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COMPUTATIONS OF STEEP GRAVITY WAVES BY A REFINEMENT OF DAVIES–TULIN'S APPROXIMATION*

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Abstract. Water waves in water of arbitrary depth and solitary waves are calculated numerically using new series truncation methods. The techniques use a refinement of Davies' approximation first proposed by Tulin. Accurate numerical solutions are obtained for all values of the steepness up to the limiting configuration with a 120° angle at the wave crest. It is shown that the proposed numerical procedure is equivalent to the method of Havelock. The method of Michell is included as a particular case. A comparison with previous numerical methods, such as boundary integral equation techniques, is given.

Key words. gravity waves, free surface flows, series truncation

AMS subject classifications. 76B15, 76E30

1. Introduction. Over the past 200 years, many analytic and numerical results on water waves have been reported. In particular, very accurate solutions were calculated for waves close to their limiting configuration (see Schwartz [1], Cokelet [2], Longuet-Higgins [3], Vanden-Broeck and Schwartz [4], Longuet-Higgins and Fox [5], Williams [6], Chen and Saffman [7], Byatt-Smith and Longuet-Higgins [8], Hunter and Vanden-Broeck [9], and references cited in those papers).

Davies [10] derived an approximate solution for water waves in water of infinite depth. His idea was to modify the free surface condition so that an exact solution can be obtained by analytic continuation. His solution is as good as the linear theory for waves of small amplitude. It has, in addition, the remarkable property of predicting the correct angle of 120° at the crest of the highest wave.

Tulin [11] found a refinement of Davies' approximation. He derived a new formulation of the water wave problem by correcting Davies' approximation so that the exact free surface condition is satisfied. To the best of our knowledge, this interesting refinement has not been used previously to compute very steep waves.

Another well-known accurate method for calculating the highest wave was introduced by Michell [12] and later used by Olfe and Rottman [13], Vanden-Broeck [14], and others. The idea is to represent the complex velocity by a power series in which the singularity associated with the 120° angle at the crest is removed by introduction of an appropriate multiplicative factor in the representation. Havelock [15] generalized Michell's method to steep waves by assuming that the same singularity as in Michell's method occurs above the fluid for waves of amplitude less than the highest. However, Grant [16] and Schwartz [1] demonstrated that an analytic continuation of the flow above the fluid produces a different singularity from the one assumed by Havelock [15]. Vanden-Broeck [14] pointed out that the assumption of the "wrong" singularity in Havelock's method does not imply that the expansion should diverge in the flow domain and on the free surface. In fact, his numerical results show that this "wrong" singularity above the fluid improves the convergence and that

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accurate results for waves of arbitrary amplitude can be obtained by Havelock's method.

In this paper we present new numerical procedures to calculate waves in water of arbitrary depth and solitary waves. The schemes are accurate and easier to implement than boundary integral equation methods.

First we compute water waves in water of infinite depth by using the refinement of Davies' approximation proposed by Tulin [11]. Our numerical method uses series truncation and collocation. Accurate results are obtained for waves of arbitrary amplitude up to the limiting configuration with a 120° angle at the crest. We show that our numerical procedure is equivalent to the method of Havelock [15]. Since we do not assume anything about singularities outside the fluid, the equivalence of the methods can be used to explain the puzzling success of Havelock's method (i.e., the fact that the introduction of the "wrong" singularity outside the fluid improves the convergence).

Our next result generalizes the procedure to compute waves in water of finite depth. The idea is to expand the cube of the complex velocity in a power series. The results are in agreement with those of Cokelet [2], Vanden-Broeck and Schwartz [4], and Williams [6].

Finally we adapt the numerical procedure to compute solitary waves of arbitrary amplitude. The scheme generalizes the method used by Hunter and Vanden-Broeck [9] to calculate the highest solitary wave. Our numerical calculations confirm the results obtained by Byatt-Smith and Longuet-Higgins [8] and by Hunter and Vanden-Broeck [9] via boundary integral equation methods.

The problem is formulated in §2. Davies' approximation, its refinement, and the computation of waves in water of infinite depth are also described in §2. The generalizations to water of finite depth and to solitary waves are presented in §§3 and 4.

2. Periodic waves in water of infinite depth.

2.1. Formulation. We consider two-dimensional periodic waves of wavelength λ and phase velocity c propagating under the influence of gravity g at the surface of a fluid of infinite depth. A frame of reference in which the waves are steady is chosen, and dimensionless variables are introduced by taking λ as the unit length and c as the unit velocity. The effects of compressibility, viscosity, and surface tension are neglected.

We introduce Cartesian coordinates with the x-axis at the mean water level and the y-axis directed vertically upward. Gravity is acting in the negative y-direction. We define the complex potential $f = \phi + i\psi$ and the complex velocity $w = u - iv$. Here ϕ is the potential function, ψ is the stream function, u is the x component of the velocity, and v is the y component of the velocity. Without loss of generality we choose $\psi = 0$ on the free surface and $\phi = 0$ at one crest.

Next we define the function $\zeta = \tau - i\theta$ by the formula

$$w = e^{\zeta} = e^{\tau - i\theta}. \tag{2.1}$$

The condition of constant pressure ($p = 0$) on the free surface can be written as

$$e^{2\tau} + \frac{4\pi}{\mu} y = 1, \tag{2.2}$$

where

$$\mu = \frac{2\pi c^2}{g\lambda}. \tag{2.3}$$

We eliminate y from (2.2) by differentiating (2.2) with respect to ϕ. Using the identity

$$\frac{\partial x}{\partial \phi} + i\frac{\partial y}{\partial \phi} = w^{-1} = e^{-\tau + i\theta} \tag{2.4}$$

we obtain

$$\frac{\partial \tau}{\partial \phi} + \frac{2\pi}{\mu} e^{-3\tau} \sin\theta = 0 \quad \text{on } \psi = 0. \tag{2.5}$$

This completes the formulation of the problem. We seek $w(f)$ as an analytic function of f, periodic with period 1, which tends to 1 as $\psi \to -\infty$ and which satisfies (2.5) on the free surface $\psi = 0$.

2.2. Davies' approximation. Following Davies [10], we use the trigonometric identity

$$\sin\theta = \frac{\sin 3\theta}{3} + \frac{4}{3}\sin^3\theta \tag{2.6}$$

to rewrite (2.5) as

$$\frac{\partial \tau}{\partial \phi} + \frac{2\pi}{3\mu} e^{-3\tau} \sin 3\theta + \frac{8\pi}{3\mu} e^{-3\tau} \sin^3\theta = 0 \quad \text{on } \psi = 0. \tag{2.7}$$

Davies' approximation consists of neglecting the last term in (2.7). Thus (2.7) becomes

$$\frac{\partial \tau}{\partial \phi} + \frac{2\pi}{3\mu} e^{-3\tau} \sin 3\theta = 0 \quad \text{on } \psi = 0. \tag{2.8}$$

From (2.6), we see that Davies' approximation replaces $\sin\theta$ by $\sin 3\theta/3$. This approximation is good for waves of small amplitude for which θ is small.

Next we rewrite (2.7) as

$$R\left[\frac{d\zeta}{df} - \frac{2\pi}{3\mu} i e^{-3\zeta}\right] = 0 \quad \text{on } \psi = 0. \tag{2.9}$$

Here R denotes the real part. Since the function in the squared brackets in (2.9) is an analytic function of f, analytic continuation implies

$$\frac{d\zeta}{df} - \frac{2\pi}{3\mu} i e^{-3\zeta} = ik \tag{2.10}$$

in $\psi < 0$. Here k is a real constant.

Using the fact that $\zeta \to 0$ as $\psi \to -\infty$, we find that $k = -(2\pi/3\mu)$.

We introduce the new function

$$F = e^{3\zeta} - 1 = w^3 - 1 \tag{2.11}$$

and rewrite (2.10) as

$$\frac{dF}{df}+\frac{2\pi}{\mu}iF=0. \tag{2.12}$$

Equation (2.12) is a simple differential equation whose solution is

$$F=Ae^{-i\frac{2\pi}{\mu}f}. \tag{2.13}$$

Here A is a constant of integration. Using (2.11) and (2.13), we obtain the exact solution

$$w=[1+Ae^{-i\frac{2\pi}{\mu}f}]^{\frac{1}{3}}. \tag{2.14}$$

The shape of the free surface can be obtained by integrating (2.4). By requiring (2.14) to be a periodic function of f with period 1, we obtain the dispersion relation

$$\mu=1. \tag{2.15}$$

The dispersion relation (2.15) agrees with the well-known linear theory. However, it does not predict the nonlinear dependence of μ on the amplitude of the wave.

The constant A in (2.14) is a measure of the amplitude of the wave. For A small, the wave is close to a linear sine wave. For $A=-1$, the wave reaches its limiting configuration with a stagnation point at the crest of the wave. A remarkable feature of Davies' approximation is that it predicts the correct angle of 120° at the crest of the limiting configuration. This can easily be checked by expanding (2.14) near $f=0$. This suggests that a refinement of Davies' approximation should provide very accurate solutions for waves of arbitrary amplitudes including the limiting configuration with a 120° angle at the crest.

2.3. Refinement of Davies' approximation. In this section we review the refinement of Davies' approximation introduced by Tulin [11]. We first rewrite the exact boundary condition (2.7) as

$$R\left[\frac{d\zeta}{df}-\frac{2\pi}{3\mu}ie^{-3\zeta}\right]+\frac{8\pi}{3\mu}e^{-3\tau}\sin^3\theta=0 \quad \text{on } \psi=0. \tag{2.16}$$

Davies' approximation neglects the last term in (2.16). We now take this term into account by defining, with analytic continuation, an analytic function $P(f)$ whose real part satisfies

$$R[P(f)]=\frac{8\pi}{3\mu}e^{-3\tau}\sin^3\theta \quad \text{on } \psi=0. \tag{2.17}$$

Then (2.16) becomes

$$R\left[\frac{d\zeta}{df}-\frac{2\pi}{3\mu}ie^{-3\zeta}+P(f)\right]=0 \quad \text{on } \psi=0. \tag{2.18}$$

Since the function in the squared brackets in (2.18) is an analytic function of f, analytic continuation implies

$$\frac{d\zeta}{df}-\frac{2\pi}{3\mu}ie^{-3\zeta}+P(f)=it \tag{2.19}$$

in $\psi<0$. Here t is a real constant. Using the condition $\zeta\to 0$ as $\psi\to-\infty$, we find from (2.19) that $R[P(f)]\to 0$ as $\psi\to-\infty$. Since $P(f)$ is defined up to an arbitrary additive imaginary constant, we can assume without loss of generality that $P(f) \to 0$ as $\psi \to -\infty$. It follows from (2.19) that $t = -(2\pi/3\mu)$.

Using (2.11), we rewrite (2.19) as

$$\frac{dF}{df} + \frac{2\pi}{\mu} iF + 3e^{3\zeta}P(f) = 0. \tag{2.20}$$

Next we introduce a change of variable from f to γ, such that (2.20) takes the same form as (2.12), namely,

$$\frac{dF}{d\gamma} + \frac{2\pi}{\mu} iF = 0. \tag{2.21}$$

Using (2.20), (2.21), and the chain rule, we obtain

$$\frac{d\gamma}{df} = 1 - \frac{3i\mu}{2\pi} \frac{(F+1)P(f)}{F}. \tag{2.22}$$

The solution to (2.21) is

$$F = Ae^{-i(2\pi/\mu)\gamma}. \tag{2.23}$$

Our problem now is to find γ such that (2.17) and (2.22) are satisfied. This is achieved numerically in the next section.

2.4. Numerical results. Since the right-hand side of (2.22) is an analytic function of f with period 1, we can represent it by a series expansion in powers of $e^{-i2\pi f}$. Thus we write

$$-\frac{3i\mu}{2\pi} \frac{(F+1)P(f)}{F} = \sum_{n=0}^{\infty} b_n e^{-i2\pi nf}. \tag{2.24}$$

Substituting (2.24) into (2.22) and integrating term by term, we obtain

$$\gamma = (1+b_0)f + \frac{i}{2\pi} \sum_{n=1}^{\infty} \frac{b_n}{n} e^{-i2\pi nf}. \tag{2.25}$$

Using (2.25) and (2.23), we see that the periodicity of F implies

$$b_0 = \mu - 1. \tag{2.26}$$

Solving (2.24) for $P(f)$ and substituting into (2.17) yield

$$\frac{8\pi}{3}\left[I\left(\frac{1}{1+F}\right)^{\frac{1}{3}}\right]^3 = R\left[\frac{i2\pi F}{3(F+1)}\left(\mu - 1 + \sum_{n=1}^{\infty} b_n e^{-2i\pi nf}\right)\right] \quad \text{on } \psi = 0. \tag{2.27}$$

Here R and I denote the real and imaginary parts, respectively.

Relations (2.25) and (2.23) give an expression for F in terms of the coefficients b_n:

$$F = A\exp\left[-i2\pi f + \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{b_n}{n} e^{-i2\pi nf}\right]. \tag{2.28}$$

Substituting (2.28) into (2.27) yields an equation for the coefficients b_n. We shall refer to this equation as the equation α. We solve the equation α numerically.

We truncate the infinite series in the equation α after $N-1$ terms and determine the $N+1$ unknowns $b_1, b_2, \dots, b_{N-1}$, μ, and A by collocation. Thus, we introduce the N collocation points

$$\phi_I = \frac{2I-1}{4N}, \qquad I = 1, 2, \dots, N. \tag{2.29}$$

We satisfy the equation α at the collocation points (2.29). This yields N nonlinear algebraic equations. Following Cokelet [2], we define the parameter ε^2 by

$$\varepsilon^2 = 1 - w(0)^2 w(1/2)^2. \tag{2.30}$$

Here $w(0)$ and $w(1/2)$ denote the values of the complex velocity at a crest and a trough, respectively. The parameter ε^2 serves as a measure of the amplitude of the wave. It ranges between 0 and 1. For ε^2 small, the wave is close to a linear sine wave. For $\varepsilon^2 = 1$, the wave reaches the limiting configuration with a stagnation point and a 120° angle at the crest. The quantities $w(0)$ and $w(1/2)$ are evaluated in terms of the unknowns from (2.11) and (2.28) with $f = 0$ and $f = 1/2$, respectively. The last equation is given by (2.30) with a prescribed ε^2. Thus for a given value ε^2, we have a system of $N+1$ equations for the $N+1$ unknowns $b_1, b_2, b_3, \dots, b_{N-1}$, μ, and A. This system is solved numerically by Newton's method.

Numerical values of μ versus ε^2 are shown in Table 1 for values of N. We also show for comparison the values obtained by Cokelet [2].

2.5. Discussion of the results. Table 1 shows that the numerical procedure in §2.4 gives accurate results for all values of ε^2 between 0 and 1. In particular, it shows that μ is not a monotonic function of ε^2 (see Longuet-Higgins [3] and Cokelet [2]). In fact Longuet-Higgins and Fox [5] showed analytically that μ oscillates infinitely often as the wave of maximum height is approached. The method of §2.4 is a refinement of Davies' approximation. It reduces to Davies' approximation if all the coefficients b_n, $n = 1, 2, \dots, N-1$ are set equal to zero. The accuracy of the results for steep waves is related to the fact that Davies' approximation already predicts an angle of 120° for the highest wave (see the end of §2.2).

Some further insight into the method can be gained by noticing that (2.28) assumes implicitly that w^3 can be expanded in powers of $e^{-2i\pi f}$. This suggests that results similar to those of §2.4 can be obtained in a simpler way by writing

$$w^3 = 1 + \sum_{n=1}^{\infty} u_n e^{-2i\pi n f} \tag{2.31}$$

TABLE 1
Values of μ for various values of ε^2 in water of infinite depth. The last column contains values previously calculated by Cokelet [2].

ε^2	$N = 60$	$N = 80$	$N = 120$	Cokelet
0.6	1.12229	1.12229	1.12229	1.12229
0.8	1.17095	1.17094	1.17093	1.17093
0.9	1.19006	1.19025	1.19019	1.19014
0.94	1.19310	1.19367	1.19408	1.19404
0.99	1.19321	1.19324	1.19332	1.19329
1.0	1.19315	1.19311	1.19309	

and finding the coefficients u_n by collocation. To check this idea, we truncate the infinite series in (2.31) after N terms and find the $N+1$ unknowns $u_1, u_2, \ldots, u_N$ and μ by satisfying (2.30) and (2.5) at the N collocation points (2.29). This leads to a system of $N+1$ nonlinear equations with $N+1$ unknowns. This system was solved by Newton's method. We found that the numerical values of μ for $N=60$, $N=80$, and $N=120$ agree with those listed in Table 1. This shows that the numerical procedure of §2.4 is equivalent to a collocation method based on (2.31). In fact, this observation gives a hint for a generalization for waves in water of finite depth, which shall be fully presented in §3.

Michell [12], Olfe and Rottman [13], and Vanden-Broeck [14] calculated the highest wave ($\varepsilon^2 = 1$) by writing

$$w = (1 - e^{-2i\pi f})^{\frac{1}{3}}\left[1 + \sum_{n=1}^{\infty} c_n e^{-2i\pi n f}\right] \tag{2.32}$$

and finding the coefficients c_n by series truncation and collocation.

Havelock [15] and Vanden-Broeck [14] calculated waves of arbitrary amplitude up to the limiting configuration by expressing w as

$$w = (1 - \beta e^{-2i\pi f})^{\frac{1}{3}}\left[1 + \sum_{n=1}^{\infty} d_n e^{-2i\pi n f}\right] \tag{2.33}$$

and finding the coefficients d_n and β by series truncation and collocation.

By taking the cube of (2.32) and (2.33), we see that the expressions (2.32) and (2.33) are particular cases of (2.31) for appropriate choices of the coefficients u_n. Therefore Havelock's method (2.33) is equivalent to the procedure in §2.4. Similarly, Michell's method (2.32) is a particular case of the procedure in §2.4 corresponding to the highest wave. This is confirmed by the fact that the results in Table 1 are essentially the same as those obtained by Vanden-Broeck [14] using (2.33) (see Table II in Vanden-Broeck [14]). Similarly, the values of μ for $\varepsilon^2 = 1$ in Table 1 are in agreement with the value 1.193072 obtained by Olfe and Rottman [13].

3. Periodic waves in water of finite depth. We now consider waves in water of finite depth. The formulation of §2.1 remains unchanged except that the flow domain is $-Q < \psi < 0$ instead of $\psi < 0$. Here $-Q$ denotes the value of the stream function on the bottom. The phase velocity c is defined as the average fluid velocity at any horizontal level completely within the fluid. It follows from the choice of the dimensionless variables that

$$\int_0^{0.5} \frac{u(\phi, \psi)}{u^2(\phi, \psi) + v^2(\phi, \psi)}\, d\phi = 0.5. \tag{3.1}$$

The value of ψ in (3.1) can take any value between $-Q$ and 0.

Following Cokelet [2] and Vanden-Broeck and Schwartz [4], we define the parameter

$$r_0 = \exp[-2\pi Q]. \tag{3.2}$$

The parameter r_0 ranges between 0 and 1. Waves in water of infinite depth correspond to $r_0 = 0$, and solitary waves correspond to $r_0 = 1$.

For a fixed value of r_0, we seek $w(f)$ as an analytic function of f in $-Q < \psi < 0$, which satisfies (2.5) on $\psi = 0$ and the kinematic condition

$$I(w) = 0 \quad \text{on } \psi = -Q. \tag{3.3}$$

Here I denotes the imaginary part.

In §2.5, we showed that the refinement of Davies' approximation in water of infinite depth essentially assumes that w^3 can be represented by the expansion (2.31). We now generalize (2.31) to water of finite depth by writing

$$w^3 = a_0 + \sum_{n=1}^{\infty} \left[a_n e^{-i2\pi nf} + b_n e^{i2\pi nf} \right]. \tag{3.4}$$

Imposing the kinematic condition (3.3), we obtain

$$b_n = a_n r_0^{2n}, \qquad n = 1, 2, \dots . \tag{3.5}$$

We truncate the infinite series in (3.4) after N terms. Using (3.5), we have $N+1$ unknown coefficients in (3.4), namely, a_n, $n = 0, \dots, N$. We satisfy (2.30) and (2.5) at the N collocation points (2.29). This leads to a system of $N+1$ nonlinear algebraic equations for the $N+2$ unknowns a_n, $n = 0, \dots, N$, and μ.

The last equation is obtained by imposing (3.1). The integral in (3.1) is evaluated by the trapezoidal rule with a uniform mesh, since for periodic integrand this is highly accurate. In most calculations the value of ψ was chosen to be $-Q/10$. We checked that the numerical results are independent of the value of $-Q < \psi < 0$.

The resulting system of $N+2$ equations with $N+2$ unknowns is solved by Newton's method.

Numerical values of μ versus ε^2 for $r_0 = 0.5$ are shown in Table 2 for various values of N. Similar results were obtained for other values of r_0. Table 2 shows that our procedure converges and gives accurate results for waves of arbitrary amplitude. We also show for comparison the numerical values of Cokelet [2] (obtained by Padé approximants) and of Vanden-Broeck and Schwartz [4] (obtained by a boundary integral equation method). The advantage of the present method is its simplicity.

Our numerical procedure gives accurate results up to the limiting configuration with a 120° angle at the wave crest. This is shown in Table 3, where we present values of the steepness (i.e., the difference of heights between a crest and a trough divided by the wavelength) for $0.1 \le r_0 \le 0.9$. We also show the values previously obtained by Williams [6].

As r_0 approaches 1, the convergence of our scheme starts to deteriorate (see bottom of Table 3). Therefore we shall adapt the numerical procedure in the next section to solve directly the case $r_0 = 1$; i.e., we shall directly compute solitary waves.

4. Solitary waves. We consider a two-dimensional solitary wave in an inviscid incompressible and irrotational fluid bounded below by a horizontal bottom. We take a frame of reference with the x-axis parallel to the bottom and moving with the phase

TABLE 2

Values of μ for $r_0 = 0.5$ and various values of ε^2. The last two columns contain values previously computed by Vanden-Broeck and Schwartz [4] *and by Cokelet* [2].

ε^2	$N = 80$	$N = 120$	$N = 200$	Vdb, Sch	Cokelet
0.4	0.666501	0.666501	0.666501	0.666501	0.666501
0.6	0.706443	0.706443	0.706443	0.706443	0.706443
0.8	0.748242	0.748230	0.748230	0.748230	0.748230
0.9	0.764411	0.764488	0.764416	0.764403	0.764403
0.95	0.766936	0.767510	0.767818	0.767750	0.767748
0.97	0.766797	0.767149	0.767481	0.767540	0.76754
0.98	0.766649	0.766833	0.767042	0.767097	0.76707
0.99	0.766504	0.766536	0.766605	0.766557	0.76648

TABLE 3
Values of μ for the highest waves corresponding to various values of r_0. The last column contains the values previously computed by Williams [6].

r_0	$N=30$	$N=60$	$N=120$	Williams
0.1	0.137825	0.137801	0.137800	0.137801
0.2	0.128523	0.128495	0.128493	0.128495
0.3	0.114474	0.114440	0.114437	0.114439
0.4	0.097421	0.097377	0.097371	0.097374
0.5	0.079140	0.079079	0.079070	0.079072
0.6	0.061086	0.060999	0.060984	0.060984
0.7	0.044138	0.044004	0.043976	0.043975
0.8	0.028541	0.028319	0.028266	0.028258
0.9	0.014323	0.013822	0.013701	0.013667

velocity c of the wave. The level $y=0$ is chosen as the undisturbed level of the free surface, and gravity is assumed to act in the negative y-direction.

As in §2.1, we introduce the complex potential $f=\phi+i\psi$ and the complex velocity $w=u-iv$. We choose $\phi=0$ at the crest and $\psi=0$ on the free surface. We denote by $-Q$ the value of ψ on the bottom. Then the undisturbed depth is defined by

$$H=\frac{Q}{c}. \tag{4.1}$$

We introduce dimensionless variables by taking H as the unit length and c as the unit velocity.

On the free surface, the Bernoulli equation yields

$$u^2+v^2+\frac{2}{F^2}y=1. \tag{4.2}$$

Here F is the Froude number defined by

$$F=\frac{c}{(gH)^{1/2}}. \tag{4.3}$$

Hunter and Vanden-Broeck [9] used a procedure by Lenau [17] to compute the highest solitary wave. We shall use the idea of expanding w^3 (see (2.31)) to generalize their procedure to calculate solitary waves of arbitrary amplitude.

We first map the flow domain into the domain $|t|<1$ by the transformation

$$f=\frac{2}{\pi}\log\frac{1+t}{1-t}-i. \tag{4.4}$$

The transformation (4.4) maps the bottom of the channel onto the real diameter $-1<t<1$ and the free surface onto the half circumference $|t|=1$ in the upper half t-plane. We use the notation $t=e^{i\sigma}$ so that the free surface is described by $r=1$, $0<\sigma<\pi$.

We now represent w^3 as an expansion in powers of t, namely,

$$w^3=1+(1-t^2)^{2\lambda}\sum_{n=1}^{\infty}a_nt^{2n-2}, \tag{4.5}$$

where λ is the smallest positive root of

$$\pi\lambda - \frac{\tan \pi\lambda}{F^2} = 0. \tag{4.6}$$

The term $(1-t^2)^{2\lambda}$ in (4.5) comes from Stokes' result, which states that the asymptotic behavior of $u - iv$ as $|\phi| \to \infty$ is

$$u - iv \approx 1 + Ae^{-\pi\lambda|\phi|}, \qquad |\phi| \to \infty. \tag{4.7}$$

Next we differentiate (4.2) with respect to σ. Using (4.4) we obtain

$$F^2[u(\sigma)u_\sigma(\sigma) + v(\sigma)v_\sigma(\sigma)] - \frac{2}{\pi}\frac{v(\sigma)}{u^2(\sigma)+v^2(\sigma)}\frac{1}{\sin\sigma} = 0. \tag{4.8}$$

We characterized the amplitude of the solitary wave by the parameter

$$\omega = 1 - F^2[u(0)]^2. \tag{4.9}$$

Here $u(0)$ is the velocity at the crest of the wave. The parameter ω is similar to the parameter ε used for periodic waves (see (2.30)). It is a measure of the amplitude of the solitary wave and $\omega = 1$ for the highest wave.

We now truncate the infinite series in (4.5) after N terms and determine the $N+2$ unknowns $a_1, a_2, \ldots, a_N$, λ, and F^2 by collocation. Thus we introduce the N collocation points

$$\sigma_I = \frac{E}{2} + (I-1)E, \qquad I = 1, \ldots, N. \tag{4.10}$$

Here $E = \frac{\pi}{N}$. We satisfy (4.8) at the collocation points (4.10). This yields N nonlinear algebraic equations. The last two equations are given by (4.6) and (4.9). Thus for a given value of ω we have a system of $N+2$ equations with $N+2$ unknowns. This system is solved by Newton's method.

Numerical values of F versus ω for various values of N are shown in Table 4. The results converge as N increases. We also show some of the numerical values previously obtained by Byatt-Smith and Longuet-Higgins [8] and by Hunter and Vanden-Broeck [9]. Our calculations confirm their values.

Finally we used our scheme to recalculate the highest solitary wave and obtained the value $F \approx 1.29091$, in agreement with the value given by Hunter and Vanden-Broeck [9].

TABLE 4

Values of the Froude number for solitary waves corresponding to various values of ω. The last two columns contain values previously calculated by Byatt-Smith and Longuet-Higgins [8] *and by Hunter and Vanden-Broeck* [9].

ω	$N=90$	$N=290$	$N=390$	B, LH	H, Vdb
0.4	1.17178	1.17310	1.17324		
0.6	1.23895	1.23944	1.23949		
0.8	1.28445	1.28470	1.28472	1.2848	
0.9	1.29373	1.29394	1.29395	1.2939	
0.95	1.29312	1.29330	1.29332		
0.98	1.29137	1.29142	1.29144		1.29145

5. Conclusions. We have presented new numerical procedures to compute steep water waves. These procedures can be used to calculate waves for all values of the depth (i.e., from infinite depth to the solitary wave limit). Tables 1–4 show that the results are accurate for all values of the steepness, including the limiting configurations. The advantage of these new procedures over other methods such as boundary integral equation techniques is their simple implementation.

Recent numerical calculations have shown that many two-dimensional free surface flows have limiting configurations with a stagnation point on the free surface and a 120° angle at it. These include, for example, jets and flows past submerged obstacles (Vanden-Broeck and Keller [18], Vanden-Broeck [19], Mekias and Vanden-Broeck [20], Dias and Vanden-Broeck [21], Lee and Vanden-Broeck [22], and Vanden-Broeck and Tuck [23]). The numerical procedures described in this paper can be used to describe some of these flows close to their limiting configurations. For example, expansions similar to (2.32) and (2.33) were used by Vanden-Broeck [19] to calculate a bubble rising in a tube and by Dias and Vanden-Broeck [21] to study flows past submerged obstacles.

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