NOTES AND CORRESPONDENCE

On the Transversal Instability of a Coupled Wind–Wave System at the Initial Stage of Wave Growth

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ABSTRACT

The evolution of the growing surface waves having a nonuniform transversal structure is discussed using the semiempirical equation for wind velocity profile in two dimensions and the expression suggested by Plant and Wright for the initial growth rate of waves. Positive feedback has been found in the coupled wind–wave system, leading to an algebraic growth of a weak transversal modulation of the wave field.

1. Introduction

A number of field observations carried out recently confirm the essential impact of the sea state on the characteristics of the atmospheric boundary layer and therefore on the momentum flux from wind to waves. As shown by Janssen (1991), the quasi-linear theory based on the Miles theory of wind–wave generation being implemented in the wave prediction model WAM (WAMDI Group 1988) describes satisfactorily this interaction and is in reasonable agreement with observations.

In the present paper we would like to point out an interesting peculiarity that arises due to such coupling at the initial stage of wave growth. It follows from the quasi-linear theory that the effective roughness parameter together with a drag coefficient is a function of the ratio of wave-induced stress to the total stress near the water surface. For a young wind sea this ratio is of the order of unity giving rise to a considerable deformation of the wind profile. Under some circumstances positive feedback can arise in this coupled wind–wave system that leads to the amplification of the growth rate with wave amplitude. If the wave field has initially an inhomogeneous structure, the magnitude of the inhomogeneity will grow in time.

As a matter of fact, however, the applicability of the quasi-linear approach is still doubtful at an early stage of wave evolution. On the other hand, there are some other empirical as well as analytical models that give a satisfactory description of initial wave growth. Among them we can refer to the paper by Plant and Wright (1977) where a simple empirical formula for growth rate was suggested on the basis of observations. Concerning the feedback effect of waves on wind, the quasi-linear theory (Fabrikant 1976; Janssen 1982, 1989, 1991) is directly applicable only to a long wave impact of waves on wind. The effect of short waves was modeled in that theory by a simple Charnock relation for the roughness parameter that is valid when short waves are already in a quasi-stationary state under rather strong wind. This approach, evidently, fails at the stage of initial growth of short waves. Instead we assume that the roughness parameter depends explicitly on the amplitude of short waves.[1]

In the present paper we are going to discuss the wind–wave interaction at the initial stage of wave growth for the case of spatially (in fact, transversally to wind direction) inhomogeneous wave field. We will find the impact of the nonuniform wave field (treated as nonuniform roughness parameter) on the wind velocity profile and also consider the evolution of the transversally nonuniform wave field making use of the semiempirical relation between local friction velocity and the growth rate of waves (Plant and Wright 1977). We will find a positive feedback in the system in the certain range of scales of transversal modulation. Therefore, the spatial nonuniformity of the wave field, if it is present at the beginning, would grow in time.

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[1] Phillips (1977) employed this approach in order to confirm the Charnock relation.

In section 2 we consider the deformation of wind profile structure above the surface with transversally nonuniform roughness. In section 3 the conditions for a positive feedback and the onset of transversal instability are obtained and some quantitative estimates are made and discussed. Section 4 gives a summary of conclusions.

2. Wind profile structure above a surface with a transversally nonuniform roughness

Before treating the coupled wind–wave problem, let us consider the structure of the wind profile blowing in the direction x over the surface with the transversally modulated roughness parameter $z_o(y)$. Within a traditional semiempirical approach (see, e.g., Monin and Yaglom 1971) neglecting the pressure gradient and molecular viscosity, and assuming that the wind flow is homogeneous along the x direction, the equation for the mean flow velocity $U(z, y)$ in a turbulent boundary layer over a rough surface takes the form

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial z}\tau_z^{\text{turb}} + \frac{\partial}{\partial y}\tau_y^{\text{turb}}, \qquad (1)$$

where $\tau_z^{\text{turb}} = \langle -u'w' \rangle$ and $\tau_y^{\text{turb}} = \langle -u'v' \rangle$ are the vertical and horizontal turbulent stresses and u', v', w' are x, y, and z components of fluctuation velocity, respectively. For turbulent stresses we will employ the mixing length hypothesis; namely,

$$\tau_z^{\text{turb}} = K\frac{\partial U}{\partial z}, \qquad (2)$$

$$\tau_y^{\text{turb}} = K\frac{\partial U}{\partial y}, \qquad (3)$$

where $K = l^2|\partial U/\partial z|$ is the turbulent viscosity coefficient and the mixing length l is given by $l = \kappa z$ ($\kappa \approx 0.41$ is the von Kármán constant). Here, we implicitly supposed that the vertical gradient of the wind velocity exceeds greatly the horizontal one and therefore provides the main contribution to the turbulent mixing that is valid for a large enough horizontal scale of transversal nonuniformity.

In the stationary state Eq. (1) takes the form

$$\frac{\partial}{\partial z}\left(z^2\left|\frac{\partial U}{\partial z}\right|\frac{\partial U}{\partial z}\right) + \frac{\partial}{\partial y}\left(z^2\left|\frac{\partial U}{\partial z}\right|\frac{\partial U}{\partial y}\right) = 0, \qquad (4)$$

and we impose the boundary condition, $U(z_o) = 0$. Assume first that the roughness parameter has a weak sinusoidal modulation:

$$z_o(y) = \bar{z}_o + z_1 \cos k_y y, \quad z_1 \ll \bar{z}_o. \qquad (5)$$

Since weak modulation of z_o leads to a weakly modulated wind profile,

$$U(z, y) = U_o(z) + u_1(z)\cos k_y y, \quad u_1 \ll U_o, \qquad (6)$$

one can linearize Eq. (4) with respect to $U_o(z)$, which, being the solution of (4), is given by the standard expression,

$$U_o(z) = \frac{u_{*o}}{\kappa}\ln\frac{z}{z_o}, \qquad (7)$$

where u_{*o} is the friction velocity of undisturbed wind flow. The substitution of (7) and linearization yield the following simple equation for velocity $u_1(z)$:

$$2\frac{d}{dz}\left(z\frac{du_1}{dz}\right) - k_y^2 z u_1 = 0. \qquad (8)$$

The solution of (8) decreasing as z tends to infinity is a modified Bessel function of zeroth order:

$$u_1 = U_1 K_o\left(\frac{k_y z}{\sqrt{2}}\right), \qquad (9)$$

where the amplitude of velocity variations U_1 can be easily found from the linearized boundary condition $u_1 + U_z'(\bar{z}_o)z_1 = 0$. As seen from (9), there are two different logarithmic "subprofiles" in the total profile of wind velocity. At a large height ($k_y z \gg 1$) the air flow does not "feel" the weak nonuniformity of the surface. At a small height ($k_y z \ll 1$) the modified Bessel function $K_o(x)$ has a logarithmic asymptotic behavior that allows one to write down the full velocity $U(z, y)$ in a logarithmic form:

$$U(z, y) \approx \frac{u_{*o}}{\kappa}\ln\frac{z}{\bar{z}_o} + U_1\ln\left(\frac{k_y z}{\sqrt{2}}\right)\cos k_y y = \frac{u_*}{\kappa}\ln\frac{z}{z_o}, \qquad (10)$$

where

$$u_* = u_{*o}\left\{1 - \frac{z_1}{\bar{z}_o}\cos k_y y\left[\ln\left(\frac{k_y \bar{z}_o}{2\sqrt{2}}\right)\right]^{-1}\right\}. \qquad (11)$$

Thus, at small z we have a logarithmic wind profile with a transversally nonuniform roughness parameter $z_o(y)$ and the corresponding friction velocity $u_*(y)$. Note that using the formulas (10), (11) one can easily treat the case of arbitrary but weak transversal distribution of the roughness by means of its Fourier decomposition.

3. Positive feedback and transversal instability

Plant and Wright (1977) suggested an empirical formula for the initial growth rate of the amplitude of surface gravity–capillary waves,

$$\gamma = k^2\left(\frac{\delta u_*^2}{\omega} - 2\nu_w\right), \qquad (12)$$

where ω and k are the frequency and the wavenumber of a surface wave related by the dispersion relation,

$$\omega = \sqrt{gk + Tk^3}, \qquad (13)$$

g is the gravity acceleration, T is the surface tension coefficient, ν_w is the kinematic viscosity of water, and $\delta = 0.1/2\pi$. It is seen from (12) that the growth rate is a sensitive function of the friction velocity u_*, and even its small deviations due to inhomogeneity of underlying water surface affect the growth rate of waves. Note that at a large enough scale of transversal inhomogeneity its effect on wave growth can simply be taken into account parametrically; therefore, we can employ a one-dimensional formula (12) with variable $u_*(y)$ in our two-dimensional problem.

Here the question arises concerning the relation between the roughness parameter z_o and the amplitude H of surface waves. A natural guess is that if the wave height is less than the roughness parameter for a wind profile over slick surface $z_{oo} \approx 0.1\nu_a/u_*$ (ν_a is the kinematic viscosity of air), then $z_o = z_{oo}$. On the other hand, if $H > z_{oo}$, for the initial stage of wave growth one can adopt, following Phillips (1977), $z_o \propto H$. Taking these arguments into account, one can write down the equation for the temporal evolution of the modulated roughness parameter

$$\frac{\partial(\bar{z}_o + z_1 \cos k_y y)}{\partial t} = \gamma(\bar{z}_o + z_1 \cos k_y y), \quad (14)$$

where the growth rate γ is described by (12) for the optimal wavenumber k_0 and frequency ω_0 for which γ reaches its maximum value:

$$\gamma = \begin{cases} k_0^2\left[\dfrac{\delta u_{*o}^2}{\omega_0} - 2\nu_w\right] \equiv \gamma_o, & \text{for} \quad z_o < 0.1\nu_a/u_{*o} \\ k_0^2\left[\dfrac{\delta u_{*o}^2}{\omega_0}\left\{1 - \dfrac{z_1}{\bar{z}_o}\cos k_y y\left[\ln\left(\dfrac{k_y \bar{z}_o}{2\sqrt{2}}\right)\right]^{-1}\right\}^2 - 2\nu_w\right], & \text{for} \quad z_o > 0.1\nu_a/u_{*o}. \end{cases} \quad (15)$$

It is evident from (15) that $\partial\gamma/\partial z_1 > 0$ for $z_o > 0.1\nu_a/u_{*o}$, and the positive feedback is larger, the greater k_y; that is, the less the transversal scale of nonuniformity.

To describe quantitatively the development of transversal modulation of roughness due to the effect under consideration, one can linearize (14) with respect to $\bar{z}_o$. Then the two following equations are found:

$$\frac{\partial \bar{z}_o}{\partial t} = \gamma_o \bar{z}_o, \quad (16)$$

$$\frac{\partial z_1}{\partial t} = \left(\frac{\partial\gamma}{\partial z_1} + \gamma_o\right) z_1. \quad (17)$$

Solving Eqs. (16) and (17) we obtain

$$\bar{z}_o = Z_o \exp(\gamma_o t) \quad (18)$$

$$z_1 = Z_1 \exp(\gamma_o t)\left[\frac{C}{C - \gamma_o t}\right]^{2A/\gamma_o}, \quad (19)$$

where $A = \delta k_0^2 u_{*o}^2/\omega_0$, $C = -\ln(k_y Z_o/2\sqrt{2})$, and $Z_{o,1}$ are initial values of $\bar{z}_o$ and z_1. Let us introduce the contrast $K = z_1/\bar{z}_o$ characterizing the magnitude of transversal modulation. When the effect of waves on the wind profile is neglected the contrast K remains constant as waves grow. However, formulas (18) and (19) show that the deformation of the wind profile leads to an algebraic (explosive) growth of the contrast:

$$K = K_o\left[\frac{C}{C - \gamma_o t}\right]^{2A/\gamma_o}, \quad (20)$$

where $K_o = Z_1/Z_o$ is the initial value of contrast. The contrast K goes to infinity in the characteristic time

$$T = -\ln(k_y Z_o/2\sqrt{2})/\gamma_o. \quad (21)$$

Of course, the validity of the expression (20), in fact, fails before the contrast goes to infinity and even to unity, but still (21) gives the estimation of the characteristic time of the development of transversal structure due to the wind profile deformation by waves. At the initial stage of evolution the contrast K grows linearly in time

$$K = K_o(1 + \Gamma t). \quad (22)$$

The growth rate

$$\Gamma = -\frac{2\delta k_0^2 u_{*o}^2}{\omega_0 \ln(k_y Z_o/2\sqrt{2})} \quad (23)$$

increases logarithmically with the transversal wavenumber k_y. It is interesting to note that the growth rate of contrast Γ also increases logarithmically with the initial roughness parameter Z_o. We have to choose $Z_o = z_{oo}$ because at lower values of Z_o, the wind profile does not "feel" nonuniformity of the wave field.

As we have just pointed out, the growth rate of the transversal instability increases with the wavenumber k_y. However, physically it seems obvious that there should be a mechanism suppressing the transversal instability at short scales. What effect could be responsible for this short-scale cutoff of the transversal instability? We can propose one possible mechanism. In fact, a periodic transversal modulation of the plane monochromatic wave indicates the presence of two additional components in the spatial Fourier spectrum of the wave field. These components correspond to a pair of oblique waves propagating at the angles $\phi_\pm = \pm\arctan(k_y/k_0)$ to the wind direction. It is well known that the growth rate for waves propagating at some angle to a wind direction is determined by the solution of the one-dimensional problem with effective wind velocity profile $U(z)\cos\phi_\pm$ (or, for logarithmic velocity profile, with effective friction velocity $u_*\cos\phi_\pm$). Therefore, the growth rate of z_1 depends on k_y not only due to the positive feedback in the coupled wind–wave system discussed above [the first term in the right-hand side of (17)] but also due to the decay of the wind input for oblique propagating wave components. At $k_y/k_0 \ll 1$ it leads to the following expression for the growth rate of the contrast

$$\Gamma = -\frac{2\delta k_0^2 u_*^2}{\omega}\left(\frac{1}{\ln(k_y z_{oo}/2\sqrt{2}} - 2\frac{k_y^2}{k^2}\right). \quad (24)$$

From (24) one can find the "most unstable" wavenumber for the transversal instability. With the logarithmic accuracy it reads

$$k_y = -\frac{k}{\ln(kz_{oo}/2\sqrt{2})}. \quad (25)$$

Let us present some estimates. Take $u_* = 0.15$ m s^{-1}, $g = 9.81$ m s^{-2}, $T = 74$ cm^3 s^{-2}, $\nu_a = 1.2 \times 10^{-5}$ m^2 s^{-1}, and $\nu_w = 10^{-6}$ m^2 s^{-1}. The wavenumber of maximally growing gravity–capillary waves determined by (12), (13) is $k_0 = 346.5$ rad m^{-1} and the corresponding growth rate $\gamma = 0.26$ s^{-1}. The undisturbed roughness parameter is $z_{oo} = 6.7 \times 10^{-6}$ m. The optimal wavenumber of transversal modulation is $k_y = 17.1$ rad m^{-1} [an approximate formula (25) gives the value of 24.4 rad m^{-1}], and the growth rate $\Gamma = 0.1$ s^{-1}. Therefore, strips 30 cm wide arise and the contrast is doubled each 10 s. Note, that, in accordance with (25), the transversal scale depends on the wavenumber of surface waves and, hence, it grows with the wave age. Moreover, even the ratio of transversal scale to the wavelength grows logarithmically with the wave age.

4. Conclusions

In this paper we have considered the possibility of the surface wave field to form a transversally inhomogeneous structure at the initial stage of wave growth. Taking the wave height as a roughness parameter one can relate the wind profile to a structure of surface wave field. Using a simple mixing-length hypothesis and assuming the fixed value of the friction velocity at a great height, we calculated the wind profile over the transversally nonuniform water surface. The wind profile deviates from the standard logarithmic profile at small heights more, the less the scale of transversal nonuniformity. Then we employed the empirical formula of Plant and Wright (1977) relating the growth rate of short gravity–capillary waves to a friction velocity and obtained the equation for the evolution of transversal modulation.

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