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PROPAGATION OF INTERNAL AND SURFACE GRAVITY WAVES IN THE APPROXIMATION OF GEOMETRICAL OPTICS*

The propagation of surface and internal gravity waves which have a modal structure is considered in an ocean whose properties vary in time and space. The problem is solved in a linear formulation by the method of geometrical optics: it is assumed that the length and period of the wave are much smaller than the corresponding spatial and temporal scales of the main motion. It was possible to reduce the equation determining the amplitude of the wave to the form of a conservation law for a certain adiabatic invariant I of the wave:

$$\frac{\partial I}{\partial t}+\operatorname{div}(\mathbf{C}_{gr}I)=0.$$

Solutions are presented for this equation and the equation determining the phase of the wave on the characteristics. Certain simple corollaries of the relationships derived are discussed in the conclusion.

1. Currents and density-field inhomogeneities that are variable in both time and space are practically always present in the real ocean. It would therefore be interesting to consider how these inhomogeneities influence the propagation of surface and internal gravity waves. In its general formulation, this problem is a highly complex one, and it is therefore convenient to consider first of all the propagation of short waves whose length and period are much smaller than the characteristic distances and times in the variability of the large-scale motions. If the latter are assumed to be of the order of 100 km and 10 hr, a rather broad class of the actually observed wave motions will fall within the range of the analysis. On the other hand, it becomes possible in this case to analyze the problem by the geometrical-optics method.

Many published papers have been devoted to this problem in a more or less general formulation. Here it is appropriate to take note of the paper [1], where the influence of density-field inhomogeneities on the propagation of internal waves was considered in the approximation of geometrical optics (the papers [2, 3] were also devoted to this problem). Wave propagation in inhomogeneous media was discussed in a general formulation in [4, 5]. However, the papers [6, 7], in which the problem was solved under general hypotheses in the case of a medium that varies slowly along the vertical coordinate, deserve special mention. The problem has also been investigated by several other authors in this formulation (see, for example, [8, 9] and their bibliographies). The results of a numerical experiment on the propagation of internal waves on a current with horizontal shear were presented in [10].

In the present paper, the propagation of internal waves with modal structure and the propagation of surface waves are analyzed in a linear formulation for an inhomogeneous, slowly varying ocean, using the geometrical-optics method. We shall treat the ocean as an incompressible fluid and use the Boussinesq approximation, i.e., we shall take account of the departure of the fluid density from unity only in the buoyancy-related terms.

2. We take the dynamic equations in the form

$$\frac{d\tilde{u}_i}{dT}=-\frac{\partial\tilde{p}}{\partial X_i},\quad i=1,2,\quad \frac{d\tilde{w}}{dT}+g\tilde{\rho}=-\frac{\partial\tilde{p}}{\partial z},$$
$$\frac{\partial\tilde{u}_i}{\partial X_i}+\frac{\partial\tilde{w}}{\partial z}=0,\quad \frac{d\tilde{\rho}}{dT}=0. \tag{1}$$

Here $\tilde{u}_i$ $(i=1, 2)$ are the horizontal velocity components, w is the vertical velocity, p and $\tilde{\rho}$ are the pressure and density, X_i and z are the horizontal and vertical coordinates (the latter increases upward), and d/dT is the total derivative. The boundary conditions have the form

$$\text{at}\quad z=\tilde{h}(X_i,T)\quad \tilde{p}=p_a(X_i,T),\quad \tilde{w}=\frac{\partial\tilde{h}}{\partial T}+\tilde{u}_i\frac{\partial\tilde{h}}{\partial X_i}, \tag{2}$$

$$\text{at}\quad z=-H(X_i)\quad \tilde{w}+\tilde{u}_i\frac{\partial H}{\partial X_i}=0, \tag{3}$$

where z = h and z = -H are the equations of the bottom and the free surface and p_a is the atmospheric pressure. We introduce the small parameter ε, which characterizes the slowness of the variations of the main motion along the horizontal coordinates and in time. Slowness of variation is not assumed along the vertical coordinate. We then represent all of the hydrodynamic fields as consisting of two components:

$$\tilde{\varphi}=\varphi^{(0)}(z,\ x_i,\ t)+a\varphi, \tag{4}$$

where $\tilde{\varphi}$ is any of the variables characterizing the motion, $\varphi^{(0)}$ for the main motion and φ for the disturbance propagating against the main motion as a background; $x_i=\varepsilon X_i$ and $t=\varepsilon T$ are the slow horizontal coordinates and time, and a is a small amplitude parameter. Since $u^{(0)}=u^{(0)}(z,\ x_i,\ t)$, it follows from the continuity equation (1) that $w^{(0)}\sim\varepsilon|u_i^{(0)}|$. We shall assume that the other hydrodynamic-field variables of the main motion are of the order of unity. We shall also assume that the atmospheric pressure and the bottom level vary slowly: $p_a=p_a(x_i,\ t)$, $H = H(x)$.

Substituting the representation (4) with consideration of the smallenss of $w^{(0)}$

$$w^{(0)}\to\varepsilon w^{(0)} \tag{5}$$

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into Eqs. (1)-(3) and separating the variables relating to the main motion, we obtain

$$\frac{\partial \mathbf{u}^{(0)}}{\partial t}+u_i^{(0)}\frac{\partial \mathbf{u}^{(0)}}{\partial x_i}+w^{(0)}\frac{\partial \mathbf{u}^{(0)}}{\partial z}=-\frac{\partial p^{(0)}}{\partial x_i}, \tag{6a}$$

$$g\rho^{(0)}=-\frac{\partial p^{(0)}}{\partial z}, \tag{6b}$$

$$\frac{\partial \rho^{(0)}}{\partial t}+u_i^{(0)}\frac{\partial \rho^{(0)}}{\partial x_i}+w^{(0)}\frac{\partial \rho^{(0)}}{\partial z}=0, \tag{6c}$$

$$\frac{\partial u_i^{(0)}}{\partial x_i}+\frac{\partial w^{(0)}}{\partial z}=0. \tag{6d}$$

The boundary conditions are the same as (2) and (3) with the tilde replaced by the superscript (0), and X_i and T by x_i and t. Equations (6a), (6c), and (6d) and the boundary conditions are exact equalities, while (6b) holds accurate to quantities of the order of ε^2.

We then find the solution of the equations for the disturbance φ in the form of the WKB expansion:

$$\varphi=[\varphi^{(1)}(z,x_i,t)+\varepsilon\varphi^{(2)}(z,x_i,t)+\ldots]\exp\left\{\frac{i}{\varepsilon}S(x_i,t)\right\},$$

where S is the phase of the wave, which determines the local wave number and the frequency in accordance with the relation $k_i=\partial S/\partial x_i$, $\omega=-\partial S/\partial t$. Now, separating quantities of the order of a from the original equations, we easily obtain an equation and boundary conditions for the vertical velocity $w^{(1)}$ of the disturbance (we shall henceforth consistently omit the superscript (1))

$$w''+\left(\frac{k^2\mu^{(0)}}{\omega_d^2}-\frac{\omega_d''}{\omega_d}-k^2\right)w=0, \tag{7}$$

$$\left(\frac{w}{\omega_d}\right)'=\frac{gk^2}{\omega_d^3}w \quad \text{at} \quad z=h^{(0)} \text{ and } w=0 \quad \text{at} \quad z=-H. \tag{8}$$

Here $\omega_d=\omega-\mathbf{k}\cdot\mathbf{u}^{(0)}$ is the Doppler frequency, which depends on z (through $\mathbf{u}^{(0)}$) and $\mu^{(0)}(z,x_i,t)=-g\partial\rho^{(0)}/\partial z$ is the square of the Brunt-Väisälä frequency. Here and below, the prime identifies the derivative with respect to z. The boundary-value problem (7), (8) gives us a set of dispersion relations for the various modes:

$$\omega=f(k_i,x_i,t) \tag{9}$$

and the corresponding solutions $w=w(z,x_i,t)$, which depend on x_i and t as parameters. All other variables characterizing the wave can be expressed in terms of w using the relations

$$\mathbf{u}=\frac{i\mathbf{k}}{k^2}\omega_d\left(\frac{w}{\omega_d}\right)'-\frac{iw}{\omega_d}\frac{\partial \mathbf{u}^{(0)}}{\partial z}, \tag{10a}$$

$$p=\frac{i\omega_d^2}{k^2}\left(\frac{w}{\omega_d}\right)', \tag{10b}$$

$$\rho=-i\frac{w}{\omega_d}\frac{\partial \rho^{(0)}}{\partial z}, \tag{10c}$$

$$h=\frac{iw}{\omega_d}. \tag{10d}$$

Then, separating quantities of the order of $a\varepsilon$ in the equations and boundary conditions, we obtain the equations

$$\begin{gathered}
i\omega_d\mathbf{u}^{(2)}-w^{(2)}\frac{\partial \mathbf{u}^{(0)}}{\partial z}-i\mathbf{k}p^{(2)}=\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial p}{\partial x_i}+u_j^{(0)}\frac{\partial u}{\partial x_j}+\\
+w^{(0)}\frac{\partial \mathbf{u}}{\partial z}+u_j\frac{\partial \mathbf{u}^{(0)}}{\partial x_j}\Leftarrow-\frac{1}{\omega_d}\frac{\partial}{\partial z}\mathbf{k},\\
i\omega_d w^{(2)}-g\rho^{(2)}+\frac{\partial p^{(2)}}{\partial z}=\frac{\partial w}{\partial t}+u_j^{(0)}\frac{\partial w}{\partial x_j}+w\frac{\partial w^{(0)}}{\partial z}+w^{(0)}\frac{\partial w}{\partial z}\Leftarrow\frac{ik^2}{\omega_d},\\
i\omega_d\rho^{(2)}-\frac{\partial \rho^{(0)}}{\partial z}w^{(2)}=\frac{\partial \rho}{\partial t}+u_j^{(0)}\frac{\partial \rho}{\partial x_j}+u_j\frac{\partial \rho^{(0)}}{\partial x_j}+w^{(0)}\frac{\partial \rho}{\partial z}\Leftarrow\frac{k^2}{\omega_d^2}g,\\
\frac{\partial w^{(2)}}{\partial z}+i\mathbf{k}\cdot\mathbf{u}^{(2)}=-\frac{\partial u_i}{\partial x_i}\Leftarrow\frac{\omega_d'}{\omega_d}+\frac{\partial}{\partial z},
\end{gathered} \tag{11}$$

and the boundary conditions

$$\begin{gathered}
p^{(2)}-g\rho^{(0)}h^{(2)}=0,\\
w^{(2)}+i\omega_d h^{(2)}=\\
\frac{\partial h}{\partial t}+u_j^{(0)}\frac{\partial h}{\partial x_j}+u_j\frac{\partial h^{(0)}}{\partial x_j}-h\left(\frac{\partial w^{(0)}}{\partial z}-\frac{\partial h^{(0)}}{\partial x_j}\frac{\partial u_j^{(0)}}{\partial z}\right) \quad \text{at} \quad z=h^{(0)},\\
w^{(2)}=-u_j\frac{\partial H}{\partial x_j} \quad \text{at} \quad z=-H.
\end{gathered} \tag{12}$$

Using the operators indicated in the right-hand side of (11) on these equations, we obtain

$$w^{(2)''}+\left(\frac{k^2\mu^{(0)}}{\omega_d^2}-\frac{\omega_d''}{\omega_d}-k^2\right)w^{(2)}=F, \tag{13}$$

where F is the result of an application of the operators to the right-hand sides. Expressing $p^{(2)}$ and $h^{(2)}$ in terms of $w^{(2)}$ in (12), we obtain the boundary conditions

$$\begin{gathered}
\omega_d\frac{\partial w^{(2)}}{\partial z}-\left(\frac{gk^2}{\omega_d}+\omega_d'\right)w^{(2)}=G_1 \quad \text{at} \quad z=h^{(0)},\\
w^{(2)}=-u_j\frac{\partial H}{\partial x_j}=G_2 \quad \text{at} \quad z=-H.
\end{gathered} \tag{14}$$

The condition for solvability of the inhomogeneous boundary-value problem (13), (14) is written in the form

$$\int_{-H}^{h^{(0)}}F\frac{iw}{k^2}dz-\frac{i}{k^2}\frac{w}{\omega_d}G_1\Big|_{z=h^{(0)}}-\frac{i}{k^2}G_2\frac{\partial w}{\partial z}\Big|_{z=-H}=0. \tag{15}$$

Here iw/k^2 was taken as the solution of the homogeneous boundary-value problem. The expressions for F and G_1 have the form

$$\begin{gathered}
F=-\frac{\mathbf{k}}{\omega_d}\frac{\partial}{\partial z}\left(\frac{\partial \mathbf{u}}{\partial t}+\nabla p+u_j^{(0)}\frac{\partial \mathbf{u}}{\partial x_j}+w^{(0)}\frac{\partial \mathbf{u}}{\partial z}+u_j\frac{\partial \dot{\mathbf{u}}^{(0)}}{\partial x_j}\right)+\\
+\frac{ik^2}{\omega_d}\left(\frac{\partial w}{\partial t}+u_j^{(0)}\frac{\partial w}{\partial x_j}+\frac{\partial}{\partial z}(ww^{(0)})\right)+\frac{k^2}{\omega_d}g\left(\frac{\partial \rho}{\partial t}+u_j^{(0)}\frac{\partial \rho}{\partial x_j}+\right.\\
\left.+u_j\frac{\partial \rho^{(0)}}{\partial x_j}+w^{(0)}\frac{\partial \rho}{\partial z}\right)-\frac{\omega_d'}{\omega_d}\frac{\partial u_j}{\partial x_j}-\frac{\partial}{\partial z}\frac{\partial u_j}{\partial x_j},
\end{gathered}$$

$$\begin{gathered}
G_1=-\omega_d\frac{\partial u_j}{\partial x_j}-\frac{gk^2}{\omega_d}\left[\frac{\partial h}{\partial t}+u_j^{(0)}\frac{\partial h}{\partial x_j}+u_j\frac{\partial h^{(0)}}{\partial x_j}-\right.\\
\left.h\left(\frac{\partial w^{(0)}}{\partial z}-\frac{\partial h^{(0)}}{\partial x_j}\frac{\partial u_j^{(0)}}{\partial z}\right)\right]-\\
-\mathbf{k}\left(\frac{\partial \mathbf{u}}{\partial t}+\nabla p+u_j^{(0)}\frac{\partial \mathbf{u}}{\partial x_j}+w^{(0)}\frac{\partial \mathbf{u}}{\partial z}+u_j\frac{\partial \mathbf{u}^{(0)}}{\partial x_j}\right).
\end{gathered}$$

Here $\nabla=(\partial/\partial x_1,\partial/\partial x_2)$. Condition (15) can be reduced by unwieldy manipulation to the form of a conservation law for a certain "adiabatic invariant" I:

$$\frac{\partial I}{\partial t}+\frac{\partial}{\partial x_i}(C_{\mathrm{gr}}^{(i)}I)=0, \tag{16}$$

where

$$I=\int_{H}^{h^{(0)}}\left(\frac{\mu^{(0)}}{\omega_d^{3}}-\frac{\omega_d''}{2\omega_d^{2}k^{2}}\right)w^{2}\,dz+\left(\frac{g}{\omega_d^{3}}+\frac{\omega_d'}{2\omega_d^{2}k^{2}}\right)w^{2}\Big|_{z=h^{(0)}}, \quad (17)$$

$$\mathbf{C}_{\mathbf{gr}}I=\int_{-H}^{h^{(0)}}\Big\{\mathbf{u}^{(0)}\left[\frac{\mu^{(0)}}{\omega_d^{3}}-\frac{\omega_d''}{2\omega_d^{2}k^{2}}\right]+\frac{1}{2\omega_d k^{2}}\frac{\partial^{2}\mathbf{u}^{(0)}}{\partial z^{2}}+\frac{\mathbf{k}}{k^{2}}\left(\frac{\mu^{(0)}}{\omega_d^{2}}-1\right)\Big\}w^{2}\,dz+$$
$$+\left[\mathbf{u}^{(0)}\left(\frac{g}{\omega_d^{3}}+\frac{\omega_d'}{2\omega_d^{2}k^{2}}\right)-\frac{1}{2\omega_d k^{2}}\frac{\partial\mathbf{u}^{(0)}}{\partial z}+\frac{g\mathbf{k}}{\omega_d^{2}k^{2}}\right]w^{2}\Big|_{z=h^{(0)}}. \quad (18)$$

Using only the boundary-value problem (7), (8), it is easily shown that the ratio of expressions (18) and (17) is indeed the group velocity $C_{\mathbf{gr}}^{(i)}=\partial f/\partial k_i$. It is important to note that Eq. (16) will not hold for arbitrary main hydrodynamic velocity, density, and other fields, but only for those that are controlled by the dynamic equations; Eqs. (6) and the boundary conditions (2), (3) were those essentially used in the calculations. If the surface waves are filtered out by using the "solid lid" boundary condition $w(0) = 0$ in the initial equations at $z = 0$, the terms to be evaluated at $z = 0$ in (17), (18) vanish.

We note that in the absence of currents, expression (17) reduces to $I=E/\omega$, where E is the energy of the wave. In the case of purely internal waves and high Richardson numbers (slow variation of the horizontal velocity with depth), we obtain $\omega_d''\ll\frac{\mu^{(0)}}{\omega_d}k^2$ and $I=\int_{-H}^{h^{(0)}}\frac{E}{\omega_d}dz$, in agreement with the usual expression for the adiabatic invariant of an internal wave in this case (here E is the energy density).

We note that the main equation (16) and expressions (17), (18) should be obtained directly from a suitable variational principle by the method proposed in [4, 5]. However, we have not yet done this.

Integrating Eq. (16) over a sufficiently large horizontal area s, we easily find that the total adiabatic invariant $\int I ds$ of the wave is conserved as it propagates.

3. Using the definitions of the wave number and frequency, we obtain an equation that determines the phase of the wave (Hamilton–Jacobi equation) from the dispersion relation (9):

$$\frac{\partial S}{\partial t}+f\left(\frac{\partial S}{\partial x_i},\ x_i,\ t\right)=0. \quad (19)$$

Equations (19) and (16) fully determine the propagation of a linear wave in the WKB approximation. The characteristics of Eq. (19) are specified with a system of ordinary differential equations:

$$\frac{dx_i}{dt}=\frac{\partial f}{\partial k_i}=C_{\mathbf{gr}}^{(i)}(k_j,x_j,t), \quad (20a)$$

$$\frac{dk_i}{dt}=-\frac{\partial f}{\partial x_i}, \quad (20b)$$

$$\frac{d\omega}{dt}=-\frac{\partial f}{\partial t}. \quad (20c)$$

The phase of the wave along a characteristic is calculated from the relation

$$S=S_0+\int_{t_0}^{t}(k_iC_{\mathbf{gr}}^{(i)}-\omega)\,dt, \quad (21)$$

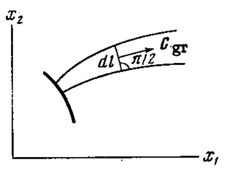

Fig. 1

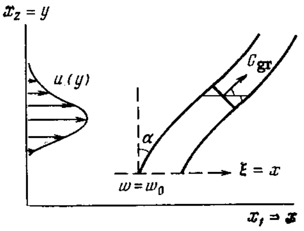

Fig. 2

Fig. 1. Illustrating definition of dl: the light lines are two closely spaced characteristic projections, and the heavy line is the original manifold $\{x_1(\xi,\ t_0),\ x_2(\xi,\ t_0)\}$.

Fig. 2. Profile of current with horizontal shear and general picture of characteristic projections onto the coordinate space in the case $\mathbf{u}^{(0)}=(u(y),0)$, $\rho^{(0)}=\rho^{(0)}(z)$.

where S_0 is its initial value. The characteristics of Eq. (16) agree with (20). From this we easily find that the quantity

$$I\frac{\partial(x_1,x_2)}{\partial(\xi,t)}=\text{const}, \quad (22)$$

where ξ is a parameter that specifies the initial manifold at $t = t_0$, is conserved along a characteristic. Relation (22) has the following geometric interpretation: if we consider two infinitesimally closely spaced characteristic projections on the coordinate space (x_1, x_2) and denote the distance between them by dl (Fig. 1), the equation

$$|\mathbf{C}_{\mathbf{gr}}I|\,dl=\text{const}. \quad (23)$$

holds along the characteristics.

We have from (20a) that the vector $\mathbf{C}_{\mathbf{gr}}$ is perpendicular to the segment dl, so that relation (23) is the law of constant flux of the adiabatic invariant along the ray tube. Thus, if equation system (20) is solved, relations (22) and (23) determine the amplitude and phase of the wave.

4. It follows from (20b) and (20c) that if the parameters of the ocean do not depend on time, ω = const along a characteristic; but if they do not depend on the x_i coordinate, we have k_i = const (Snell's law).

Let us consider, for example, a plane-parallel current: $\{u_1^{(0)},u_2^{(0)}\}=\{u(y),0\}$, $\rho^{(0)}=\rho^{(0)}(z)$. In this case, the characteristics will be a family of parallel curves (Fig. 2), whence follows conservation not only of ω and k_x, but also of

$$C_{\mathbf{gr}}^{(y)}I=\text{const}. \quad (24)$$

If we put $\mu^{(0)}=N_0^2=\text{const}$, we obtain from (24) and (18)

$$\frac{k_y}{k^2}\left(\frac{N_0^2}{(\omega-k_xu)^2}-1\right)w^2=\text{const},$$

from which

$$\frac{w}{w_0}=R\left(\frac{\cos^2\alpha}{R-\sin^2\alpha}\right)^{1/4}.$$

Here

$$R=\left(\frac{N_0^2}{\omega^2}-1\right)\Big/\left(\frac{N_0^2}{(\omega-k_x u)^2}-1\right); \quad k_x=\frac{\pi n}{H}\left(\frac{N_0^2}{\omega^2}-1\right)\sin\alpha;$$

H is the depth of the ocean, n is the mode number, and ω and w_z are the initial frequency and amplitude (on the quiet water). If $uk_x > 0$, then we have as $|u|$ increases $k_y \to 0$, $C_{gr}^{(y)} \to 0$ and $w \to \infty$ at the point where $R=\sin^2\alpha$. But if $uk_x < 0$, we have as $|u|$ increases $\omega-k_x u \to N_0$ as $k_y \to \infty$, $w \to \infty$. It is interesting that in the latter case the wave arrives at the critical level after an infinite time $(t \sim |u-u_{cr}|^{-1/2})$, while in the former case this happens after a finite time, but since $k_y \to 0$, the limits of validity of geometrical optics are crossed sooner.

Let us consider the one-dimensional problem of propagation of short surface waves on deep water with a current that does not have vertical shear and whose velocity depends on x (this can be brought about by varying the relief of the distant bottom).

With $\mu^{(0)}=0$, $u^{(0)}=u(x)$, $H=\infty$, the boundary-value problem (7), (8) yields the solution

$$w=w_0 e^{kz}, \quad (\omega-ku)^2=gk.$$

In this case ω = const and

$$C_{gr}I=-\frac{w_0^2}{k}\int_{-\infty}^{0} e^{2kz}\,dz+$$

$$\left(u\frac{g}{(\omega-ku)^3}+\frac{g}{k(\omega-ku)^2}\right)w_0^2=\frac{w_0^2}{2k^2}\frac{\omega+uk}{\omega-uk}=\text{const}.$$

This result agrees with the expression obtained in [11].

Let us consider the propagation of a short linear wave against the background of a nonlinear stationary wave traveling along the x axis at velocity C. In this case.

$$\omega=f(k_i,\ x_i,\ t)=f(k_i,\ x-Ct).$$

Here it follows from Eqs. (20) that the quantities $\omega-k_x C$ =const, k_y=const are conserved along a characteristic (the first of them is the Doppler frequency in the coordinate system of the stationary wave). These three equations can be solved graphically (Fig. 3). As the traveling coordinate x' varies (i.e., on transition to a different phase of the stationary wave), the dispersion curves will undergo some deformation, while the straight lines remain in place. We see that if at a certain moment the phase velocity of the wave $C_p=\omega/k<C$ (Fig. 3a), there will always be a point of intersection no matter how the dispersion curve changes. The short wave will at all times lag the stationary wave, since $C_{gr} < C$. But if $C_p > C$, the picture will be somewhat different. If at some time the parameters of the wave correspond to point 1 (Fig. 3b), then

$$\frac{d}{dt}(x-Ct)>0,$$

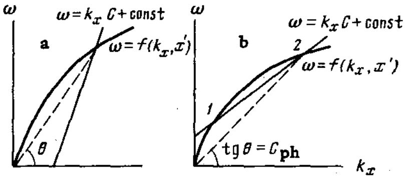

Fig. 3. Graphical construction for the determination of the parameters of a short wave. The dispersion curves are drawn for a certain fixed "traveling coordinate" x' = x - Ct. The points of intersection correspond to the parameters of the wave.

and the short wave will overtake the stationary wave. If the dispersion curve is strongly enough deformed when this happens, points 1 and 2 will merge at a certain time, after which the wave parameters will correspond to point 2 and the short wave will cease to lag the stationary wave:

$$\frac{d}{dt}(x-Ct)<0.$$

If the stationary wave is periodic, points 1 and 2 will again merge at a corresponding time, the wave will "cross" to point 1, and this process will be successively repeated. While the short wave spends time in all phases of the stationary wave in the case of Fig. 3a, there are in the case of Fig. 3b certain forbidden zones that the short wave does not penetrate. In both cases, the wave parameters vary in such a way that $\Delta\omega/\Delta k_x=C$.

It seems reasonable to assume that Eq. (16) will also hold in a more general formulation of the problem, as when the rotation of the earth is taken into account or the Boussinesq approximation is dispensed with. Then the expressions for the adiabatic invariant and its flux are easily surmised if we note that their ratio must be equal to the group velocity, a relationship for which is simple to obtain from the usual boundary-value problem, and by using the limit transition to the case under consideration.

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