{L}}{\partial a} = 0, \quad \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \gamma} - \frac{\partial}{\partial \mathbf{x}}.\frac{\partial \mathscr{L}}{\partial \boldsymbol{\beta}} = 0, \tag{12}$$

which lead to

$$\frac{\partial}{\partial t}(H+a) + \frac{\partial}{\partial \mathbf{x}}.\left[(\mathbf{U}+\boldsymbol{\beta})(H+a) + \sum_{n>0}\frac{\mathbf{k}_n}{\omega_n - \mathbf{U}.\mathbf{k}_n} 2ga_n a_n^*\right] = 0, \tag{13a}$$

$$\frac{\partial \boldsymbol{\beta}}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\left[\mathbf{U}.\boldsymbol{\beta} + ga + \sum_{n>0}\frac{k_n}{\sinh 2k_n H} 2ga_n a_n^*\right] = 0, \tag{13b}$$

correct to second order.

These equations describe the well-known influence of waves on the mean current and water level (Longuet-Higgins & Stewart 1961). They could be derived directly from conservation of mass and momentum as discussed by Whitham (1967) for $N = 1$. The quantities a and $\boldsymbol{\beta}$ can be determined from (13) if the linear approximation to the wave field is known. In the following, for simplicity we redefine H and $\mathbf{U}$ to be the *total* mean depth and current, respectively. We must then remember, however, that the mean state cannot be regarded as independent of the wave field.

† Up to terms which do not contribute to the following variations.

The equations for the variations with respect to a_{mn}, A_{mn} and A_n can be used to express these quantities in terms of the linear amplitudes a_n:

$$A_n = \frac{\omega_n - \mathbf{U}.\mathbf{k}_n}{k_n \tanh k_n H} a_n\{1 + \sum_{m>0} R(\mathbf{k}_n, \mathbf{k}_m, \omega_n, \omega_m) a_m a_m^*\} + O(\epsilon^5), \tag{14a}$$

$$a_{mn} = P(\mathbf{k}_n, \mathbf{k}_m, \omega_n, \omega_m) a_m a_n + O(\epsilon^4), \tag{14b}$$

$$A_{mn} = Q(\mathbf{k}_n, \mathbf{k}_m, \omega_n, \omega_m) a_m a_n + O(\epsilon^4). \tag{14c}$$

The a_n variation gives the dispersion law

$$(\omega_n - \mathbf{k}_n.\mathbf{U})^2 = gk_n \tanh(k_n H) \times \{1 + 4\sum_{m>0} S(\mathbf{k}_n, \mathbf{k}_m, \omega_n, \omega_m) a_m a_m^*\} + O(\epsilon^4). \tag{15}$$

The coefficients P, Q and S are given in appendix B.

The linear part of (15) is the well-known relation for small amplitude waves. The nonlinear part represents, besides the influence of changes in H and $\mathbf{U}$, the frequency shift due to the finite amplitude of the wave components. This can be compared with a result of Longuet-Higgins & Phillips (1962), who calculated the change in phase velocity produced by tertiary nonlinear wave interactions. In fact, the two results turn out to be identical.† This is not surprising if one looks at (9d), which states that only those quadruples of wave components for which the conditions $s_j + s_l = 0$ and $s_m + s_n = 0$ hold contribute to the averaged Lagrangian. Because of (4) this is equivalent to

$$\left.\begin{aligned} \mathbf{k}_j + \mathbf{k}_l = \mathbf{k}_m + \mathbf{k}_n = 0, \\ \omega_j + \omega_l = \omega_m + \omega_n = 0, \end{aligned}\right\} \tag{16}$$

which from the viewpoint of weak interaction theory is a special case of the general resonance condition for a four-wave process:

$$\left.\begin{aligned} \mathbf{k}_j + \mathbf{k}_l + \mathbf{k}_m + \mathbf{k}_n = 0, \\ \omega_j + \omega_l + \omega_m + \omega_n = 0. \end{aligned}\right\} \tag{17}$$

Longuet-Higgins & Phillips (1962) found that the interactions (16) change the phase velocity but do not transfer energy among different wave components. This result is valid if the wave field is strictly steady and homogeneous. In the following it will be shown how these interactions contribute to the energy transport in an unsteady and/or inhomogeneous wave field.

The last variational equation is given by the variation of (11) with respect to the phases s_n, which occur in the averaged Lagrangian only through their derivatives $\mathbf{k}_n$ and ω_n:

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \omega_n} - \frac{\partial}{\partial \mathbf{x}}.\frac{\partial \mathscr{L}}{\partial \mathbf{k}_n} = 0. \tag{18}$$

This represents the conservation of action density, and by defining a transport velocity $\mathbf{u}_n = -(\partial\mathscr{L}/\partial\mathbf{k}_n)/(\partial\mathscr{L}/\partial\omega_n)$ may be expressed as

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \omega_n} + \frac{\partial}{\partial \mathbf{x}}.\left(\mathbf{u}_n \frac{\partial \mathscr{L}}{\partial \omega_n}\right) = 0. \tag{19}$$

† Apart from a slight misprint in the cited paper.

We find that

$$\frac{\partial \mathscr{L}}{\partial \omega_n} = -\frac{2ga_n a_n^*}{(gk_n \tanh k_n H)^{\frac{1}{2}}}\{1+2\sum_{m>0} T(\mathbf{k}_n, \mathbf{k}_m, \omega_m, \omega_n)\, a_m a_m^*\}, \quad (20\,a)$$

$$\mathbf{u}_n = \mathbf{U}+\frac{\partial}{\partial \mathbf{k}_n}(gk_n \tanh k_n H)^{\frac{1}{2}} + \sum_{m>0}\left(B_{mn}\frac{\mathbf{k}_n}{k_n}+C_{mn}\frac{\mathbf{k}_m}{k_m}\right) a_m a_m^*. \quad (20\,b)$$

In the linear approximation $\mathbf{u}_n$ is identical with the group velocity while the action density is given by the ratio of the wave energy to the intrinsic frequency, a general result for linear waves (Bretherton & Garrett 1968).

Equations (15) and (19) constitute a system of $2N$ partial differential equations for the $a_n(\mathbf{x}, t)$ and $s_n(\mathbf{x}, t)$ which in principle could be solved with given initial values. This would be practical, however, only for a *very* small number N of wave components. For oceanic conditions N must be taken as a large number; it is therefore reasonable to consider the limit $N \to \infty$ and to adopt a continuous representation of the wave field. This can be done in the usual way and leads to the radiation balance equation (Hasselmann 1968). It is first necessary, however, to look in more detail at the appropriate nonlinear group velocity.

3. Nonlinear group velocity and radiation transfer equation

It is convenient to rewrite (15) and (19) by introducing the dependent variables $I_n(\mathbf{x}, t) = \partial \mathscr{L}/\partial \omega_n$ and $\mathbf{k}_n(\mathbf{x}, t)$ instead of $a_n(\mathbf{x}, t)$ and $s_n(\mathbf{x}, t)$. Using (4), we then have

$$\frac{\partial I_n}{\partial t}+\frac{\partial}{\partial \mathbf{x}}.\mathbf{u}_n I_n = 0, \quad \frac{\partial \mathbf{k}_n}{\partial t}+\frac{\partial \omega_n}{\partial \mathbf{x}} = 0, \quad (21\,a, b)$$

where ω_n is obtained from (15) as

$$\omega_n = \Omega(\mathbf{k}_n; \mathbf{k}_1, \ldots, \mathbf{k}_N, a_1, \ldots, a_N; \mathbf{x}, t). \quad (22)$$

Since the amplitudes in (22) occur only at second order in ϵ, we may replace a_n by I_n, using the linear part of (20 *a*).

In the limit of small amplitude, the system (21) is decoupled for different indices, and each pair of equations has one double characteristic velocity, which is equal to the corresponding linear group velocity $\partial \omega_n/\partial \mathbf{k}_n$.

For finite amplitudes, however, (21) turns out to be hyperbolic and in general has $2N$ different characteristic velocities, which are the correct generalizations of group velocity for nonlinear waves, as shown by Whitham (1965 *b*) for $N = 1$. Consequently, disturbances in wavenumber and amplitude will propagate at different speeds. The quantity $\partial \omega_n/\partial \mathbf{k}_n$ must be defined carefully, since its value depends on which measure of amplitude is kept constant during the differentiation with respect to wavenumber. As pointed out by Lighthill (1965), $\partial \omega_n/\partial \mathbf{k}_n$ is always the energy propagation velocity if the amplitude measure $\mathscr{L}/\omega$ is kept constant.

In contrast, in a random wave field consisting of many components with different wavelengths, the propagation of each component is again governed by only one velocity, provided that the energy spectrum is sufficiently smooth.

It is shown below that in this case the velocity $\partial\omega_n/\partial\mathbf{k}_n$ tends to a unique value, regardless of the amplitude measure which is kept constant.

The dispersion relation (22) can be written more explicitly as

$$\omega_n = \Omega^{(0)}(\mathbf{k}_n) + \sum_m \hat{S}(\mathbf{k}_m, \mathbf{k}_n)\, a_m a_m^*, \tag{23}$$

where

$$\Omega^{(0)}(\mathbf{k}_n) = \mathbf{U}.\mathbf{k} + (gk \tanh kH)^{\frac{1}{2}},$$

$$\hat{S}(\mathbf{k}_m, \mathbf{k}_n) = 2(gk \tanh kH)^{\frac{1}{2}}\, S(\mathbf{k}_m, \mathbf{k}_n, \Omega^{(0)}(\mathbf{k}_m), \Omega^{(0)}(\mathbf{k}_n)).$$

Let $d_m = a_m a_m^*/\phi(\mathbf{k}_m)$ be an amplitude measure for the mth component, $\phi(\mathbf{k})$ being an arbitrary positive function. The gradient of frequency with respect to wavenumber, keeping d_m constant, is calculated from (23):

$$\left(\frac{\partial\omega_n}{\partial\mathbf{k}_n}\right)_{d_m} = \frac{\partial\Omega^{(0)}(\mathbf{k}_n)}{\partial\mathbf{k}_n} + \sum_{m>0}\left[\frac{\partial\hat{S}(\mathbf{k}_m, \mathbf{k}_n)}{\partial\mathbf{k}_n}\phi(\mathbf{k}_m) + \hat{S}\,\frac{\partial\phi(\mathbf{k}_m)}{\partial\mathbf{k}_n}\right] d_m. \tag{24}$$

Since

$$\frac{\partial\phi(\mathbf{k}_m)}{\partial\mathbf{k}_n} = \frac{\partial\phi(\mathbf{k}_n)}{\partial\mathbf{k}_n}\,\delta_{mn},$$

it follows that

$$\left(\frac{\partial\omega_n}{\partial\mathbf{k}_n}\right)_{d_m} = \frac{\partial\Omega^{(0)}(\mathbf{k}_n)}{\partial\mathbf{k}_n} + \sum_{m>0}\frac{\partial\hat{S}(\mathbf{k}_m, \mathbf{k}_n)}{\partial\mathbf{k}_n}\, a_m a_m^* + \hat{S}(\mathbf{k}_n, \mathbf{k}_n)\,\frac{1}{\phi(\mathbf{k}_n)}\,\frac{\partial\phi(\mathbf{k}_n)}{\partial\mathbf{k}_n}\, a_n a_n^*. \tag{25}$$

Equation (25) shows explicitly the dependence of $\partial\omega_n/\partial\mathbf{k}_n$ on the choice of $\phi(\mathbf{k})$, at $O(\epsilon^2)$.

We assume now that the number of wave components is large, each contributing only a small fraction to the sum in (25), which is to say that the spectrum must not be sharply peaked at some dominant wavenumber. The sum can then be replaced by an integral with a finite value, whereas the last term in (25) tends to zero, regardless of the form of $\phi(\mathbf{k})$. Physically, neglect of this term corresponds to the neglect of self-interactions compared with interactions among different wave components. Therefore, in the continuous limit, which seems to be a reasonable approximation for oceanic surface waves, the velocity $\partial\omega_n/\partial\mathbf{k}_n$ is defined uniquely even for finite amplitudes, and as one can show by similar considerations, its value coincides with the transport velocity $\mathbf{u}_n$ defined in (19).

Now there are no difficulties in writing (19) in a continuous form. The wavenumber spectrum $E(\mathbf{k}, \mathbf{x}, t)$ is defined in the usual way as the Fourier transform of the autocorrelation of the vertical elevation $\zeta(\mathbf{x}, t)$ and is the power spectrum in the statistical sense rather than the spectrum of physical energy.

From (3*a*) and the assumption that the initial values of the phases are independent and equally distributed over $(0, 2\pi)$ we obtain the relation between $E(\mathbf{k}, \mathbf{x}, t)$ and the amplitudes a_n:

$$E(\mathbf{k}, \mathbf{x}, t)\, d\mathbf{k} \simeq \sum_n^{d\mathbf{k}} \{2a_n a_n^* + 4 \sum_{m>0} (P_{m,n-m}^2 + P_{m,m-n}^2)\, a_m a_m^* a_n a_n^*\}, \tag{26}$$

with the coefficients P_{mn} from (14*b*). The superscript $d\mathbf{k}$ on the first summation sign indicates that the sum is taken only over values of n for which $\mathbf{k}_n(\mathbf{x}, t)$ lies in the range $d\mathbf{k}$ around the fixed wavenumber $\mathbf{k}$. The nonlinear term in (26) represents the distortion of the wave form due to sum and difference wavenumbers; it corresponds to higher harmonics in a Stokes wave. For one-dimensional frequency spectra this term has been calculated by Tick (1961).

Details of the limiting process $N \to \infty$ can be found in appendix A; a more elegant proof, valid for linear wave fields, is given by Dewar (1970).

We end up with the radiation transfer equation

$$\frac{D}{Dt}\left\{\frac{E(\mathbf{k},\mathbf{x},t)}{(gk\tanh kH)^{\frac{1}{2}}}(1+J(\mathbf{k},\mathbf{x},t))\right\} = 0, \tag{27}$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t}+\frac{\partial\Omega}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{x}}-\frac{\partial\Omega}{\partial\mathbf{x}}\cdot\frac{\partial}{\partial\mathbf{k}};$$

the frequency Ω is given by (15), which in continuous form reads

$$\Omega(\mathbf{k},\mathbf{x},t) = \mathbf{k}\,.\,\mathbf{U}+(gk\tanh kH)^{\frac{1}{2}}\left\{1+\int S(\mathbf{k},\mathbf{k}')\,E(\mathbf{k}')\,d\mathbf{k}'\right\}. \tag{28}$$

The function $J(\mathbf{k},\mathbf{x},t)$ summarizes the nonlinear contributions in (20a) and (26); it has the form

$$J = \int E(\mathbf{k}')\left\{T(\mathbf{k},\mathbf{k}')-P^2(\mp\mathbf{k}',\mathbf{k}\pm\mathbf{k}')\frac{E(\mathbf{k}\pm\mathbf{k}')}{E(\mathbf{k})}\right\}d\mathbf{k}', \tag{29}$$

where the sum over both signs is to be taken. The linear part of (27),

$$\left(\frac{\partial}{\partial t}+\frac{\partial\Omega^{(0)}}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{x}}-\frac{\partial\Omega^{(0)}}{\partial x}\frac{\partial}{\partial\mathbf{k}}\right)\frac{E(\mathbf{k},\mathbf{x},t)}{(gk\tanh kH)^{\frac{1}{2}}} = 0, \tag{30}$$

was given in this form by Hasselmann (1970). It describes the propagation and refraction of small amplitude wave fields on varying currents and depths and includes the energy transfer between waves and the mean current via the radiation-stress mechanism (Longuet-Higgins & Stewart 1961, 1962; Bretherton & Garrett 1968).

The nonlinear terms in (27) have the following effects.

(a) Owing to the change in Ω and therefore $\partial\Omega/\partial\mathbf{k}$, the waves propagate with a different velocity which depends on the total wave height spectrum. In particular, the direction of $\partial\Omega/\partial\mathbf{k}$ is generally different from the wavenumber direction even if there is no mean current. It should be noted that the change in group velocity is different in both magnitude and direction from Stokes' mass transport velocity.

(b) The refraction term, proportional to $\partial\Omega/\partial\mathbf{x}$, may be non-vanishing even if the mean current and depth are horizontally homogeneous; e.g. in a fetch area where $\partial E/\partial\mathbf{x} \neq 0$, according to (28) we also have $\partial\Omega/\partial\mathbf{x} \neq 0$. This can be interpreted as follows: at second order the properties of the waveguide depend on the wave field and therefore an inhomogeneous wave field causes refraction even if the mean state is homogeneous.

(c) The change in action density expressed by the function $J(\mathbf{k},\mathbf{x},t)$ in (29) leads to an additional 'energy'† flow between the waves and the mean field, thus representing a higher-order correction to the radiation-stress effects.

† 'Energy' in the previous sense.

4. Quantitative results

To estimate quantitatively the influence of the nonlinear corrections to (27) on wave propagation, they must be compared either with their linear values, or if possible, with other contributions to the energy balance (1). We consider first the energy transfer due to resonant interactions of the general form (17), which have been found to be responsible for the major part of the total energy transfer during wave generation (Hasselmann *et al.* 1973). Its contribution to the source function is of the form

$$S_{\mathrm{r.i.}} = \int d\mathbf{k}'\, d\mathbf{k}'' \ldots E(\mathbf{k})\, E(\mathbf{k}')\, E(\mathbf{k}''), \tag{31}$$

where the dots stand for a kernel function which does not concern us here. The main feature of (31) is that it contains a triple product of energy spectra; in terms of our perturbation parameter ϵ, the integral is $O(\epsilon^6)$.

The nonlinear terms in (27) contain products of two spectra. Furthermore, they contain space or time derivatives which are assumed to be small; their order of magnitude is given by $O(\eta\epsilon^4)$. The relevant ratio is therefore η/ϵ^2. Taking ϵ to be a typical wave steepness and putting $\epsilon \approx 0{\cdot}1$, we conclude that the additional energy flow given by (27) is of the same order as the resonant energy transfer if $\eta \approx 10^{-2}$, i.e. if the scale of inhomogeneity is about 100 typical wavelengths or wave periods. This order-of-magnitude calculation is very crude; nevertheless it should provide a first rough estimate of whether or not the nonlinear terms in (27) can influence the evolution of the wave field in a concrete situation.

The changes in frequency, according to (28), and thus the change in group velocity, as well as the function J from (29), have been computed numerically for two cases.

(*a*) Fully developed wind waves in oceanic conditions, which can be described by a Pierson–Moskowitz (1964) spectrum.

(*b*) Waves in shallow water (depth $\approx$ 7 m) with a strong onshore wind ($\sim$ 9 bft), measured by Schrader (1968) in the Elbe estuary.

In both cases the details of the directional distribution turned out to be of minor importance, and a $\cos^4$-law was assumed.

In figures 1 and 2 the relative change in group velocity is shown. In the oceanic case, it increases monotonically with wavenumber, which means that it is mainly short waves that feel the change in propagation properties due to finite amplitude. The direction of the group-velocity change (not shown) is essentially the mean wind direction (maximum deviation about 20°). This means that the interactions (16) tend to sharpen the directional distribution. Since the magnitude does not exceed a few per cent, however, it is not likely that this effect could be observed.

In case (*b*), for high wavenumbers the curves behave similarly, whereas in the range near the spectral maximum, where the waves are essentially in shallow water, the behaviour is different. The relative magnitude is considerably larger than in deep water, e.g. for waves with wavelength one-third of those of the waves with maximum energy the change in group velocity is more than 10 %.

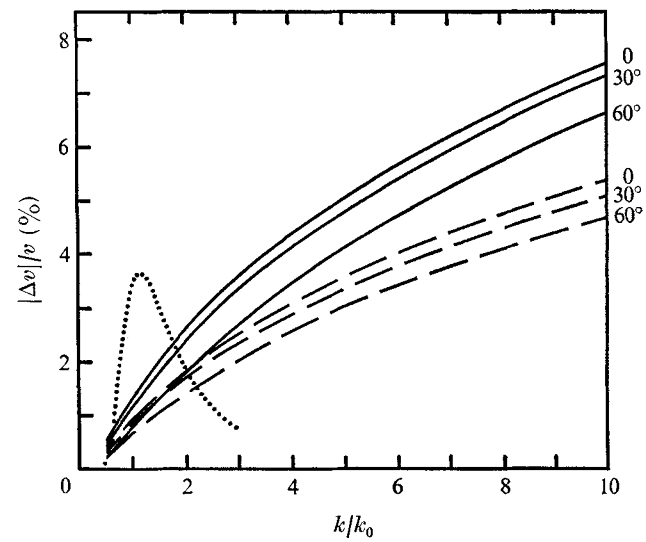

FIGURE 1. Relative change in group velocity according to (28) for three wavenumber directions relative to mean wind direction for an oceanic wave field. ——, $\cos^4$ directional distribution; ---, directional distribution

$$S(\Phi) = \begin{cases} \pi^{-1} & \text{for } |\Phi| < \tfrac{1}{2}\pi; \\ 0 & \text{otherwise}; \end{cases}$$

······, scalar Pierson–Moskowitz spectrum. The curves do not depend on the value k_0 of the maximum-energy wavenumber.

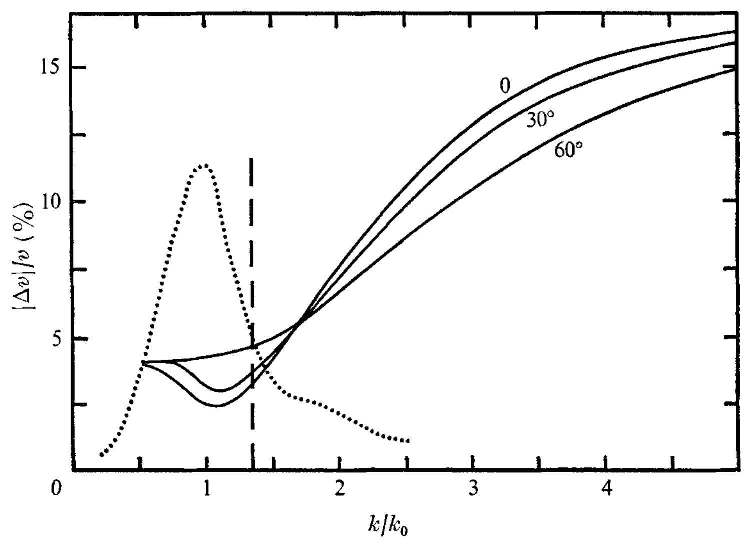

FIGURE 2. Relative change in group velocity for shallow-water waves. ······, scalar spectrum constructed from the frequency spectrum measured by Schrader (1968). ---, $kH = 1$. The maximum energy wavenumber k_0 corresponds to a wavelength $\approx$ 60 m.

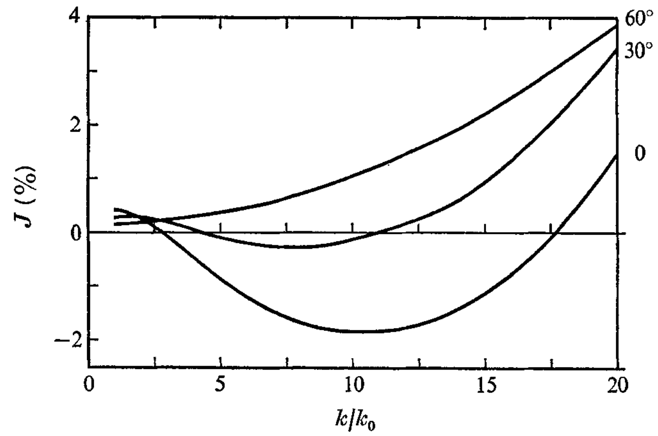

FIGURE 3. Relative change in effective action density according to (29) for oceanic waves.

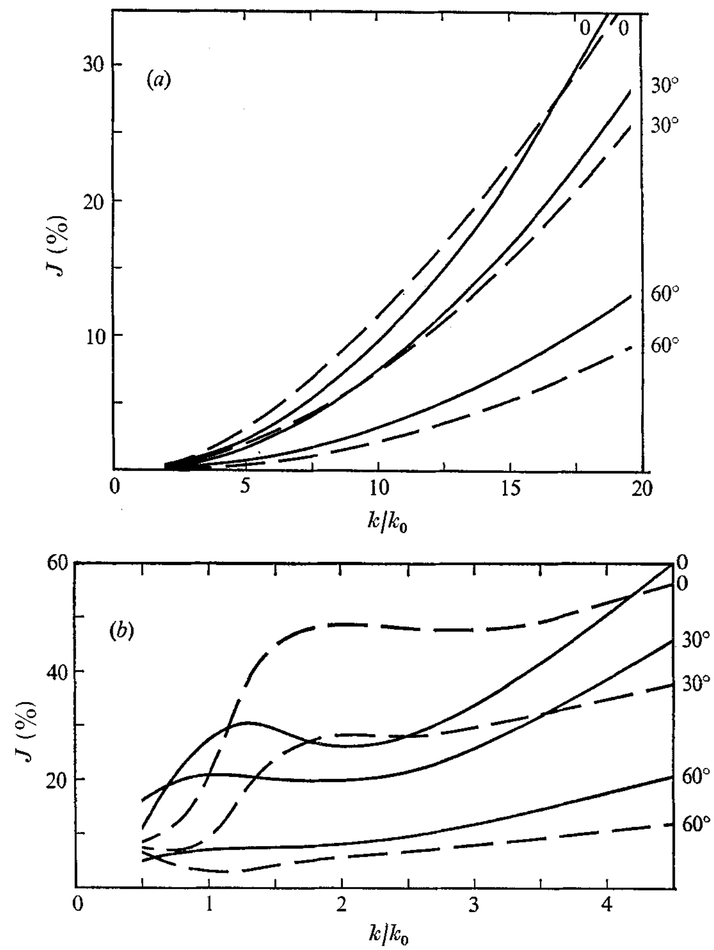

FIGURE 4. Contributions to the integral in (29). ——, first summand; ----, second summand. (*a*) Oceanic waves; the difference between these curves is shown in figure 3. (*b*) Shallow-water waves.

The relative change in action density as defined by (29) is shown in figure 3 for the oceanic case. In the interesting range it is smaller than a few per cent and clearly negligible. It is nevertheless interesting to look more carefully at (29). The integrand is a sum of two terms: the first expresses the fact that the conserved action density is no longer proportional to the square of the amplitude, while the second gives the change in the spectrum due to higher harmonics as discussed above. In spite of their quite different algebraic structures, these terms tend to cancel each other, as seen from figure 4 (*a*).

In case (*b*), the magnitude of each term is even higher than in deep water, up to 50 % of the linear value. Again they have different signs and nearly the same magnitude (figure 4 (*b*)). The difference is not shown here, since it is quite sensitive to the accuracy of the numerical evaluation of the integrals; it is also not clear how strongly it depends on the details of the energy spectrum. However, it may be of considerable magnitude and should be calculated carefully in a concrete shallow-water situation.

We may conclude therefore that the nonlinear corrections to the energy transport equation (27) can be neglected for oceanic waves, whereas they should be taken into account for shallow-water waves. However, as long as the important dissipation mechanisms are not quantitatively understood, it is not clear how strongly these nonlinearities influence the actual behaviour of shallow-water waves.

I thank Professor Klaus Hasselmann for helpful discussions.

Appendix A. Derivation of the transport equation (27)

We start from (21) and introduce the spectrum of action density in the form

$$A(\mathbf{k},\mathbf{x},t)\,d\mathbf{k} \simeq \sum_n^{d\mathbf{k}} I_n(\mathbf{x},t), \tag{A 1}$$

where the summation is performed as in (26). Instead of the index n, we may use, for example, the initial values $\mathbf{p}_n = \mathbf{k}_n(\mathbf{x},0)$ of the wavenumbers to identify the different wave components:

$$I_n = I(\mathbf{p}_n,\mathbf{x},t), \quad \mathbf{k}_n = \mathbf{k}(\mathbf{p}_n,\mathbf{x},t). \qquad \text{(A 2), (A 3)}$$

Without loss of generality, we may assume the p_n to be independent of $\mathbf{x}$. Equation (A 1) is equivalent to

$$A(\mathbf{k}_n,\mathbf{x},t)\,d\mathbf{k} = A(\mathbf{k}_n,\mathbf{x},t)\,D(\mathbf{p}_n,\mathbf{x},t)\,dp = I(\mathbf{p}_n,\mathbf{x},t)\,d\mathbf{p}, \tag{A 4}$$

where $d\mathbf{p}$ is a volume element in $\mathbf{p}$ space corresponding to $d\mathbf{k}$ in $\mathbf{k}$ space, and $D(\mathbf{p}_n,\mathbf{x},t) = \partial\mathbf{k}_n/\partial\mathbf{p}_n$ is the Jacobian of (A 3).

Equation (21 *a*) therefore leads to

$$\frac{\partial}{\partial t}(A(\mathbf{k}_n,\mathbf{x},t)\,D(\mathbf{p}_n,\mathbf{x},t)) + \frac{\partial}{\partial \mathbf{x}}.(\mathbf{u}_n A(\mathbf{k}_n,\mathbf{x},t)\,D(\mathbf{p}_n,\mathbf{x},t)) = 0. \tag{A 5}$$

Now from (21 *b*) and (4) it may easily be shown that the time evolution of the Jacobian is governed by

$$\frac{\partial D(\mathbf{p}_n,\mathbf{x},t)}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot\left(\frac{\partial \omega_n}{\partial \mathbf{k}_n} D(\mathbf{p}_n,\mathbf{x},t)\right) = 0, \tag{A 6}$$

and since we have argued that the transport velocity $\mathbf{u}_n$ can be identified with $\partial\omega_n/\partial\mathbf{k}_n$, it follows that

$$\frac{\partial A(\mathbf{k}_n,\mathbf{x},t)}{\partial t}+\frac{\partial\omega_n}{\partial\mathbf{k}_n}\cdot\frac{\partial A(\mathbf{k}_n,\mathbf{x},t)}{\partial\mathbf{x}}=0. \tag{A 7}$$

The $\mathbf{x}$ and t derivatives in (A 7) are taken with a fixed value of n, i.e. $\mathbf{p}_n$. Introducing now derivatives for fixed $\mathbf{k}_n(\mathbf{x},t)$, denoted by a subscript $\mathbf{k}_n$, (A 7) can be expanded to give

$$\left(\frac{\partial A}{\partial t}\right)_{\mathbf{k}_n}+\frac{\partial\omega_n}{\partial\mathbf{k}_n}\left(\frac{\partial A}{\partial x}\right)_{\mathbf{k}_n}+\frac{\partial A}{\partial\mathbf{k}_n}\left[\frac{\partial\mathbf{k}_n}{\partial t}+\frac{\partial\omega_n}{\partial\mathbf{k}_n}\cdot\frac{\partial\mathbf{k}_n}{\partial x}\right]=0. \tag{A 8}$$

From (21 *b*) we have

$$\frac{\partial\mathbf{k}_n}{\partial t}+\frac{\partial\omega_n}{\partial\mathbf{k}_n}\cdot\frac{\partial\mathbf{k}_n}{\partial\mathbf{x}}=-\left(\frac{\partial\omega_n}{\partial\mathbf{x}}\right)_{\mathbf{k}_n}, \tag{A 9}$$

and dropping the indices, we finally obtain

$$\frac{\partial A}{\partial t}+\frac{\partial\Omega}{\partial\mathbf{k}}\cdot\frac{\partial A}{\partial\mathbf{x}}-\frac{\partial\Omega}{\partial\mathbf{x}}\cdot\frac{\partial A}{\partial\mathbf{k}}=0, \tag{A 10}$$

A and Ω being functions of $(\mathbf{k},\mathbf{x},t)$.

Furthermore, from (20 *a*), (26) and (29) we conclude that

$$A(\mathbf{k},\mathbf{x},t)=-\frac{E(\mathbf{k},\mathbf{x},t)}{(gk\tanh kH)^{\frac{1}{2}}}(1+J(\mathbf{k},\mathbf{x},t))+O(\epsilon^6), \tag{A 11}$$

thus completing the proof of (27).

Appendix B. Explicit expressions for the nonlinear coefficients

Let us introduce the following definitions:

$$\tau(k)=\tanh kH,\quad \tau'=\tau(k'),\quad \sigma=(gk\tau)^{\frac{1}{2}},\quad \sigma'=(gk'\tau')^{\frac{1}{2}},$$

$$\Delta(\mathbf{k},\mathbf{k}')=(\sigma+\sigma')^2-g|\mathbf{k}+\mathbf{k}'|\tanh|\mathbf{k}+\mathbf{k}'|H,$$

$$D(\mathbf{k},\mathbf{k}')=\frac{1}{2}\left\{\sigma^2+\sigma'^2+\sigma\sigma'\left(1-\frac{\mathbf{k}.\mathbf{k}'}{kk'\tau\tau'}\right)\right\},$$

$$E(\mathbf{k},\mathbf{k}')=\frac{1}{2}\left\{\frac{\sigma k}{\tau}+\frac{\sigma'k'}{\tau'}+\mathbf{k}.\mathbf{k}'\left(\frac{\sigma}{k\tau}+\frac{\sigma'}{k'\tau'}\right)\right\},$$

$$Q(\mathbf{k},\mathbf{k}')=\{gE-(\sigma+\sigma')D\}/\Delta.$$

The coefficients $P(\mathbf{k},\mathbf{k}')$, $S(\mathbf{k},\mathbf{k}')$ and $T(\mathbf{k},\mathbf{k}')$ appearing in (28) and (29) are then given as follows:

$$P(\mathbf{k},\mathbf{k}')=\{(\sigma+\sigma')E-|\mathbf{k}+\mathbf{k}'|\tanh(|\mathbf{k}+\mathbf{k}'|H)D\}/\Delta,$$

$$S(\mathbf{k},\mathbf{k}')=\frac{1}{g}\left\{-D_\pm P_\pm+E_\pm Q_\pm+\frac{\sigma\sigma'}{kk'\tau\tau'}(k\tau+k'\tau')\mathbf{k}.\mathbf{k}'+\frac{1}{2}\left(\frac{\sigma^2k}{\tau}+\frac{\sigma'^2k'}{\tau'}\right)\right\},$$

$$T(\mathbf{k},\mathbf{k}')=\frac{1}{g}\left\{D_\pm P_\pm+E_\pm Q_\pm+(\sigma^2-\sigma'^2)P_\pm+2\sigma P_\pm Q_\pm -\frac{\sigma}{k\tau}(k^2\pm\mathbf{k}.\mathbf{k}')Q_\pm+\frac{1}{2}\left(\frac{\sigma'^2k'}{\tau'}-\frac{\sigma^2k}{\tau}\right)\right\}.$$

Here $D_\pm=D(\mathbf{k},\pm\mathbf{k}')$, etc., and summation over both signs is understood.

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