

Transformation of wave profile over a sloping bottom*

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Abstract: Changes of profiles of a solitary wave and a sine wave over a sloping beach were considered theoretically, solving the linear equation by RIEMANN's method. Validity of a linear equation is confined to the case when the waves do not reach so shallow place as wave height and wave steepness attain the same order of quantity compared with the water depth and the bottom slope. In the case of a sine wave, the decrease in the wave length is great but the wave height shows little change compared with the case of a solitary wave.

1. Proposition of the problem.

The equation of the non-linear shallow water wave theory has been derived by J. J. STOKER, assuming that the horizontal velocity of water particles is independent of vertical coordinate,† and the hydrostatical pressure condition is applied. The equation of motion is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial \eta}{\partial x}, \quad (1.1)$$

where η means the elevation of water surface from the mean level, and u the x -component of particle velocity, g the acceleration of gravity. The density of water is assumed unity.

Now, let us consider water waves progressing shoreward (positive direction of x axis) on a sloping beach. The surface and the bottom condition yield the formula,

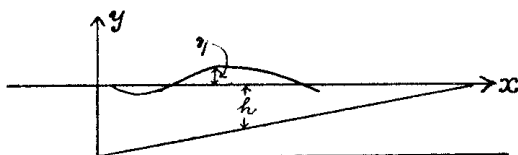


Fig. 1. x -axis agrees with the mean water level, η means surface elevation from this level.

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† According to the actual observation of the breakers, this assumption is not always valid. The horizontal velocity of water particles decreases to the bottom. (IVERSEN, 1952).

$$\frac{\partial}{\partial x}[u(\eta+h)] = -\frac{\partial \eta}{\partial t} \quad (1.2)$$

Actual laboratory experiments of shallow water waves (Rep. Trans. Tech. Res. Inst.) show the validity of neglecting the term uu_x compared with u_t and $g\eta_x$ (u_x means $\frac{\partial u}{\partial x}$ etc.), for example, the maximum particle velocity u is about 25 cm/sec, and u_x is 0.1 c.g.s., while u_t is 60 cm/sec, and $g\eta_x$ in 39 c.g.s.⁽³⁾ Then, from (1.1)

$$u_t = -g\eta_x \quad (1.3)$$

If we consider a wave whose surface elevation η is small compared with depth h and wave steepness η_x is also small compared with beach slope h_x , we can get from (1.2) and (1.3)

$$gh \frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \eta}{\partial t^2} + gh_x \frac{\partial \eta}{\partial x} = 0 \quad (1.4)$$

By the shallow water wave theory on a sloping beach, we cannot derive a linearized equation, unless the wave steepness and η are small compared with the bottom slope and the water depth respectively. In the case of a tank observation (Rep. Trans. Tech. Res. Inst.) carried out by Transportation Technical Research Institute, the mean uniform depth of water is 30 cm, and the surface elevation due to wave motion is only a few centimeter, and η_x (wave steepness) is about 0.04 while h_x is 1/10~1/15: we are allowed to use the linearized equation (1.4) so far as water over a sloping beach is not so shallow compared with wave elevation.††

†† η_x sometimes becomes greater than beach slope, especially near the breaking point.

2. Solution of the fundamental equation.

If an initial wave profile is given, transformation of the wave profile at time t over a sloping beach can be obtained from (1.4), using the RIEMANN's integration method of a hyperbolic partial equation.

Because the equation (1.4) is hyperbolic, the characteristic equation is given by

$$gh\varphi_x^2 - \varphi_t^2 = 0, \quad (2.1)$$

or,

$$\left. \begin{aligned} \sqrt{gh} \psi_x &= \psi_t, \\ -\sqrt{gh} \varphi_x &= \varphi_t. \end{aligned} \right\} \quad (2.2)$$

Introducing new independent variables,

$$\left. \begin{aligned} \xi &= \varphi(x, t), \\ \tau &= \psi(x, t) \end{aligned} \right\} \quad (2.3)$$

(1.4) is transformed into

$$\begin{aligned} 2(gh\varphi_x\psi_x - \varphi_t\psi_t)\eta_{\xi\tau} + (gh\varphi_{xx} - \varphi_{tt})\eta_{\xi} \\ + (gh\psi_{xx} - \psi_{tt})\eta_{\tau} + gh_x(\varphi_x\eta_{\xi} + \psi_x\eta_{\tau}) = 0 \end{aligned} \quad (2.4)$$

From (2.2) we can get

$$\frac{\partial \psi}{\partial x} \sqrt{gh} = \frac{\partial \psi}{\partial t}, \quad -\frac{\partial \varphi}{\partial x} \sqrt{gh} = \frac{\partial \varphi}{\partial t}$$

therefore,

$$\left. \begin{aligned} \xi = \varphi &= \frac{2}{\alpha} \sqrt{\frac{h}{g}} - t, \\ \tau = \psi &= \frac{2}{\alpha} \sqrt{\frac{h}{g}} + t. \end{aligned} \right\} \quad (2.5)$$

Then,

$$\varphi_x = -\frac{1}{\sqrt{gh}}, \quad \psi_x = -\frac{1}{\sqrt{gh}}, \quad \psi_t = 1,$$

$$\varphi_t = -1, \quad \varphi_{xx} = -\frac{\alpha}{2\sqrt{gh^3}},$$

$$\psi_{xx} = -\frac{\alpha}{2\sqrt{gh^3}},$$

where $h = h_0 - \alpha x$. Putting these relations into (2.4) we can get

$$\eta_{\xi\tau} + \frac{\alpha}{8} \sqrt{\frac{g}{h}} (\eta_{\xi} + \eta_{\tau}) = 0, \quad (2.6)$$

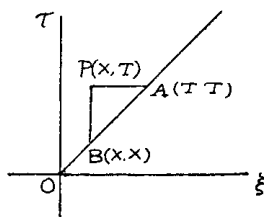


Fig. 2. $\xi\tau$ -diagram.

or,

$$\eta_{\xi\tau} + \frac{1}{2(\xi + \tau)} (\eta_{\xi} + \eta_{\tau}) = 0 \quad (2.7)$$

because from (2.5), $\xi + \tau = \frac{4}{\alpha} \sqrt{\frac{h}{g}}$.

If we denote a RIEMANN's function of equation (2.7) by V , the adjoint equation⁽⁴⁾ of (2.7) is given by

$$\frac{\partial^2 V}{\partial \xi \partial \tau} - \frac{1}{2(\xi + \tau)} \left(\frac{\partial V}{\partial \xi} + \frac{\partial V}{\partial \tau} \right) + \frac{V}{(\xi + \tau)^2} = 0. \quad (2.8)$$

Putting

$$\begin{aligned} v &= \frac{(X + \tau)^{\frac{1}{2}} (T + \xi)^{\frac{1}{2}}}{\xi + \tau} V, \\ \zeta &= \frac{(\xi - X)(T - \tau)}{(X + \tau)(T + \xi)} \end{aligned}$$

(2.8) is transformed into

$$\zeta(1 + \zeta) \frac{d^2 v}{d\zeta^2} + (1 + 2\zeta) \frac{dv}{d\zeta} + \frac{v}{4} = 0 \quad (2.9)$$

where X and T mean values of ξ , η at P . In the $\xi\tau$ -diagram OBA is an initial line and P represents a point (X, T) to any value of x and t .

Solving (2.9)

$$v = F\left(\frac{1}{2}, \frac{1}{2}; 1; -\zeta\right) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \zeta \sin^2 \phi}}$$

where $F\left(\frac{1}{2}, \frac{1}{2}; 1; -\zeta\right)$ means a hypergeometric function. Therefore, the RIEMANN's function is given by

$$\begin{aligned} V(X, T; \xi, \tau) &= \frac{\xi + \tau}{(\tau + X)^{\frac{1}{2}} (T + \xi)^{\frac{1}{2}}} \\ &\times F\left[\frac{1}{2}, \frac{1}{2}; 1; -\frac{(\xi - X)(T - \tau)}{(X + \tau)(T + \xi)}\right] \end{aligned} \quad (2.10)$$

Thus, η at P can be solved as an expression,

$$\begin{aligned} 2\eta(X, T) &= V(X, T; T, T) f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} T^2\right) \\ &+ V(X, T; X, X) f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} X^2\right) \\ &+ \int_X^T V(X, T; \xi, \xi) f_2\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} \xi^2\right) d\xi \\ &+ \int_X^T \left(\frac{\partial V}{\partial \xi} - \frac{\partial V}{\partial \tau} \right) f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} \xi^2\right) d\xi, \end{aligned} \quad (2.11)$$

where the initial condition is expressed by

$$\eta = f_1(x), \quad \frac{\partial \eta}{\partial t} = f_2(x) \quad \text{at } t=0, \quad \text{and}$$

$$X = \frac{2}{\alpha} \sqrt{h_0 - \alpha x} - t, \quad T = \frac{2}{\alpha} \sqrt{h_0 - \alpha x} + t.$$

The value of V expressed by (2.10) being inserted in this equation, we get

$$\begin{aligned} 2\eta(X, T) = & \sqrt{\frac{2T}{T+X}} f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} T^2\right) \\ & + \sqrt{\frac{2X}{T+X}} f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} X^2\right) \\ & + \int_x^T \frac{2\xi}{(\xi+X)^{\frac{1}{2}}(\xi+T)^{\frac{1}{2}}} F\left(\frac{1}{2}, \frac{1}{2}; 1; -\zeta_0\right) \\ & \times f_2\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} \xi^2\right) d\xi + \int_x^T \frac{\xi(T-X)}{(\xi+X)^{\frac{3}{2}}(\xi+T)^{\frac{3}{2}}} \\ & \times F\left(\frac{1}{2}, \frac{1}{2}; 1; -\zeta_0\right) f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} \xi^2\right) d\xi \\ & + \int_x^T \frac{\xi^2(T^2-X^2)}{(\xi+X)^{\frac{5}{2}}(\xi+T)^{\frac{5}{2}}} F\left(\frac{3}{2}, \frac{3}{2}; 2; -\zeta_0\right) \\ & \times f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} \xi^2\right) d\xi \end{aligned} \quad (2.12)$$

where

$$\zeta_0 = \frac{(\xi-X)(T-\xi)}{(\xi+X)(T+\xi)} < 1$$

If $\alpha=0$ (non sloping beach), $\frac{h_0}{\alpha} - \frac{g\alpha}{4} \left(\frac{T^2}{X^2}\right)$

$$= x - \frac{g\alpha}{4} t^2 \mp t\sqrt{g(h_0 - \alpha x)} \quad \text{becomes } x \mp \sqrt{gh_0}t,$$

so $f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} T^2\right)$ and $f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} X^2\right)$ show

progressing and retreating waves to the positive direction of x axis respectively.*

Now, we consider a progressing wave only, namely the wave pushes from the offing toward the shore over sloping bottom, whose energy propagates entirely with the individual wave velocity not by group velocity. The term corresponding to the progressing

$$\text{wave is } \sqrt{\frac{2T}{T+X}} f_1\left(\frac{h_0}{\alpha} - \frac{g\alpha}{4} T^2\right) \quad \text{in (2.12),}$$

therefore the elevation of the water surface at any time and position due to the progress ing wave can be represented as follows:

* When $\alpha=0$, sum of the first and second term of right hand members of expression (2.12) becomes $f_1(x - \sqrt{gh_0}t) + f_1(x + \sqrt{gh_0}t)$, and the third term becomes $\frac{1}{\sqrt{gh_0}} \int_{x-\sqrt{gh_0}t}^{x+\sqrt{gh_0}t} f_2(\xi) d\xi$.

Remaining terms reduce to 0.

$$\begin{aligned} \eta(X_1, T_1) = & \sqrt{\frac{2T_1}{X_1+T_1}} f_1\left[\frac{h_0}{\alpha} \left(1 - \frac{T_1^2}{4}\right)\right] \\ & + \frac{1}{2\alpha} \sqrt{\frac{h_0}{g}} \int_{x_1}^{T_1} \frac{2\xi_1}{(\xi_1+X_1)^{\frac{1}{2}}(\xi_1+T_1)^{\frac{1}{2}}} \\ & \times F\left(\frac{1}{2}, \frac{1}{2}; 1; -\zeta_0\right) f_2\left[\frac{h_0}{\alpha} \left(1 - \frac{\xi_1^2}{4}\right)\right] d\xi_1 \\ & + \frac{1}{2} (T_1^2 - X_1^2) \int_{x_1}^{T_1} \frac{\xi_1^2}{(\xi_1+X_1)^{\frac{3}{2}}(\xi_1+T_1)^{\frac{3}{2}}} \\ & \times F\left(\frac{3}{2}, \frac{3}{2}; 2; -\zeta_0\right) f_1\left[\frac{h_0}{\alpha} \left(1 - \frac{\xi_1^2}{4}\right)\right] d\xi_1 \\ & + \frac{1}{2} (T_1 - X_1) \int_{x_1}^{T_1} \frac{\xi_1}{(\xi_1+X_1)^{\frac{5}{2}}(\xi_1+T_1)^{\frac{5}{2}}} \\ & \times F\left(\frac{1}{2}, \frac{1}{2}; 1; -\zeta_0\right) f_1\left[\frac{h_0}{\alpha} \left(1 - \frac{\xi_1^2}{4}\right)\right] d\xi_1 \end{aligned} \quad (2.13)$$

Here, non dimensional variables were introduced like that,

$$\left. \begin{aligned} \alpha \sqrt{\frac{g}{h_0}} t = t_1, \quad \frac{x}{h_0} = x_1, \quad \alpha \sqrt{\frac{g}{h_0}} \xi \\ = \xi_1, \quad \alpha \sqrt{\frac{g}{h_0}} X = X_1 \\ \alpha \sqrt{\frac{g}{h_0}} T = T_1, \quad X_1 = 2\sqrt{1 - \alpha x_1} \\ -t_1, \quad T_1 = 2\sqrt{1 - \alpha x_1} + t_1 \end{aligned} \right\} \quad (2.14)$$

3. Numerical examples.

(a) Solitary wave.

Let us consider the transformation of the profile of a solitary wave represented by equations,

$$\left. \begin{aligned} \eta = f_1(x) = A \operatorname{sech}^2 \frac{x^{**}}{2B} \\ \frac{\partial \eta}{\partial t} = f_2(x) = 0 \end{aligned} \right\} \quad (3.1)$$

at time $t=0$, where A is an amplitude and $B^2 = \frac{h_0^2(A+h_0)}{3A}$, h_0 being water depth in the uniform bottom region. We assume that at time $t=0$ the wave crest situated at the end of the slope where the origin of x -axis exists. Now, taking $\alpha = \frac{1}{10}$, $h_0 = 30$ cm and $A = 8.7$ cm, consequently $B = 36.5$ cm, wave profiles at $t=0.5$ sec and $t=1.0$ sec computed from (2.13) are given in Fig. 3. About 1 sec after since the wave entered into sloping region the wave heights are not always small compared with water depth, therefore the linear theory cannot be applicable. The wave

** LAMB, "Hydrodynamics", 6th Ed., p. 425

height increases gradually little greater than that of given by Airy's theory in which the slope of beach was not taken into account. Computed wave heights and those of given by Airy's theory are contained in Table 1.

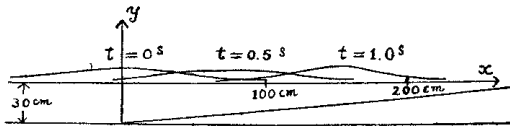


Fig. 3. Change of solitary wave profile $\eta = 8.7 \operatorname{sech}^2(0.0137x)$; $t=0$ progressing to the positive direction of x -axis. Beach slope is $\frac{1}{10}$ and the mean water level agrees with x -axis.

(b) Sine wave.

The initial wave character follows:*

$$\begin{aligned} \text{wave amplitude} & A = 3.2 \text{ cm} \\ \text{wave length} & \lambda = 110 \text{ cm} \\ \text{beach slope} & \alpha = 1/15 \\ \text{uniform water depth} & h_0 = 30 \text{ cm} \\ \eta = f_1(x) &= A \sin \frac{2\pi}{\lambda} x \quad (-\lambda < x < 0) \\ &= 0 \quad (x > 0 \text{ and } x < -\lambda) \\ \frac{\partial \eta}{\partial t} &= 0 \end{aligned}$$

$\eta(x, t)$ is given from (2.13) and (2.14) like that

$$\begin{aligned} \eta(x_1, t_1) &= A \sqrt{1 + \frac{t_1}{2\sqrt{1-\alpha x_1}} \sin \frac{\pi h_0}{2\lambda \alpha}} \\ &\quad (4\alpha x_1 - t_1^2 - 4t_1\sqrt{1-\alpha x_1}) \\ &\quad + A t_1 \int_{x_1}^{t_1} \frac{\xi}{(\xi + X_1)^{\frac{3}{2}} (\xi + T_1)^{\frac{3}{2}}} \\ &\quad \times F\left(\frac{1}{2}, \frac{1}{2}; 1; -\xi_0\right) \sin \frac{2\pi h_0}{\lambda \alpha} \left(1 - \frac{\xi^2}{4}\right) d\xi \\ &\quad + 4A t_1 \sqrt{1-\alpha x_1} \int_{x_1}^{t_1} \frac{\xi^2}{(\xi + X_1)^{\frac{3}{2}} (\xi + T_1)^{\frac{3}{2}}} \\ &\quad \times F\left(\frac{3}{2}, \frac{3}{2}; 2; -\xi_0\right) \sin \frac{2\pi h_0}{\lambda \alpha} \left(1 - \frac{\xi^2}{4}\right) d\xi \end{aligned} \quad (3.2)$$

TABLE 1

time (sec)	wave height by our theory (cm)	wave height by Airy's theory (cm)	water depth (cm)
0.0	8.70	8.70	30.0
0.5	9.44	9.42	21.8
1.0	10.40	10.33	15.1

* This wave example is one of the tank experiment carried out by Transportation Technical Research Institute.

The change of profile during the time $t=0$ and $t=0.67$ sec ($t_1=0.572$) computed from (3.2) is given in Fig. 4, where the

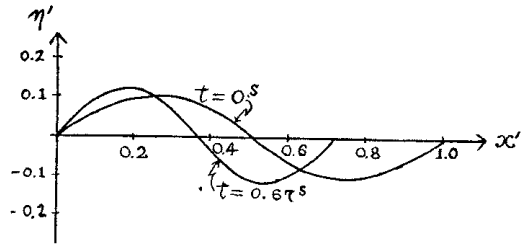


Fig. 4. Profiles of sine wave at $t=0$ and $t=0.67$ sec; $x' = \frac{x}{\lambda_0}$ ($\lambda_0 = 110$ cm) is taken as an abscissa, and $\eta' = \frac{\eta}{h_0}$ ($h_0 = 30$ cm) an ordinate. Beach slope is $\frac{1}{15}$

abscissa is $x' = \frac{x}{\lambda}$, and the ordinate $\eta' = \frac{\eta}{h_0}$. Decrease in the wave length during 0.67 sec is from 110 cm to about 80 cm, while increase in the wave amplitude is from 3.2 cm to about 3.72 cm. (Fig. 5)

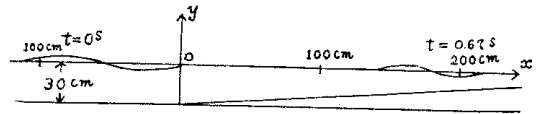


Fig. 5. Sine wave $\eta = 3.2 \sin \frac{2\pi}{110} x$ ($t=0$) advancing over the sloping beach (slope $\alpha = \frac{1}{15}$) toward positive x direction.

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