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Water Waves Induced by a Fluctuating Tangential Stress*

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Abstract: In the framework of linear wave theory both the effects of tangential and normal stresses on the water waves are discussed without the assumption of irrotational water motion. A formal solution initially at rest with level surface and developed under the actions of the surface stresses depending arbitrarily on time and sinusoidally on space is obtained.

A progressive wave type tangential stress whose wave number and frequency are satisfying the dispersion relation of the water waves is shown to be equivalent to the normal stress of the same type on the growth of the waves except the phase relations between the stresses and the water motion. The growth rate of the waves induced by the tangential stress is also shown to be quite insensitive to the actual value of the viscosity.

The rotational part of the water motion can dominate only in the early stage of wave generation and becomes negligible with the growth of the waves relative to the irrotational part of the motion even in the case where the motion is induced by the tangential stress alone. Therefore it is not reasonable to neglect effect of the tangential stress on wind waves even if the developed wind waves seem to be irrotational.

1. Introduction

A sight seen in near shore region of the sea in the wind blowing seaward from the land or wind waves in a wind-wave tunnel, give us the impression that the water surface is dragged by the wind. The wind blows horizontally over the water surface, so it is natural to consider that the wind stress also mainly acts horizontally on the surface.

It is well known that a wind blowing over a ocean surface makes a drift current which is inevitably rotational motion of the water and can not be generated without the effect of the viscosity of the water at the surface. Therefore the momentum of the flow is transferred form the air to the water by the effect of the viscosities of the fluids. If it be so, it seems unlikely that the momentum is uniformly transferred from the air to the water rather than not uniformly. For example, it is natural to consider that the presence of ups and downs of the water surface makes some differences between the magnitudes of time rate of the momentum transfer at the ups and the downs. In this case, the ununiformity of the magnitude of the momentum flux propagates on the water surface. If the ups and downs take the form of a wave train, the ununiformity propagates just with the phase velocity of the wave train independently of the wind velocity blowing over it. It is interesting that whether the phase velocity of the wave train is larger than the wind velocity or not, the situation mentioned above seems to be quite unchanged. This suggests a possibility of an exitation mechanism of a wave train whose phase velocity is faster than the wind velocity blowing over it.

Here a question arises whether a propagating ununiformity of time rate of the momentum transfer from the air to the water by the viscosities has some effects on wave generation and growth or not.

Available theories are classified into two groups *i.e.* some of them stand on the hypothesis of irrotational motion of water, accordingly the effect of a fluctuating tangential stress can not be included in them, and the others do not. Some of the former are as follows.

It was known that in certain circumstances Kelvin-Helmholts instability may arise at the surface of separation of two fluids when there is a finite difference between the velocities on the two sides of the surface on the hypothesis of irrotational motion for frictionless fluids (LAMB, 1932, pp. 373-375). The predicted

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critical value of the velocity of the wind (minimum wind velocity to generate wind waves) relative to the water is in disagreement with observations (JEFFREYS, 1924).

A model of a resonance mechanism between a component of random surface pressure distribution associated with the onset of a turbulent wind and the free surface wave having the same wave number as the component, was proposed by PHILLIPS (1957). In a parallel shear flow over a water surface disturbed by waves, MILES (1957) made the calculation of the wave induced pressure component in phase with the wave slope on the basis of the inviscid Orr-Sommerfeld equation. A combined effect of these two kinds of pressure fluctuations owing to turbulent wind and induced by waves on generation and growth of the water waves was considered by PHILLIPS (1969). It was shown that the growth rates predicted from these theories are smaller than the observations executed by SNYDER and COX (1966) and BARNETT and WILKERSON (1967). The observed spectra of ocean waves have respectable amount of energy in the components whose phase velocities are faster than mean wind velocity, clearly' this can not be explained by these theories.

The extension works of these theories in which the effect of the wave induced variation of the turbulence of wind and weak wave-wave interaction studied by HASSELMANN (1962) are introduced to overscome the weak points of them, have been developing (IMASATO, 1976; ICHIKAWA and IMASATO, 1976).

As the second group, JEFFREYS (1924) introduced at first, sheltering concept to illustrate pressure distribution relative to water surface elevation, and next, he introduced the hypothesis of skin friction. In both cases the motion of water was not assumed irrotational. In the former case the tangential stress was set zero at the surface and the value of sheltering coefficient estimated from critical wind velocity leads to overestimate of growth rate of wind waves in comparison with observations. In the latter case the surface pressure was given by Bernoulli's equation on the assumption of irrotational air flow and the tangential stress was set to be proportional to the square of the air velocity at the surface. Though he denied himself the latter theory because of the disagreement of the critical wind velocity estimated by the theory with observations, his theory does not mean that the tangential stress can not have any effect on the waves but means that the stress can have some.

It was shown by LAMB (1932, pp. 625-631) that in an equilibrium state, both of normal stress and tangential stress can independently maintain a train of waves in the form of $e^{i(\kappa x-\sigma t)}$ respectively, where $i=\sqrt{-1}$, κ is the wave number, σ the frequency. The stresses necessary to maintain a train η are

$$\tau_{yy}=-4\,i\mu\kappa\sigma\eta,\ \tau_{xy}=4\,\mu\kappa\sigma\eta \qquad (1.1)$$

where τ_{yy} and τ_{xy} are normal and tangential stresses in the form of $e^{i(\kappa x-\sigma t)}$ respectively, μ the coefficient of viscosity of water.

Starting from a linear system of equation of motion and boundary conditions, HAMADA *et al.* (1963) calculated a characteristic value of complex frequency of a water wave component under the assumption that each magnitude of fluctuating tangential and normal stresses is proportional to surface elevation with a complex factor. Their calculation shows that a tangential stress τ_{xy} has a equivalent effect on waves to the normal stress $\tau_{yy}=-i\tau_{xy}$.

By adopting an approximation about the boundary-layer beneath a water surface, LONGUET-HIGGINS (1968) showed that a variable tangential stress τ_{xy} is precisely equivalent to a normal stress $\tau_{yy}=-i\tau_{xy}$ in quadrature with the tangential stress as to the effect on waves, and the corresponding rate of growth of the waves is less than $|\tau_{xy}|/2\,\rho c$, where ρ is the density of water and c the phase velocity of the waves. Under the assumptions that the horizontal component of the orbital velocity in the waves at the surface is much larger than the additional velocity in the boundary layer due to the fluctuating tangential stress, and the dissipation due to the viscosity is negligibly smaller than the work done by the tangential stress at the surface, he gave the relation

$$\frac{\partial|\eta|}{\partial t}=|\tau_{xy}|/2\,\rho c \qquad (1.2)$$

The conclusive results about the effect of a tangential stress fluctuation on waves have not been obtained yet, so let us consider only the effect of a fluctuating tangential stress on waves in contradistinction to that of normal one out of many mechanisms which are concerned in momentum transfer between the atmosphere and the sea. When we come to the conclusion that the effect of a fluctuating tangential stress can have only negligibly smaller effect than that of normal one, the hypothesis of irrotational water motion can be an excellent approximation to the problems of wind waves, and it is also effective to simplify the analysis of the problems. On the other hand, when the reverse is the case, it requires sufficient grounds for the justification of the each introduction of the irrotational-hypothesis to the problems of wind waves. In this case, evaluation of the magnitude of rotational part of the water motion induced by the fluctuating tangential stress and its variation with time also seem to be worthwhile subject to investigate for better understanding about the physical processes occurring at the air-water interface and for the interpretation of the experimental results measuring particle velocity of the water accompanied with wind waves.

Restating the end of this study, among the many mechanisms which take a part of momentum transfer from the air to the water, only the effect of a fluctuating tangential stress on wave generation and growth is clarified in comparison with the effect of a normal one.

2. Fundamental equations

To investigate the problem mensioned above, let us consider the water motion which is assumed in two dimensions, of which one (x') is horizontal, and the other (y') is drawn vertically upwards from the undisturbed water surface and to be induced by a tangential stress fluctuation and normal one acting on the surface, where the prime mean the dimensional. Let us also suppose that there are no limits to the water and no changes of the amplitudes of the stress fluctuations in the direction of x' and the depth of the water is infinite, and the motion is infinitely small and has been generated originally from rest.

Linear equations of the motion nondimensionalized using wave number κ and frequency σ with incompressibility assumption of water, reduce to the forms

$$\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+a\nabla^2 u$$
$$\frac{\partial v}{\partial t}=-\frac{\partial p}{\partial y}+a\nabla^2 v-g \tag{2.1}$$

with

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.2}$$

where $x=\kappa x'$, $y=\kappa y'$, $t=\sigma t'$, $u=(\kappa/\sigma)u'$, $v=(\kappa/\sigma)v'$, $p=(p'/\rho)(\kappa/\sigma)^2$, $a=\nu\kappa^2/\sigma$, $g=g'\kappa/\sigma^2$ and t' is time, u' the horizontal velocity component, v' the vertical one, ν the kinematic coefficient of viscosity of water, p' the pressure, ρ the density of water and g' the acceleration of gravity.

Hereafter constants and variables are dimensionless unless they are used with notice. The above equations are satisfied by

$$u=-\frac{\partial \Phi}{\partial x}+\tilde{u},\ v=-\frac{\partial \Phi}{\partial y}+\tilde{v} \tag{2.3}$$

and

$$p=\frac{\partial \Phi}{\partial t}-gy \tag{2.4}$$

provided

$$\nabla^2\Phi=0 \tag{2.5}$$

$$\frac{\partial \tilde{u}}{\partial t}=a\nabla^2\tilde{u} \text{ and } \frac{\partial \tilde{v}}{\partial t}=a\nabla^2\tilde{v} \tag{2.6}$$

where Φ is a scalar function of x, y and t, $\tilde{u}$ the rotational part of horizontal velocity component and $\tilde{v}$ the vertical one. The equation

$$\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{v}}{\partial y}=0 \tag{2.7}$$

is also satisfied as a result of Eqs. (2.2), (2.3) and (2.5) (LAMB, 1932, pp. 625–626).

Neglecting second order terms, kinematic surface condition becomes

$$\frac{\partial \eta}{\partial t}=v, \quad \text{at } y=0 \tag{2.8}$$

where η is surface elevation. If P_{yy} denote normal stress and P_{xy} the tangential stress, the dynamical conditions at the surface are

$$P_{yy}=-p+2a\frac{\partial v}{\partial y}-T\frac{\partial^2\eta}{\partial x^2} \tag{2.9}$$

and

$$P_{xy}=a\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \qquad (2.10)$$

where $T=T'\kappa^3/\rho\sigma^2$ and T' is the surface tension. If the depth of water be infinite, the condition of no motion at the bottom are

$$\tilde{u}=\tilde{v}=\Phi=0, \text{ as } y\to-\infty \qquad (2.11)$$

The condition that the motion is initially at rest gives

$$\tilde{u}=\tilde{v}=\eta=\Phi=0 \text{ at } t=0 \qquad (2.12)$$

3. Formal solutions

Using Green's formula, the solution of Eq. (2.6) satisfying the initial condition (2.12) can be expressed by the normal derivative of it at the boundary. The expression becomes

$$\tilde{u}(x, y, t)=\int_{-\infty}^{\infty}\int_0^t a\Psi\frac{\partial\tilde{u}(\xi,\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} d\xi\, d\tau \qquad (3.1)$$

where Ψ is the Green's function of the second kind satisfying

$$\frac{\partial\Psi}{\partial\zeta}=0 \text{ at } \zeta=0 \text{ and } \Psi=0 \text{ as } \zeta\to-\infty \qquad (3.2)$$

The Green's function Ψ, the solution of

$$\left(\frac{\partial}{\partial t}+\nabla^2\right)\Psi=-\delta(\boldsymbol{R}_1-\boldsymbol{R}_2)$$

where $\boldsymbol{R}_1=(\xi,\zeta,\tau)$, $\boldsymbol{R}_2=(x,y,t)$ and δ the Dirac's delta function, which is satisfying the boundary condition (3.2) is obtained by the reflection method of constructing Green's function. It is

$$\Psi=\begin{cases} 0, \text{ for } t-\tau<0, \\ \dfrac{1}{4\pi a(t-\tau)}\left\{e^{-\frac{(x-\xi)^2+(y-\zeta)^2}{4a(t-\tau)}}+e^{-\frac{(x-\xi)^2+(y+\zeta)^2}{4a(t-\tau)}}\right\} \\ \text{for } t-\tau>0 \end{cases} \qquad (3.3)$$

This expression changes Eq. (3.1) into

$$\tilde{u}(x, y, t)=\int_{-\infty}^{\infty}\int_0^t\frac{\exp\left\{-\dfrac{(x-\xi)^2+y^2}{4a(t-\tau)}\right\}}{2\pi(t-\tau)} \times\frac{\partial\tilde{u}(\xi,\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} d\xi\, d\tau \qquad (3.4)$$

In application to simple-harmonic motion, the equations are shortened if we assume complex factor such as e^{ix}, and in the end let us reject imaginary parts of our expressions. If $\partial\tilde{u}/\partial y$, at $y=0$, is assumed to have the form

$$\frac{\partial\tilde{u}}{\partial y}=e^{ix}f(t)$$

the expression (3.4) gives

$$\tilde{u}(x, y, t)=\frac{\sqrt{a}}{\sqrt{\pi}}e^{ix}\int_0^t\frac{\exp\left\{-a(t-\tau)-\dfrac{y^2}{4a(t-\tau)}\right\}}{\sqrt{t-\tau}} \times f(\tau)\, d\tau \qquad (3.5)$$

Using the continuity relation (2.7) and the boundary conditions at the bottom (2.11), this gives

$$\frac{\partial\tilde{u}}{\partial x}=-\frac{\partial\tilde{v}}{\partial y} =i\frac{\sqrt{a}}{\sqrt{\pi}}e^{ix}\int_0^t\frac{\exp\left\{-a(t-\tau)-\dfrac{y^2}{4a(t-\tau)}\right\}}{\sqrt{t-\tau}}f(\tau)\, d\tau$$

$$\tilde{v}=-i\frac{\sqrt{a}}{\sqrt{\pi}}e^{ix}\int_0^t\frac{e^{-a(t-\tau)}}{\sqrt{t-\tau}}f(\tau)d\tau \times\int_{-\infty}^{y}\exp\left\{-\frac{y^2}{4a(t-\tau)}\right\}dy$$

and

$$\frac{\partial\tilde{v}}{\partial x}=\frac{\sqrt{a}}{\sqrt{\pi}}e^{ix}\int_0^t\frac{e^{-a(t-\tau)}}{\sqrt{t-\tau}}f(\tau)d\tau \times\int_{-\infty}^{y}\exp\left\{-\frac{y^2}{4a(t-\tau)}\right\}dy \qquad (3.6)$$

On the surface these become

$$\tilde{u}=\frac{\sqrt{a}}{\sqrt{\pi}}e^{ix}\int_0^t\frac{e^{-a(t-\tau)}}{\sqrt{t-\tau}}f(\tau)d\tau$$

$$\tilde{v}=-iae^{ix}\int_0^t e^{-a(t-\tau)}f(\tau)d\tau$$

$$\frac{\partial\tilde{u}}{\partial x}=-\frac{\partial\tilde{v}}{\partial y}=i\frac{\sqrt{a}}{\sqrt{\pi}}e^{ix}\int_0^t\frac{e^{-a(t-\tau)}}{\sqrt{t-\tau}}f(\tau)d\tau$$

and

$$\frac{\partial\tilde{v}}{\partial x}=ae^{ix}\int_0^t e^{-a(t-\tau)}f(\tau)d\tau \qquad (3.7)$$

respectively.

At the water surface Eq. (2.4) is denoted by

$$p=\frac{\partial \Phi}{\partial t}-g\eta \tag{3.8}$$

After elimination of p from the normal stress condition (2.9) using the relation (3.8), partial differentiation with respect to t transforms the stress condition into

$$\frac{\partial^2\Phi}{\partial t^2}+2a\frac{\partial^3\Phi}{\partial t\partial y^2}+g\frac{\partial \Phi}{\partial y}-T\frac{\partial^3\Phi}{\partial x^2\partial y}$$
$$=(g+T)\tilde{v}+2a\frac{\partial^2\tilde{v}}{\partial y\partial t}-\frac{\partial P_{yy}}{\partial t},\ \text{for}\ y=0$$

where the relation (2.3) and (2.8) are used. Since the system of equations is linear, all variables have space factor e^{ix}, provided one of which has the factor. Therefore, notations

$$P_{yy}=e^{ix}p_{yy}(t),\ P_{xy}=e^{ix}p_{xy}(t)$$
$$\text{and}\ \Phi=e^{ix+y}\phi(t) \tag{3.10}$$

are allowed, considering that Φ must satisfy Laplace's equation. Using these notations, relation (3.9) is written in the form

$$\frac{\partial^2\phi}{\partial t}+2a\frac{\partial \phi}{\partial t}+G\phi=h(t) \tag{3.11}$$

where $G=g+T$ and $h(t)$ is time dependent part of the right hand side of Eq. (3.9), and is expressed by $f(t)$ with aid of Eq. (3.7),

$$h(t)=-\frac{\partial p_{yy}}{\partial t}-iaG\int_0^t e^{-a(t-\tau)}f(\tau)d\tau$$
$$-2ia\frac{\sqrt{a}}{\sqrt{\pi}}\frac{\partial}{\partial t}\int_0^t\frac{e^{-a(t-\tau)}}{\sqrt{t-\tau}}f(\tau)d\tau$$

The solution to (3.11) satisfying the initial conditions

$$\phi=\frac{\partial\phi}{\partial t}=0\ \text{at}\ t=0$$

appropriate to a state of rest, is expressed in the form

$$\phi(t)=\frac{1}{\omega}\int_0^t h(\tau)e^{-a(t-\tau)}\sin\{\omega(t-\tau)\}d\tau \tag{3.12}$$

where $\omega=\sqrt{G-a^2}$.

The tangential stress condition (2.10) is also transformed by using Eqs. (2.3), (3.7) and (3.10) into

$$\phi(t)=-\frac{i}{2}f(t)-\frac{ia}{2}\int_0^t e^{-a(t-\tau)}f(\tau)d\tau$$
$$+\frac{i}{2a}p_{xy} \tag{3.13}$$

Equating $\phi(t)$ appearing in (3.12) to that of (3.13) leads to an integral equation that $f(t)$ must satisfy (KODAIRA, 1974),

$$2ia\int_0^t\left\{-iaG\int_0^\theta e^{-a(\theta-\tau)}f(\tau)d\tau-2ia\frac{\sqrt{a}}{\sqrt{\pi}}\frac{\partial}{\partial\theta}\right.$$
$$\left.\times\int_0^\theta\frac{e^{-a(\theta-\tau)}}{\sqrt{\theta-\tau}}f(\tau)d\tau-\frac{\partial p_{yy}(\theta)}{\partial\theta}\right\}e^{-a(t-\theta)}$$
$$\times\sin\{\omega(t-\theta)\}d\theta-a^2\omega\int_0^t e^{-a(t-\tau)}f(\tau)d\tau$$
$$-a\omega f(t)+\omega p_{xy}=0 \tag{3.14}$$

where p_{yy} and p_{xy} should be given externally. The solution $f(t)$ to the equation (3.14) for given external stresses p_{yy} and p_{xy} is connected with the water motion through Eqs. (3.5), (3.6) and (3.12). To find the solution $f(t)$, the Laplace transformation will be useful because integrals appearing in the equation have the form of convolution.

The Laplace transformation of Eq. (3.14) is

$$2ia\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\}\left\{-iaG\frac{L(f)}{s+a}-\frac{2ia\sqrt{a}\,s}{\sqrt{s+a}}L(f)\right.$$
$$\left.-L\left(\frac{\partial p_{yy}}{\partial t}\right)\right\}-a^2\omega\frac{L(f)}{s+a}-a\omega L(f)$$
$$+\omega L(p_{xy})=0 \tag{3.15}$$

where s is a parameter of the Laplace transformation and

$$L(f)=\int_0^\infty e^{-st}f(t)dt$$

In Eq. (3.15), the term including $1/\sqrt{s+a}$ appears, this comes from

$$L\left(\frac{e^{-at}}{\sqrt{t}}\right)=\frac{\sqrt{\pi}}{\sqrt{s+a}}$$

It is a double-valued function of s, when s is considered in the complex plane. To remove double-valued character from the function, let cut the plane of s along the straight line from the point, $s=-a$, with the angle θ between the line and the real exis from the point to $+\infty$, the cut line must be set not to cross the domain in which the Laplace transformation and the inversion,

$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}L(f)e^{st}ds,\ \gamma>0$$

are defined. Hence the relation

$$\frac{\pi}{2}<\theta<\frac{3}{2}\pi \tag{3.16}$$

must be satisfied, and the argument of $s+a$ should lie in the range

$$\theta>\mathrm{Arg}\,(s+a)>\theta-2\pi \tag{3.17}$$

as $1/\sqrt{s+a}$ must be real and positive when s is real and positive.

From Eq. (3.15) the expression

$$L(f)=\frac{(s+a)\left[2iaL\left(\frac{\partial p_{yy}}{\partial t}\right)-\{(s+a)^2+\omega^2\}L(p_{xy})\right]}{as\{-(s+a)^2-2a(s+a)+4a\sqrt{a}\sqrt{s+a}-2a^2-\omega^2\}} \tag{3.18}$$

is given. Neglecting space factor e^{ix}, the expressions of the other variables at the surface are given from Eq. (3.18). They are

$$L(\tilde{u})=\frac{\sqrt{s+a}\left[2iaL\left(\frac{\partial p_{yy}}{\partial t}\right)-\{(s+a)^2+\omega^2\}L(p_{xy})\right]}{\sqrt{a}s\{-(s+a)^2-2a(s+a)+4a\sqrt{a}\sqrt{s+a}-2a^2-\omega^2\}}$$

$$L(\tilde{v})=\frac{2aL\left(\frac{\partial p_{yy}}{\partial t}\right)+i\{(s+a)^2+\omega^2\}L(p_{xy})}{s\{-(s+a)^2-2a(s+a)+4a\sqrt{a}\sqrt{s+a}-2a^2-\omega^2\}}$$

$$L(\phi)=\frac{(s+2a)L\left(\frac{\partial p_{yy}}{\partial t}\right)+i(2\sqrt{as}\sqrt{s+a}+a^2+\omega^2)L(p_{xy})}{s\{-(s+a)^2-2a(s+a)+4a\sqrt{a}\sqrt{s+a}-2a^2-\omega^2\}}$$

and

$$L(\eta)=\frac{-L\left(\frac{\partial p_{yy}}{\partial t}\right)+i(s+a-2\sqrt{a}\sqrt{s+a}+a)L(p_{xy})}{s\{-(s+a)^2-2a(s+a)+4a\sqrt{a}\sqrt{s+a}-2a^2-\omega^2\}} \tag{3.19}$$

where the relation (3.7), (3.13) and

$$L(\eta)=L\left(\int_0^t v\,dt\right)=\frac{1}{s}L(v)=\frac{1}{s}\{L(\tilde{v})-L(\phi)\}$$

are used. Eqs. (3.19) will be also obtained by direct operation of the Laplace transformation on Eqs. (2.3), (2.4), (2.5) and (2.6), and on boundary conditions. The inversions of Eqs. (3.18) and (3.19) give f, $\tilde{u}$, $\tilde{v}$, ϕ and η with arbitrarily time dependent stresses, p_{yy} and p_{xy}.

4. Examples

To weigh the effect of tangential stress of a progressive wave type on the water motion against that of a normal stress of the same type, two cases are examined. In the first, the normal stress alone is assumed to act on the surface, *i.e.*

$$p_{yy}=\alpha e^{i(x-t)},\ p_{xy}=0 \tag{4.1}$$

where α is a constant. In the second, the acting stress is interchanged,

$$p_{yy}=0,\ p_{xy}=\alpha e^{i(x-t)} \tag{4.2}$$

Applying the conditions (4.1) and (4.2) to the last of Eqs. (3.19), the two expressions

$$L(\eta_I)=\frac{-i\alpha}{(r-a+i)(r-a)(r^2+2ar-4a\sqrt{a}\sqrt{r}+2a^2+\omega^2)}$$

and

$$L(\eta_{II})=\frac{-i\alpha(r-2\sqrt{a}\sqrt{r}+a)}{(r-a+i)(r-a)(r^2+2ar-4a\sqrt{a}\sqrt{r}+2a^2+\omega^2)} \tag{4.3}$$

are obtained respectively, where the suffix, I, denotes the first case and, II, does the second, and $r=s+a$. The space factor e^{ix} is neglected in the notation of η and, hereafter, it is also neglected in the notations of $\tilde{u}$ and $\tilde{v}$. Since both of the denominators and the numerators of Eqs. (4.3) are polinominals of $\sqrt{r}$ and the orders of $\sqrt{r}$ in the latters are less than those in the formers, 8, they can be decomposed in the form of sum of their partial fractions,

$$L(\eta_I)=\sum_{j=1}^{8}\frac{C_{I,j}}{\sqrt{r}-\beta_j}$$

and

$$L(\eta_{II})=\sum_{j=1}^{8}\frac{C_{II,j}}{\sqrt{r}-\beta_j} \qquad (4.4)$$

where $C_{I,j}$ and $C_{II,j}$ are constants given by

$$C_{I,j}=\frac{-i\alpha}{\prod_{m,m\neq j}(\beta_j-\beta_m)}$$

and

$$C_{II,j}=\frac{-i\alpha(\beta_j^2-2\sqrt{a}\beta_j+a)}{\prod_{m,m\neq j}(\beta_j-\beta_m)} \qquad (4.5)$$

and $\beta_{j\,(j=1,\dots,8)}$ are the solutions of

$$(x^2-a+i)(x^2-a)(x^4+2ax^2 -4a\sqrt{a}x+2a^2+\omega^2)=0$$

The first four of them are

$$\beta_1=\sqrt{a-i},\quad \beta_2=-\sqrt{a-i},$$
$$\beta_3=\sqrt{a},\quad \beta_4=-\sqrt{a}$$

Since both of the numerators of the expressions (4.3) do not have the term including $\sqrt{r}^{\,7}$, the equalities

$$\sum_{j=1}^{8}C_{I,j}=\sum_{j=1}^{8}C_{II,j}=0 \qquad (4.6)$$

are given, then η can be written in the form

$$\eta_I=\sum_{j=1}^{8}C_{I,j}D_j(t)e^{-at}$$

and

$$\eta_{II}=\sum_{j=1}^{8}C_{II,j}D_j(t)e^{-at} \qquad (4.7)$$

by the equalities (4.6), where the factor e^{-at} appears due to the relation $r=s+a$ and

$$D_j(t)=\beta_j e^{\beta_j^2 t}+\frac{\beta_j^2}{\sqrt{\pi}}e^{\beta_j^2 t}\int_0^t\frac{e^{-\beta_j^2 t}}{\sqrt{t}}dt$$

To find the behaviour of η as $t\to\infty$, $\beta_{j\,(j=5,\dots,8)}$ must be known, but it is expected that when the relation between the wave number and the frequency of the acting stress is approximated by the dispersion relation of water waves, the stress makes a significant effect on waves. For the waves whose frequency is 1.0 Hz,

$$a=\nu\kappa^2/\sigma\simeq 2.6\times 10^{-6}$$

is obtained using the values $\sigma=2\pi(s^{-1})$, $\kappa=(2\pi)^2/980\,(cm^{-1})$ and $\nu=10^{-2}\,(cm^2\,s^{-1})$. So we can assume $a\ll 1\simeq\omega$. Under the condition, the relations

$$|Re(\beta_j)|<|Im(\beta_j)|,\ j=5,\dots,8 \qquad (4.8)$$

are obtained as follows. Since $\beta_{j\,(j=5,\dots,8)}$ are solutions of

$$x^4+2ax^2-4a\sqrt{a}x+2a^2+\omega^2=0 \qquad (4.9)$$

the substitution in the above equation for $x=\alpha+i\beta$, where α and β are real, gives the equations

$$\alpha^4-6\alpha^2\beta^2+2a\alpha^2-2a\beta^2 -4a\sqrt{a}\alpha+2a^2+\omega^2=0 \qquad (4.10)$$

and

$$\beta(\alpha^3-\alpha\beta^2+a\alpha-a\sqrt{a})=0$$

which must be simultaneously satisfied by real α and β. $\beta=0$ is not a solution, because under the condition, Eq. (4.10) can not be satisfied by real α, therefore

$$\alpha(\alpha^2-\beta^2)+a(\alpha-\sqrt{a})=0$$

must be satisfied. The left hand side of this equation can not be zero for a negative value of α and for a value of α larger than $\sqrt{a}$, provided $\alpha^2>\beta^2$.

A condition under which there is no solution whose absolute value is less than $\sqrt{2}\sqrt{a}$, *i.e.* $0<\alpha\leqq\sqrt{a}$, obtained by the substitution in Eq. (4.9) for $x=\lambda\sqrt{2}\sqrt{a}e^{i\gamma}$, where λ and γ are real and $0<\lambda\leqq 1$, is $a^2(6+2\sqrt{2})<\omega^2$.

This is sufficiently satisfied in our cases, after all the relation (4.8) is satisfied. The relation (4.8) allows the notation

$$\beta_j^2=-b+ci,\quad \text{for}\quad j=5,\dots,8$$

where b and c are real and $b>0$. Using the notation it is shown that

$$\lim_{t\to\infty}\left|e^{(-b+ci)t}\int_0^t \frac{e^{(b-ci)t}}{\sqrt{t}}dt\right|$$
$$\leqq \lim_{t\to\infty} e^{-bt}\int_0^t \frac{e^{bt}}{\sqrt{t}}dt$$
$$=\lim_{t\to\infty} 2\sqrt{t}e^{-bt}\int_0^1 e^{bty^2}dy$$
$$<\lim_{t\to\infty}\frac{2(1-e^{-bt})}{b\sqrt{t}}=0 \qquad (4.11)$$

in other words, $\lim_{t\to\infty} D_j(t)=0$, for $j=5, \ldots, 8$.

After all η as $t\to\infty$ is described by $D_j(t)$ $(j=1,\ldots,4)$ alone as follows,

$$\eta_I \underset{t\to\infty}{=} \frac{\alpha e^{-it}}{\omega^2-1-4ia-4a\sqrt{a}\sqrt{a-i}+5a^2}-\alpha$$

and

$$\eta_{II} \underset{t\to\infty}{=} \frac{-\alpha(i+2\sqrt{a}\sqrt{a-i}-2a)e^{-it}}{\omega^2-1-4ia-4a\sqrt{a}\sqrt{a-i}+5a^2} \qquad (4.12)$$

The amplitudes of oscillating parts of them largely depend on the value of ω^2. For only neighbourhood of $\omega^2=1$, they become significant value and the case is of our interest (the dispersion relation of free water waves is correctly satisfied by the wave number and the frequency of the acting stress when $\omega^2=1-a^2$). Since the difference of the value of ω^2 of the order a^2 makes no virtual effect on the amplitude of η, (Fig. 1), let us set

$$\omega^2=1-2a^2 \qquad (4.13)$$

for the simplicity of our calculation.

The behaviours of the other variables as $t\to\infty$ under the condition (4.13) are given along the same way denoted above, they are

$$f_I=\alpha\left\{-\frac{i}{2a}+\frac{\sqrt{2}}{4\sqrt{a}}(1-i)+\frac{5}{8}\right\}e^{-it}$$
$$\tilde{u}_I=\alpha\left\{\frac{\sqrt{2}}{4\sqrt{a}}(1-i)+\frac{1}{2}\right\}e^{-it}$$
$$\tilde{v}_I=-\alpha\frac{i}{2}e^{-it}$$
$$\phi_I=\alpha\left\{-\frac{1}{4a}-\frac{\sqrt{2}}{8\sqrt{a}}(1+i)-\frac{9}{16}i\right\}e^{-it}$$
$$f_{II}=\alpha\left[\left\{\frac{1}{2a}+\frac{\sqrt{2}}{4\sqrt{a}}(1+i)+\frac{3}{8}i\right\}e^{-it}-i\right]$$
$$\tilde{u}_{II}=\alpha\left[\left\{\frac{\sqrt{2}}{4\sqrt{a}}(1+i)+\frac{i}{2}\right\}e^{-it}-i\right]$$
$$\tilde{v}_{II}=\alpha\left(\frac{1}{2}e^{-it}-1\right)$$

and

$$\phi_{II}=\alpha\left[\left\{\frac{i}{4a}+\frac{\sqrt{2}}{8\sqrt{a}}(1-i)+\frac{7}{16}\right\}e^{-it}-1\right] \qquad (4.14)$$

where coefficients are expressed in the form of power series of $\sqrt{a}$ to the order 1, and the difference of the value of ω^2 of the order a^2 affects on the third order terms of each variables.

As the next step, let us calculate the motion for early stage, $at\ll 1$. In the calculation, β_j $(j=5,\ldots,8)$ which are four solutions of

$$x^4+2ax^2-4a\sqrt{ax}+1=0 \qquad (4.15)$$

where the relation (4.13) is assumed, are required, but unfortunately they cannot be expressed in a simple form applicable to the expression (4.7), so that let us make an approximation. Consider the equation

$$x^4+2ax^2+yx+1=0$$

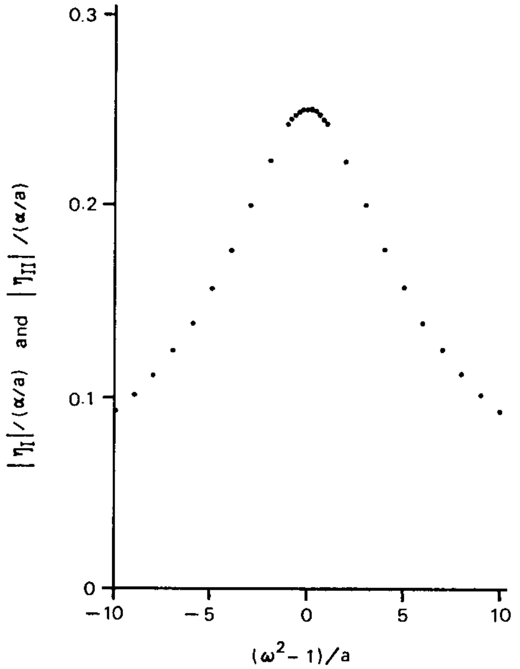

Fig. 1. The changes of amplitudes of water elevation in the equilibrium state with the value of ω^2-1. The difference between η_I and η_{II} is so small that it cannot be expressed in this figure. This is the case where $a=10^{-6}$, but as far as the condition $a\ll 1$ is satisfied, there is no substantial change with value of a.

Let x_0 be a solution of

$$x^4+2ax^2+1=0$$

then, $y=0$, for $x=x_0$ and $y'(x_0)=-4(x_0+a)\neq 0$, when $a<1$. Therefore in the neibourhood of $y=0$, inverse relation

$$x=g(y)=x_0+\sum_{j=1}^{\infty} b_j y^j \qquad (4.16)$$

is uniquely determined, it is shown that if a is less than 0.1 the expression (4.16) converges at least in the area $|y|<2.6a$ in virtue of Rouche's Theorem. Since, in our case,

$$y=-4a\sqrt{a}$$

the requirement of convergence is sufficiently satisfied. After substitution in Eq. (4.15) for the expression

$$\beta_j=\sum_{n=0}^{\infty} b_n(\sqrt{a})^n$$

b_n are determined by setting each coefficient of $(\sqrt{a})^n$ to be zero as follows:

$$b_0=(-1)^{1/4},\ b_1=0,\ b_2=-\frac{1}{2}b_0^{-1},$$

$$b_3=b_0^{-2},\ b_4=\frac{1}{8}b_0^{-3},\ b_5=0,\ \ldots\ldots$$

But the relation (4.13) has an effect on the value of b_4, so neglecting heigher order terms more than a^2, β_j corresponding four values of $b_0(e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$ and $e^{7\pi i/4})$ become

$$\beta_5=\ \frac{\sqrt{2}}{2}(1+i)-\frac{\sqrt{2}}{4}(1-i)a-ia\sqrt{a}$$

$$\beta_6=-\frac{\sqrt{2}}{2}(1-i)+\frac{\sqrt{2}}{4}(1+i)a+ia\sqrt{a}$$

$$\beta_7=-\frac{\sqrt{2}}{2}(1+i)+\frac{\sqrt{2}}{4}(1-i)a-ia\sqrt{a}$$

$$\beta_8=\ \frac{\sqrt{2}}{2}(1-i)-\frac{\sqrt{2}}{4}(1+i)a+ia\sqrt{a}$$

respectively.

The coefficients of partial fractions in the expressions (4.4) are given in the form of power series of $\sqrt{a}$ by substitution in the relation (4.5) for these β_j. The functions $e^{\beta_j^2 t}$ and $e^{-\beta_j^2 t}$ appearing in the expressions (4.7) are also given in the form of Maclaurin series in the aid of $at\ll 1$, except the parts e^{it} and e^{-it}. By these representations approximated solutions for the early stage will be obtained, they are

$$\eta_I=\alpha\left\{\left(\frac{3}{4}+\frac{it}{2}\right)e^{-it}+\frac{1}{4}e^{it}-1\right\}$$

and

$$\eta_{II}=\alpha\left\{\left(-\frac{i}{4}+\frac{t}{2}\right)e^{-it}+\frac{i}{4}e^{it}\right\} \qquad (4.17)$$

The solutions for other variables at the early stage will be obtained by the same method from the expression (3.19), they are

$$f_I=\alpha\left\{\left(-\frac{1}{2}-it\right)e^{-it}+\frac{1}{2}e^{it}\right\}$$

$$\tilde{u}_I=\alpha 0(\sqrt{a})$$

$$\tilde{v}_I=\alpha 0(a)$$

$$\phi_I=\alpha\left\{\left(\frac{i}{4}-\frac{t}{2}\right)e^{-it}-\frac{i}{4}e^{it}\right\}$$

$$f_{II}=\alpha\left\{\frac{1}{a}e^{-it}+\left(\frac{i}{2}-t\right)e^{-it}+\frac{i}{2}e^{it}-i\right\}$$

$$\tilde{u}_{II}=\alpha\left\{\frac{1}{\sqrt{a}}I(-i)e^{-it}\right\}$$

$$\tilde{v}_{II}=\alpha\{e^{-it}-1\} \text{ and}$$

$$\phi_{II}=\alpha\left\{\left(\frac{3}{4}+\frac{it}{2}\right)e^{-it}+\frac{1}{4}e^{it}-1\right\} \qquad (4.18)$$

where

$$I(x)=\int_0^t \frac{e^{-xt}}{\sqrt{\pi t}}dt$$

$0(a)$ and $0(\sqrt{a})$ are neglected terms of order a and $\sqrt{a}$ respectively. The expressions (4.17) and (4.18) are shown to the order 1, and these are not affected by a change of value of ω^2 in the order a^2.

The changes of the amplitudes of the variables with time expressed by Eqs. (4.12) and (4.14) for the equilibrium state and by Eqs. (4.17) and (4.18) for the early stage are shown schematically in Fig. 2. The evaluation of $D_j(t)_{(j=5,\ldots,8)}$ using the inequality (4.11) shows that as far as $t>10^4/a^2\pi$ is satisfied, the magnitudes of the neglected terms in Eqs. (4.12) and (4.14) consisting of $C_jD_j(t)_{(j=5,\ldots,8)}$, where $C_{j(j=5,\ldots,8)}$ are constants, are smaller than one percent of the magnitudes of the leading terms of Eqs. (4.12) and (4.14), respectively. In case I, $\tilde{u}$ and $\tilde{v}$ are much smaller than η, f and ϕ, but all of them,

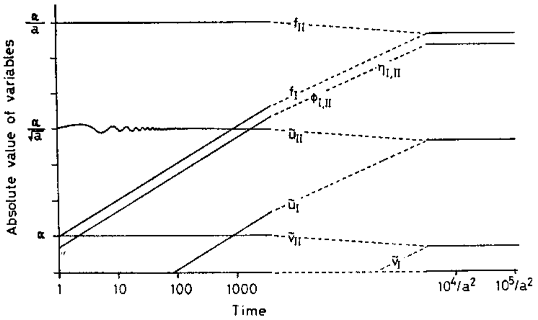

Fig. 2. Schematic behaviors of amplitudes of variables for the early stage and the equilibrium state in the case where $a=10^{-6}$. Dashed lines show the area where the solutions are not obtained in the explicit form.

$\tilde{u}, \tilde{v}, \eta, f$ and ϕ, largely change their magnitudes from the early stage to the equilibrium state respectively. On the other hand, in case II, the variables which largely change their magnitudes after $t=1$ are η and ϕ alone.

Some characteristic features of the water motions in the early stage $at \ll 1$ expressed in Eqs. (4.17) and (4.18) are as follows.

(I) In both cases where a normal stress of progressive wave type is applied on the water surface and a tangential stress of the same type is, the wave amplitudes develop linearly with time because of constancy of the magnitudes of the stresses α, and the time rates of the developments of the wave amplitudes are $\alpha/2$. The relation between the tangential stress and induced wave is in phase, but the normal stress is lagging in space 90° behind the wave induced by the applied normal stress. These are in consistent with the results given by LONGUET-HIGGINS (1968).

(II) In the former case where the water motion is induced only by the normal stress, the water motion is almost completely composed of irrotational motion and it is substantially consistent with a solution obtained under hypothesis of irrotational water motion.

(III) In the latter case where only the tangential stress is acting, the rotational part of the water motion, especially $\tilde{u}$, is dominant before $t=2/\sqrt{a}$ over the irrotational motion ϕ, but $\tilde{u}$ does not grow with time after $t=1$. Therefore the ratio of it to the total water motion decreases with the development of the waves.

In the equilibrium state the water motion is described by Eqs. (4.17) and (4.18) which also seem to be good approximation for $t>10^4/a^2\pi$, Some characteristic features in this stage are as follows.

(I) The amplitudes of variables f, $\tilde{u}$, $\tilde{v}$, ϕ and η have same magnitude in both cases respectively (Fig. 2). The relations of phase between applied stress and induced water motions are same as in the early stage.

(II) Since the ratio of $|\tilde{u}|$ to $|\phi|$ is $\sqrt{a}$ and the ratio of $|\tilde{v}|$ to $|\phi|$ is a in both cases, the motion can be regarded as potential flow approximately notwithstanding that in one case the acting stress is only normal one and in the other the stress is only the tangential.

(III) The relation between an applied stress and the surface elevation becomes approximately

$$\eta_I = \frac{i}{4a} p_{yy}, \text{ for the first case or}$$

$$\eta_{II} = \frac{1}{4a} p_{xy}, \text{ for the second.}$$

These two relations are equivalent to Eqs. (1.1) given by Lamb.

In the same stage $t>10^4/a^2\pi$, the time rate of work per unit distance done by the surface stress is calculated using the leading terms of Eqs. (4.14). It is

$$\overline{p_{yy}v} \simeq \overline{p_{yy}(-\partial\phi/\partial y)} = \alpha^2/8a, \text{ for case I or}$$

$$\overline{p_{xy}u} \simeq \overline{p_{xy}(-\partial\phi/\partial x)} = \alpha^2/8a, \text{ for case II,}$$

where the over bar means space average. The value $\alpha^2/8a$ of the energy flux coming through the irrotational water motion into the water from the air (hereafter it is called ir-flux) is equivalent to nondimensional form of the energy dissipation $2\mu\kappa^3\alpha^2c^2$ (Lamb's notation) in a free wave motion of the wave number κ and the amplitude α which is equal to dimensional value of the amplitude of η_I or η_{II} in the stage (LAMB, 1932, p. 624).

In the early stage, $100\,a<at\ll 1$, it is

$$\overline{p_{yy}v} \simeq \overline{p_{yy}(-\partial\Phi/\partial y)} = \alpha^2 t/4, \text{ for case I or}$$

$$\overline{p_{xy}u} \simeq \overline{p_{xy}(-\partial\Phi/\partial x + \tilde{u})} \simeq \alpha^2 t/4 + \alpha^2/2\sqrt{2}\sqrt{a}, \text{ for case II,}$$

where the value $\alpha^2 t/4$ of ir-flux is acculate in the range $0<at\ll 1$, but the value $\alpha^2/\sqrt{2}\sqrt{a}$

of the energy flux transferred from the air to the water through rotational part of the motions (hereafter it is called ro-flux) can not be so accurate for $t<100$, because the approximation

$$I(-i)=e^{\pi i/4} \tag{4.19}$$

used in the calculation will bring $\pm 5\%$ error on the absolute value and $\pm 3\%$ on the value of argument of the calculated value of the flux for a value of t near 100, and they decrease with the increase of t. In case I, since the rotational motion is negligibly small in comparison with the irrotational motion, energy is supplied to the water only through the irrotational motion. On the other hand in case II $\tilde{u}$ have significant value, so that energy is regarded as to be supplied through both the irrotational and rotational motions. The ir-flux in case I and that in case II have the same value $\alpha^2 t/4$ which is equal to the magnitude necessary to make the wave grow with the growth rate $\alpha/2$. Therefore ro-flux $\alpha^2/2\sqrt{2}\sqrt{a}$ in case II seems to be not able to contribute to the irrotational water motion.

5. Discussion

In the situation same as our case II and by adopting an approximation about a boundary-layer under a action of a fluctuating tangential stress, LONGUET-HIGGINS (1968) obtained the growth rate which is equivalent to ours as a result of two assumptions that $|\tilde{u}|\ll|\phi|$, *i.e.* to neglect ro-flux, and energy dissipation due to irrotational motion is negligibly smaller than ir-flux. It is found from Eqs. (4.18) that his first assumption should be relevant for the latter stage $t\gg 2/\sqrt{a}$, because when t is less than $2/\sqrt{a}$, $|\tilde{u}|$ is larger than $|\phi|$. The earlier the stage, his second one should be the more relevant. Because in the quasi-equilibrium state of the motion, $t>10^4/\pi a^2$, the time rate of energy dissipation due to irrotational motion almost equal ir-flux and the time rate of the energy dissipation is directly proportional to the square of the wave amplitude but ir-flux linearly depend on the amplitude. Under these two assumptions he obtained the growth rate equal to ours. But Eqs. (4.18) give the same value $\alpha/2$ of growth rate even for $t<2/\sqrt{a}$ where his first one cannot be assumed. Moreover if we take into account that in case I the growth rate is the same value $\alpha/2$, though almost all energy flux is ir-flux in the case, we come to a conclusion that ro-flux of case II can not contribute to the growth of waves. Accordingly his first assumption is not essential.

To make sure this, let us calculate horizontally averaged time rate of energy dissipation ε due to the water motion in the early stage. It is expressed in the form

$$\varepsilon=\int_{-\infty}^{0}\overline{a\left\{2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\}}dy$$

In the stage, this can be written by

$$\varepsilon=\int_{-\infty}^{0}\overline{a\left(\frac{\partial \tilde{u}}{\partial y}\right)^2}dy$$

because the conditions

$$\left|\frac{\partial u}{\partial y}\right|\gg\left|\frac{\partial u}{\partial x}\right|=\left|\frac{\partial v}{\partial y}\right|>\left|\frac{\partial v}{\partial x}\right| \text{ and } \left|\frac{\partial \tilde{u}}{\partial y}\right|\gg\left|\frac{\partial \Phi}{\partial y}\right|$$

are satisfied in the stage.

The vertical distribution of $\tilde{u}_{II}$ is obtained by a numerical integration of the expression (3.5) with the next relation

$$f_{II}=\frac{\alpha}{a}e^{-it}$$

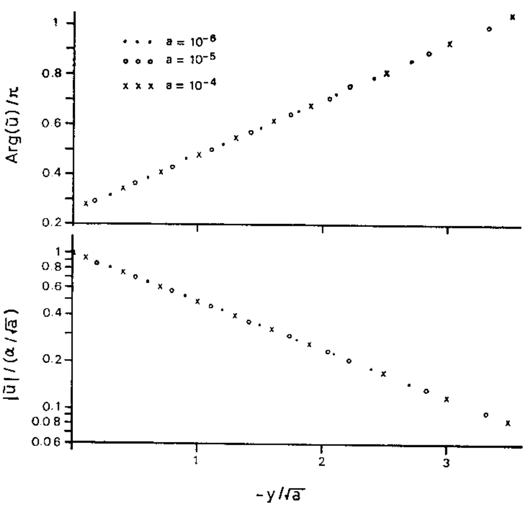

Fig. 3. Mean vertical distribution of $\tilde{u}'$ from $t=80$ to 100 in the thin layer beneath the surface. These are obtained by a numerical integration of Eq. (3.5) with the condition (4.19) at the surface in the early stage.

given by Eqs. (4.18). The absolute value of $\tilde{u}_{II}$ at the surface is not a constant but fluctuate around $|\tilde{u}_{II}|=\alpha/\sqrt{a}$ as shown in Fig. 2, so does the value of $|\tilde{u}_{II}|$ in the water and the fluctuation decreases with time. As the amplitude of the fluctuation is up to several percent of $|\tilde{u}_{II}|$ at time about 100, each value of $|\tilde{u}_{II}|$ and Arg $(\tilde{u}_{II})$ shown in Fig. 3 is averaged from $t=80$ to $t=100$ (about three cycles) to decrease the uncertainty of the value. For the value of time beyond 1,000 the fluctuation becomes negligibly small and there is no noticeable difference between $\tilde{u}_{II}$ at $t\simeq 1{,}000$ and the averaged value of $\tilde{u}_{II}$. Three vertical distributions of $\tilde{u}_{II}$ are calculated for three values of a, 10^{-6}, 10^{-5} and 10^{-4}, and in Fig. 3 log$(|\tilde{u}_{II}|)$ and Arg$(\tilde{u}_{II})$ are plotted against y. For all three values of a, plotted points of each log$(|\tilde{u}_{II}|)$ and Arg$(\tilde{u}_{II})$ make a straight line, so an approximate expression

$$\tilde{u}_{II}=\tilde{u}_{II}(x,t)\,e^{(im+n)y} \tag{5.1}$$

where m and n are constans, seems to be realized at least in the thin layer beneath the surface where $\tilde{u}_{II}$ and $\partial\tilde{u}_{II}/\partial y$ have significant magnitudes. The numerical integrations give the expressions

$$m=0.707/\sqrt{a} \quad\text{and}\quad n=0.708/\sqrt{a} \tag{5.2}$$

for all three values of a. Using the approximation (4.19) for the value of $\tilde{u}_{II}$ at the surface, the expression (5.1) becomes

$$\tilde{u}_{II}=Re\left[\frac{\alpha}{\sqrt{a}}\exp\left\{i\left(x-t+\frac{\pi}{4}+my\right)+ny\right\}\right]$$

This gives

$$\left(\frac{\partial\tilde{u}_{II}}{\partial y}\right)^2=\frac{\alpha^2}{a}(m^2+n^2)\cos^2(x-t+my+\gamma)\,e^{2ny}$$

where γ is a constant, and ε becomes

$$\begin{aligned}\varepsilon&=\frac{\alpha^2}{2}(m^2+n^2)\int_{-\infty}^{0}e^{2ny}\\&\times\{1-\overline{\cos(2x-2t+2\gamma)}\cos(2my)\\&+\overline{\sin(2x-2t+2\gamma)}\sin(2my)\}\,dy\\&=\frac{\alpha^2(m^2+n^2)}{4n}\simeq\alpha^2/2\sqrt{2}\sqrt{a}\end{aligned}$$

by use of the values (5.2). This value of ε is equal to ro-flux. Substantially all of ro-flux is considered to be dissipated by the viscosity of water in the shear layer of $\tilde{u}_{II}$ beneath the surface and can not contribute to the growth of the wave. This result is also expected from the fact that the rotational part of the water motion substantially does not develop after $t=1$ and speedily comes into quasi-equilibrium state under the action of the fluctuating tangential stress as shown in Fig. 2. Therefore it is confined that the first assumption of LONGUET-HIGGINS is needless.

The important points of our results are as follows. A fluctuating tangential stress and a normal one have equivalent effect on the generation and growth of waves except that the fluctuating tangential stress induces a much more large rotational water motion than the normal one does in the early stage and the phase relation between the water motion and the fluctuating tangential stress is different from that between the motion and the normal one. In other words, the fact that the atmospheric pressure is heigher on the wind-ward side of wave crest than on the lee side has equivalent effect on waves to the fact that tangential stress is larger on the crest of waves than on the troughs. Both of these two facts have the effect to make water masses at the crests move faster *i.e.* there are excesses of momentum transfer from wind to water at the crests of wind waves, and it is not a question whether the momentum transfer is beared by tangential stress or normal one. If we take the problem in the sence of the response of water surface elevation to a fluctuation of time rate of momentum transfer from wind to water, we need not distinguish a tangential stress from a normal one in the problems of wind wave generation and growth.

Let us think about the connection between our results and experiments or observations. In general, when the viscosity of a liquid is small, a rotational motion of the liquid cannot be brought into existence with significant magnitude without an action of a tangential stress, this is consistent with our results shown above. Therefore if a rotational water motion having correlation to waves were observed, the observation itself is a proof of existence of a tangential stress acting correlately on waves. In the early stage the fluctuating tangential stress is substantially $a\partial\tilde{u}/\partial y$ at the surface, so that the existence of it will be confirmed and the magnitude of it will be estimated by measurements of horizontal rotational velocity $\tilde{u}$ of water at or beneath the surface.

By the measurements of particle velocity of water in a wind wave tunnel, OKUDA *et al.* (1976) showed an existence of a viscous skin flow in the wind-ward sides of the wind wave crests. OKUDA *et al.* (1977) reported an existence of skin friction whose intensity varies greatly along the surface of wind waves as a function of the phase angle, and they also reported that the skin friction is so large that it can be considered to bear most of the shearing stress of wind.

These experiments together with our study not only confirm us the existence of a fluctuating tangential stress correlated to a water surface elevation of the actual wind waves but also present a question whether the hypothesis of irrotational water motion, which proved great success in the interpretation of the propagation phenomena of water waves, is applicable to all problems concerning water waves, especially to the problems of generation of wind waves.

Eqs. (4.18) show that $|\phi|=|\tilde{u}|$ when $t=2/\sqrt{a}$ and $|\tilde{u}|>|\phi|$ before $t=2/\sqrt{a}$. a is estimated as 2.6×10^{-6} for waves whose period is one second and for waves whose period is ten seconds a becomes very small value 2.6×10^{-9}. On the sea surface there are many wave components whose period is smaller than one or ten seconds and also exist many irregular movements of water whose characteristic lengths are smaller than the wave length of water wave whose period is one or ten seconds. Therefore effective value of a is much larger than above mensioned values of a obtained by use of the viscosity of water $\nu=10^{-2}$ ($cm^2\,s^{-1}$). Yanagino obtained the values of eddy diffusion coefficient by scattering processes of semi-neutral particles beneath wind waves in a wind wave tunnel (personal communication). His values are 20 $cm^2\,s^{-1}$ at the depth 1.5~3 cm and 3.8 $cm^2\,s^{-1}$ at 3~4 cm below the surface in the wind whose velocity is 6.2 m s^{-1}. Using his minimum value 3.8 $cm^2\,s^{-1}$, a and T are reevaluated. For a wave train whose period is one second,

$a=9.8\cdot10^{-4}$, $T=64$, $T'=10$ seconds and

$a=9.8\cdot10^{-7}$, $T=2.0\cdot10^3$, $T'=54$ minutes for a wave train whose period is ten seconds, where T is time when $|\tilde{u}|=|\phi|$, and $T'=T/\sigma$. These values of T' may be over estimate for sea waves, because the eddy diffusion coefficient used in the estimation of T is munimum value and more over the value is obtained by experiments in a wind wave tunnel not in the sea. Still, these values of T' are the beginings of developments of wave components whose periods are one and ten seconds respectively. In addition to that, Eqs. (5.1) and (5.2) show that the rotational motion $\tilde{u}$ consentrates in the thin layer beneath the surface unlike the irrotational motion Φ. These make it difficult to detect the rotational motion in the sea. And the measurements that a water motion due to wind waves in the sea is practically irrotational, cannot be a precise proof that the waves are generated and developed by an action of normal stress alone.

The main points of this study have already stated, in addition to those let us give wings to our imagination and think about wave breaking. The most familiar sights appear on the sea surface in moderate breeze or in stronger wind, are white horses. They are beautiful in the sunshine and threatening under dark rain clouds. We have not been successful in the quantitative description of this very familiar phenomenon of wave breaking, so we would like to make a trial of qualitative analysis of it. For the simplicity of our analysis, three stages of wave breaking are assumed. The first; a crest of wave is sharp but still water mass at the crest is moving as a part of wave. The second; the small mass of water which was at the crest in the first, completely flies out into the air. The third; the water mass has come down to the wave (Fig. 4).

(I) The wave energy of the second is smaller than that of the first, because the water mass in the air has potential and kinetic energies given from the wave motion of the first. The water mass has momentum too and its direction is that of the wind.

(II) When the water mass falls on the wave surface, not all but most of the energy of the

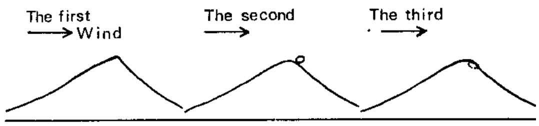

Fig. 4. Typical three stages of wave breaking.

water mass becomes the energy of turbulent motion, but the momentum of it is transfered to adjacent water of it. At the place where the water mass fall, the momentum is locally transfered to water near the falling point within small reach of the crest. After all the transition from the second to the third means the existence of excess of momentum transfer near the crest. This phenomenon occurs intermittently both in time and in space on the sea surface. In a word, though the momentum transfer is intermittent, there are excess of the momentum transfer at the crests which are propagating with the characteristic velocity of water waves. Of course the transfered momentum makes current but our results mensioned above show that a part of it can be the momentum of waves again.

The transition from the first to the second is equivalent to the outflow of energy from waves. And the transition from the second to the third is inflow of the energy to the current and waves. As the form of wind waves is not sinusoidal, they are considered to be sum of many component waves. The flying water mass in the second receives the energy from wave motion and the amount that each component wave contribute to the energy depends on the wave form near the crest. The amount of energy that each wave component receives the energy when the water mass fall, depends on the distribution of the area on the sea surface where wave breakings occur. So that it is likely that the relative magnitudes of the outflow energy from component waves are different from those of the inflow energy. These considerations show that wave breaking can be one of wave-wave interactions taking with energy dissipation, and this wave-wave interaction clearly different from that studied by HASSELMANN (1962). To separate clearly three stages of wave greaking, we assumed that the water mass completely fly up in the air but even in case where the water mass does not completely fly up, the same kind of mechanism of wave-wave interaction can be expected.

As early as in 1964, MITSUYASU and KIMURA experimentally suggested the existence of wave-wave interaction accompanied wave breaking by the measurements of wave spectrum of wind waves in decay area in a wind wave tunnel (MITSUYASU and KIMURA, 1964).

Consequently, it is shown that besides an atmospheric pressure fluctuation, a fluctuating tangential stress can make wind waves grow, and besides week wave-wave interaction, there can be another mechanism of wave-wave interaction. Therefore it is very dangerous to close the right hand side of the energy balance equation

$$\frac{DF}{Dt}=S_1+S_2+S_3+\ldots.$$

where F is the spectrum of wind waves and $S_1, S_2, S_3, \ldots$ are source functions representing net transfer of energy to the spectrum, in the form of including nothing but terms whose quantitative description have already given.

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接線応力で誘起される水の波について

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要旨: 線形の理論の適用範囲において，水面における接線応力と法線応力の波におよぼす効果が水の運動の非回転性を仮定しないで議論された．

空間的には正弦曲線的に，時間的には任意に変動する接線および法線応力のもとで引き起こされた初期に静止していた水の運動の形式的な解が得られた．

波数と周波数が水の波の分散関係を満たす進行波の形を持つ接線応力は，同じ形の法線応力と波の成長に関しては，波とそれぞれの応力との位相関係を除けば同じ効果を持つこと，また，この時の波の成長率は水の粘性係数にはほとんど依存しないことが示された．

接線応力の変動成分の作用のみで水の運動が起こされた場合でも，水の運動の回転成分は運動全体に比べると波の発達の初期にしか卓越した大きさを持たない．よって発達した風波に伴う水の運動が非回転に見えるからといって接線応力の作用を無視するのは合理的でない．

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