The Influence of Unsteady Depths and Currents of Tides on Wind-Wave Propagation in Shelf Seas

H. L. TOLMAN

Faculty of Civil Engineering, Delft University of Technology

(Manuscript received 3 August 1989, in final form 8 February 1990)

ABSTRACT

The influence of unsteady depths and currents on wind wave propagation on the scale of shelf seas such as the North Sea is investigated. The attention is focused on depth and current variations due to tides, which are essentially stationary at the scale of a single wave but unsteady at the scale of wave propagation. Analytical solutions are derived for changes of monochromatic unidirectional linear waves due to a one-dimensional tide in water with a constant bottom level. It is shown that the change of absolute frequency due to variations of depth and current in time is of the same order of magnitude as the Doppler shift, and that it is not negligible, as assumed in a quasi-stationary approximation. The common quasi-stationary approximation leads to large errors in the predicted change of wave parameters in all situations considered here.

1. Introduction

In the present study the influence of unsteady depths and currents on wind-generated surface gravity waves is investigated. Interactions between waves and currents, in particular the influence of currents on waves, have been studied extensively in past decades. Their importance is generally recognized and the subject is treated in numerous textbooks (e.g. Whitham 1974; Phillips 1977; Mei 1983), review papers (e.g. Peregrine 1976) and reports (e.g. Peregrine and Jonsson 1983). Wave-current interactions are usually considered in small scale (coastal) areas where depth and current are treated as inhomogeneous but stationary. In such cases the absolute wave frequency ω, which is observed in a fixed frame of reference, remains constant in space and time. This property is exploited in numerical wave propagation models for stationary depths and currents (e.g. Tayfun et al. 1976) and in calculations of spectral transformations due to stationary currents (e.g. Hedges et al. 1985).

If wave-current interactions are considered at the larger scale of shelf seas such as the (southern) North Sea, the assumption of stationarity is no longer valid as the time scale of variations in depth and current (typically 12 hours) is not large compared to the travel time of the waves through shelf seas (of the order of days). The assumption that the absolute frequency remains constant is therefore no longer valid. This is in conflict with the suggestion in some studies that deal with large-scale depth and current fields, e.g., Burrows and Hedges (1985), Chen and Wang (1983). In the latter study the assumption of constant absolute frequency is used, even though it is stated explicitly that depth and current are assumed to be unsteady ("nonstationary"). Note that the assumption of constant absolute frequency is obviously correct for more stationary large scale currents like the Gulf Stream. The subject of unsteady depths and currents is addressed more properly by, e.g., Unna (1941), Barber (1949), Longuet-Higgins and Stewart (1960) and Christoffersen (1982). However, the need to incorporate the influence of unsteadiness of depth and current in wave models does not seem to have been studied before.

The potential importance of wave-current interactions at larger scales is illustrated here with measurements acquired in the southern North Sea (Fig. 1). These results, which are not unusual for the location considered, show a wave height modulation of up to 50% with a period of approximately 12 hours. Although some of these modulations can be related to variations of the local wind velocity, this is generally not the case. The modulation period suggests a tidal influence. Since the tide induced depth modulation in the measurement area is only about 5%, current variations (in space and time) rather than depth variations are expected to be responsible for the observed wave height modulation.

Within the linear theory of surface gravity waves, the propagation of short-crested, random waves across unsteady depths and currents is properly described with the evolution of the two-dimensional action (or energy) density spectrum. The evolution of such spectra is commonly modeled with a discrete spectral balance equation (e.g., the SWIM group 1985, no currents;

Corresponding author address: Dr. Hendrik L. Tolman, Code 911, NASA/Goddard Space Flight Center, Greenbelt, MD 20771.

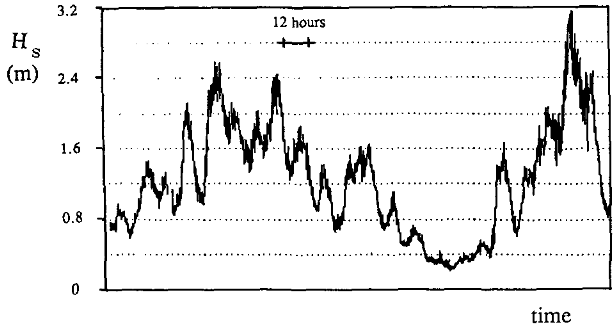

FIG. 1. Measured significant wave heights H_s at the southern North Sea, platform Euro-0, 50 km west of the entrance to the port of Rotterdam, water depth 26 m.

Tayfun and Dalrymple 1976, with stationary currents), where the effects of unsteadiness of depths and currents are represented by migration of action (or energy) across the frequency domain, analogous to migration across directions to represent refraction. In such an approach, interference due to the short-crested randomness of refracting waves may mask the specific effects of unsteadiness. Since the effects of unsteadiness are the main subject of this study, such interference should be avoided. Therefore, this study deals with the development of a single, nonrefracting wave component. The results thus obtained are directly applicable to all spectral components as they are formulated within the domain of the linear wave theory. To obtain analytical solutions this study furthermore considers a one-dimensional tide (constant bottom level). Analytical expressions for wave parameters are obtained using three approximations. The first is the common quasi-stationary approximation, in which the change of absolute frequency due to variations of depth and current in time is neglected. The second is a less common quasi-homogeneous approximation in which the change of wavenumber due to variations of depth and current in space is neglected. The third is a small perturbation approximation accounting for changes of both absolute frequency and wavenumber. Results for more realistic situations will be published elsewhere.

2. Basic equations

Consider monochromatic unidirectional waves as described with the linear (Airy) theory for surface gravity waves. They propagate over a depth and current field in which variations occur at space and time scales which are large compared to the wave length and period, but which are not necessarily large compared to the residence time of waves in shelf seas. The current velocity ($\mathbf{U}$) is taken to be constant over the vertical and the influence of waves on mean depth (d) and current is neglected.

Such unidirectional monochromatic waves are characterized with wavenumber (k), absolute frequency (ω), amplitude (a) and direction (θ). If waves propagating on currents are considered, it is convenient to make a distinction between a frame of reference fixed to the bottom, in which the absolute frequency ω is observed, and a frame of reference moving with the local current velocity $\mathbf{U}$, in which the relative or intrinsic frequency σ is observed. As depth and current are approximately stationary and homogeneous on the scale of a single wave, the absolute frequency ω, the relative frequency σ and the wavenumber k ($=|\mathbf{k}|$, k has direction θ) are interrelated by the dispersion relation:

$$\omega = \sigma + \mathbf{k}\cdot\mathbf{U} \tag{1}$$

where

$$\sigma^2 = gk\tanh(kd). \tag{2}$$

Waves can thus be described locally with three parameters, i.e., the amplitude a, direction θ and one of the three phase parameters ω, σ or k.

To describe the changes of the above wave parameters a method of characteristics is used in which wave energy is followed as it propagates over varying depths and currents. The propagation velocity of the wave energy c_g (direction θ), as observed in a frame of reference moving with the local current velocity $\mathbf{U}$, is given by the linear theory as:

$$c_g = \frac{\partial\sigma}{\partial k} = n\frac{\sigma}{k} \tag{3}$$

in which

$$n = \frac{1}{2} + \frac{kd}{\sinh(2kd)}. \tag{4}$$

In the fixed frame the propagation velocity of the energy ($\mathbf{c}_w$) is (e.g., Phillips 1977):

$$\mathbf{c}_w = \mathbf{c}_g + \mathbf{U} = \frac{d\mathbf{x}}{dt}. \tag{5}$$

The corresponding rate of change of absolute frequency ω, relative frequency σ, wavenumber k and direction θ (denoted as $d\omega/dt$, $d\sigma/dt$, dk/dt and $d\theta/dt$ respectively) can be determined using Eq. (1) and the following kinematic consistency relation [e.g., Whitham 1974, his Eq. (1.31)], which is a conservation equation for the density of the waves:

$$\frac{\partial\mathbf{k}}{\partial t} + \nabla\omega = 0. \tag{6}$$

The equations for these rates of change are (e.g., Christoffersen 1982; Mei 1983, p. 96):

$$\frac{d\omega}{dt} = \frac{\partial\sigma}{\partial d}\frac{\partial d}{\partial t} + \mathbf{k}\cdot\frac{\partial\mathbf{U}}{\partial t} \tag{7}$$

$$\frac{dk}{dt} = -\frac{\partial\sigma}{\partial d}\frac{\partial d}{\partial s} - \mathbf{k}\cdot\frac{\partial\mathbf{U}}{\partial s} \tag{8}$$

$$\frac{d\sigma}{dt} = \frac{\partial\sigma}{\partial d}\left[\frac{\partial d}{\partial t} + \mathbf{U}\cdot\nabla d\right] - c_g\mathbf{k}\cdot\frac{\partial\mathbf{U}}{\partial s} \qquad (9)$$

$$\frac{d\theta}{dt} = -\frac{1}{k}\left[\frac{\partial\sigma}{\partial d}\frac{\partial d}{\partial m} - \mathbf{k}\cdot\frac{\partial\mathbf{U}}{\partial m}\right] \qquad (10)$$

in which s is a coordinate in the direction θ and m is a coordinate perpendicular to s. The operator d/dt is defined as:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{c}_g + \mathbf{U})\cdot\nabla. \qquad (11)$$

Since Eqs. (1) and (2) interrelate ω, σ and k, only one of the equations, (7) through (9), needs to be integrated to obtain expressions for changes of ω, σ and k. Of these the rates of change of absolute frequency and wavenumber are convenient in the following analysis as they become zero in (quasi-) stationary or (quasi-) homogeneous conditions, respectively.

The determination of the wave amplitude for propagation over currents is based on the conservation of wave action (e.g., Whitham 1974; Phillips 1977). For monochromatic unidirectional waves the (Eulerian) action conservation equation is

$$\frac{\partial A}{\partial t} + \nabla\cdot[(\mathbf{c}_g + \mathbf{U})A] = 0 \qquad (12)$$

in which A is the action density (i.e. action per unit area):

$$A = E/\sigma \qquad (13)$$

and E is the energy density:

$$E = \frac{1}{2}\rho g a^2. \qquad (14)$$

3. Measure of unsteadiness

To quantify the relative importance of unsteadiness in the following analysis, consider a depth or current field with variations at length scale L_{dc} (in the propagation direction of the waves) and time scale T_{dc}. The travel time of wave energy over such a distance L_{dc} is L_{dc}/c_w, where c_w is the wave propagation velocity. If this travel time is small compared to the time scale T_{dc}, the depth and current field is quasi-stationary with respect to wave propagation. Otherwise the depth and current field is essentially unsteady. Similarly, the travel distance in a time interval T_{dc} is c_wT_{dc}. If this travel distance is small compared to the length scale L_{dc}, the depth and current field is quasi-homogeneous. The following parameter can therefore be used as a measure of the unsteadiness (and inhomogeneity) of depth and current with respect to wind wave propagation:

$$\Psi = \frac{L_{dc}}{c_wT_{dc}} = \frac{\text{travel time}}{\text{time scale}} = \frac{\text{length scale}}{\text{travel distance}}. \qquad (15)$$

The depth and current field is unsteady and inhomogeneous if $\Psi = O(1)$, quasi-stationary and inhomogenous if $\Psi \ll 1$ and unsteady and quasi-homogeneous if $\Psi \gg 1$. For stationary, homogenous conditions Ψ is not defined as both L_{dc} and T_{dc} then approach infinity.

The time scale T_{dc} of depth and current variations in shelf seas is obviously the tidal period T_t, but the length scale L_{dc} is less readily estimated. This length scale is determined by the tidal wave length L_t and by characteristic length scales of the bathymetry. The latter length scales are dominant in nearshore areas, whereas the tidal wave length is dominant away from the coast.

To estimate the magnitude of Ψ for wind waves on tides away from the coast, consider a simple situation with unidirectional monochromatic waves propagating in water with a constant bottom level at an angle θ relative to the propagation direction of a one-dimensional tide. Taking the tidal period T_t and $L_t/|\cos\theta|$ as estimates of the time scale T_{dc} and the length scale L_{dc} and using the propagation velocity of the tide $c_t = L_t/T_t$, the value of Ψ is

$$\Psi = \frac{c_t}{c_w|\cos\theta|}. \qquad (16)$$

As the tidal velocity $c_t \approx \sqrt{gd}$ and as the wave propagation velocity c_w as estimated by Eq. (5) is always smaller than $\sqrt{gd}$ (since $c_g < \sqrt{gd}$ and tidal currents usually have a small Froude number, i.e., $U \ll \sqrt{gd}$), Ψ is always larger than 1, so that tides are never stationary with respect to waves. In many cases $\Psi = O(1)$, in which case the tide is unsteady and inhomogeneous, whereas for some situations $\Psi \gg 1$ (i.e., in extremely deep water where $c_t \gg c_w$ or for $|\cos\theta| \ll 1$), in which case the tide is homogeneous (and unsteady).

The unsteadiness parameter does not of course indicate the magnitude of current influences.

4. One-dimensional waves on a one-dimensional tide

To illustrate the influence of (space and) time derivatives of depth and current on waves, consider monochromatic unidirectional waves. The tide is represented by a one-dimensional long wave, propagating in the positive x-direction in water with a constant bottom level:

$$d(x, t) = d_0 + A_d \sin\chi(x, t) \qquad (17)$$

$$U(x, t) = A_U \sin\chi(x, t) \qquad (18)$$

$$\chi(x, t) = Kx - \Omega t \qquad (19)$$

$$c_t = \Omega/K = \sqrt{gd_0} \qquad (20)$$

$$A_U/A_d = \sqrt{g/d_0}. \qquad (21)$$

In these equations χ is the tidal phase, c_t is the tidal propagation velocity, K and Ω are the wavenumber and frequency of the tide and A_U and A_d are the current and depth amplitude, respectively (the current velocity is assumed to be constant over the depth).

To determine changes of wave parameters a method of characteristics is used in which the waves are followed with the propagation velocity of the wave energy $\mathbf{c}_w$. The analysis starts at a location and time where $\chi = 0$, which implies that in the initial situation the depth equals d_0 and that there are initially no current. The wave parameters in this initial situation are indicated with suffix zero (e.g., ω_0).

Three different approximations can be used to determine changes of wave parameters. The first is a quasi-stationary approximation in which time derivatives of depth and current are neglected. The second is a quasi-homogeneous approximation in which space derivatives of depth and current are neglected. The third is an unsteady inhomogeneous approximation in which both time and space derivatives of depth and current are accounted for.

a. Phase parameters

In the following changes of the phase parameters ω, σ and k are determined and discussed for all three approximations separately. Changes are expressed as absolute changes from the initial situation, e.g., $\Delta\omega = \omega - \omega_0$.

1) QUASI-STATIONARY APPROXIMATION

In a quasi-stationary approximation partial time derivatives of depth and current are assumed to be zero, so that $d\omega/dt$ in Eq. (7) is zero. The absolute frequency ω ($=\omega_s$ in this approximation) therefore remains constant and equal to the initial frequency ω_0. The wavenumber (k_s) can be determined at any location (time) along ray paths from ω_0, (local) depth d, (local) current velocity U and direction θ. Using Eqs. (1) and (2) the changes of absolute frequency and wavenumber are

$$\Delta\omega_s = 0 \tag{22}$$

$$\Delta k_s = k_s - k_0. \tag{23}$$

As the absolute frequency remains constant, the change of relative frequency $\Delta\sigma_s$ is opposite to the Doppler shift kU_p (where U_p is the current velocity in the propagation direction of the waves, $U_p = U\cos\theta$), which for small variations in wavenumber ($\Delta k/k \ll 1$) approximately equals k_0U_p:

$$\Delta\sigma_s \approx -k_0U_p. \tag{24}$$

2) QUASI-HOMOGENEOUS APPROXIMATION

In a quasi-homogeneous approximation partial space derivatives rather than time derivatives of depth and current are assumed to be zero. It follows then from Eq. (8) that wavenumber k ($=k_h$ in this approximation) rather than absolute frequency ω remains constant. The frequency can be determined at any time (location) along the ray path from k_0, the (local) depth d and the (local) current velocity U. Using Eqs. (1) and (2) the changes of absolute frequency and wavenumber are

$$\Delta k_h = 0 \tag{25}$$

$$\begin{aligned}\Delta\omega_h &= \sqrt{gk_0\tanh(k_0d)} - \omega_0 + k_0U_p \\ &= \Delta\sigma_h + k_0U_p.\end{aligned} \tag{26}$$

The change of absolute frequency $\Delta\omega_h$ consists of the Doppler shift k_0U_p, due to current variations only, and the change of relative frequency $\Delta\sigma_h$, due to depth variations only. For arbitrary depth and current variations the ratio between those two contributions to $\Delta\omega_h$ is determined using Eq. (2) and a truncated Taylor series expansion for $\sqrt{\tanh(\Delta d/d_0)} \ll 1$):

$$\frac{\Delta\sigma_h}{k_0U_p} = \frac{\Delta d}{d_0}\frac{\sqrt{gd_0}}{U_p}\frac{\sqrt{k_0d_0}}{2\cosh^2(k_0d_0)\sqrt{\tanh(k_0d_0)}}. \tag{27}$$

For the depth and current field considered here, Δd and U ($=\Delta U$ since $U = 0$ initially) are interrelated [Eqs. (17), (18) and (21)], so that this ratio becomes a function of k_0d_0 and θ only:

$$\frac{\Delta\sigma_h}{k_0U_p} = (\cos\theta)^{-1}\frac{\sqrt{k_0d_0}}{2\cosh^2(k_0d_0)\sqrt{\tanh(k_0d_0)}}. \tag{28}$$

In Fig. 2 this ratio is plotted as a function of k_0d_0 for several angles θ. This figure shows that the depth induced variation of absolute frequency ($\Delta\sigma_h$) in Eq. (26) is only relevant compared to the current induced variation of the absolute frequency (k_0U_p) for relatively shallow water (e.g., $k_0d_0 < 1$), and for waves traveling in directions almost perpendicular to the propagation direction of the tide (e.g., $80° < |\theta| < 110°$). In the latter case, however, Doppler shifts become negligible as $\cos\theta$ approaches 0, so that the entire change of absolute frequency becomes negligible. For shelf seas

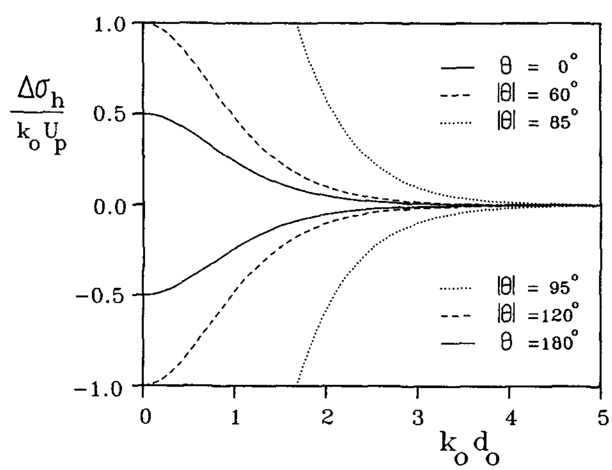

FIG. 2. The ratio between the depth induced and current induced contributions to the change of absolute frequency $\Delta\omega_h$ in quasi-homogeneous approximation ($\Delta\sigma_h$ and k_0U_p, respectively).

away from the coast, where relative water depths $k_0 d_0$ are usually not extremely small, Eq. (26) thus becomes

$$\Delta\omega_h \approx k_0 U_p. \tag{29}$$

3) UNSTEADY INHOMOGENEOUS APPROXIMATION

If both space and time derivatives of depth and current are accounted for in the basic equations of section 2, both wavenumber k and absolute frequency ω will change. To obtain the change of phase parameters (e.g., ω), one would have to integrate Eqs. (5), (10) and one of the three equations, (7) through (9), simultaneously (in time). To simplify the derivation of analytical solutions, a small perturbation approach is used in which (i) changes in k, ω and σ are assumed to be relatively small ($|\Delta k/k_0| \ll 1$ etc.), (ii) effects of depth variations are neglected (which away from the coast are negligible in the other two approximations), and (iii) the change $\Delta\theta$ of direction is neglected (i.e., tide induced refraction is neglected). Using such an approximation Eq. (7) becomes

$$\frac{d\omega}{dt} = \left.\frac{\partial\sigma}{\partial d}\right|_{k=k_0} \frac{\partial d}{\partial t} + k_0 \cos\theta \frac{\partial \mathrm{U}}{\partial t}. \tag{30}$$

Since $\partial d/\partial t$ and $\partial \mathrm{U}/\partial t$ are periodic functions of the tidal phase χ, $d\omega/dt$ and ω vary with χ only. It is therefore convenient to integrate the rate of change $d\omega/d\chi$ in χ (instead of $d\omega/dt$ and $d\chi/dt$ in t). Since

$$\frac{d\omega}{d\chi} = \frac{1}{-\Omega + c_w \cos\theta K} \frac{d\omega}{dt} \tag{31}$$

[from Eqs. (11) and (19)], the change of absolute frequency becomes

$$\Delta\omega(\chi) = \frac{1}{-\Omega + c_w \cos\theta K} \int_0^\chi F(\chi')d\chi' \tag{32}$$

where $F(\chi)$ is the rate of change $d\omega/dt$ as given by Eq. (30). Equation (32) holds for any Ω and K, not only for Ω and K related as in Eq. (20). The integral at the right-hand side of this equation can be determined by considering a quasi-homogeneous situation for which $c_w \cos\theta K/\Omega$ approaches 0. In such a situation $\Delta\omega(\chi)$ approaches $\Delta\omega_h(\chi)$ [Eq. (29)], so that the integral at the right-hand side of Eq. (32) equals $-\Omega\Delta\omega_h(\chi)$. Using Eqs. (20) and (29), Eq. (32) then becomes

$$\Delta\omega = \frac{1}{1 - c^* \cos\theta} k_0 U_p \tag{33}$$

in which the propagation ratio c^* is defined as

$$c^* = \frac{c_w}{c_t} = n_0 \left[\frac{\tanh(k_0 d_0)}{k_0 d_0}\right]^{0.5}. \tag{34}$$

Since c^* is a function of $k_0 d_0$ only, it can be interpreted as an alternative relative depth parameter (instead of $k_0 d_0$).

The change of absolute frequency $\Delta\omega$ relative to the Doppler shift $k_0 U_p$ is shown in Fig. 3 as a function of direction θ and relative depth $k_0 d_0$. This figure shows that for waves traveling in directions opposite to the tide (i.e. $90° < \theta < 270°$), $\Delta\omega$ is smaller than the Doppler shift, with a minimum of $0.5 k_0 U_p$. For waves traveling in the same direction as the tide ($-90° < \theta < 90°$), $\Delta\omega$ is larger than the Doppler shift and for most relative depths and directions of the same order of magnitude as the Doppler shift, except for waves in extremely shallow water, traveling in approximately the same direction as the tide (e.g. $k_0 d_0 < 1$ and $|\theta| < 20°$). In the latter case the change of absolute frequency becomes an order of magnitude larger than the Doppler shift.

An expression for the change of relative frequency is simply found by subtracting $\omega_0 = \sigma_0$ from Eq. (1) and by using Eq. (33):

$$\Delta\sigma = \frac{c^* \cos\theta}{1 - c^* \cos\theta} k_0 U_p. \tag{35}$$

The normalized change of relative frequency $\Delta\sigma/k_0 U_p$ is also shown in Fig. 3. This figure shows that the change of relative frequency $\Delta\sigma$ usually differs largely from the Doppler shift $k_0 U_p$. For waves traveling in directions opposite to the tide ($90° < \theta < 270°$), $\Delta\sigma$ has a sign opposite to that of the Doppler shift and is in absolute value smaller than the Doppler shift by a factor of 2 or more. For waves traveling in the same direction as the tide ($-90° < \theta < 90°$), $\Delta\sigma$ and $k_0 U_p$ show the same sign, and $\Delta\sigma$ can be either much smaller, much larger or of the same order of magnitude as the Doppler shift.

An expression for the change of wavenumber can be obtained from Eq. (8) using an approximation similar to that of the above derivation for $\Delta\omega$ [now invoking a quasi-stationary situation where $c_w \cos\theta K/\Omega$ approaches infinity instead of zero and using Δk_s as given by Eq. (23)]. The result is

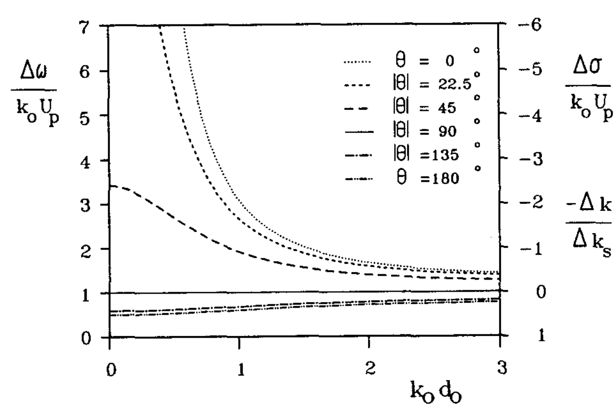

FIG. 3. Normalized change of absolute frequency $\Delta\omega/k_0 U_p$, relative frequency $\Delta\sigma/k_0 U_p$ and wavenumber $\Delta k/\Delta k_s$ as a function of relative depth $k_0 d_0$ for waves traveling on a one-dimensional tide at angle θ.

$$\Delta k = \frac{-c^* \cos\theta}{1 - c^* \cos\theta} \Delta k_s. \quad (36)$$

Finally the normalized change of wavenumber $\Delta k / \Delta k_s$ is shown in Fig. 3. This figure shows that Δk always differs largely from Δk_s, as Δk is either smaller than Δk_s by a factor of 2 or more ($90° < \theta < 270°$, wave traveling in directions opposite to the tide), or shows an opposite sign ($-90° < \theta < 90°$, waves traveling in the same direction and the tide).

4) ANALYSIS OF RESULTS

For all three approximations a current velocity U causes a local difference between the absolute frequency ω and the relative frequency σ, which equals the Doppler shift $k_0 U_p$. In the quasi-stationary approximation all variations in the Doppler shift are balanced by changes in the relative frequency, whereas in a quasi-homogeneous approximation all variations in the Doppler shift are balanced by changes of the absolute frequency. In the more realistic unsteady inhomogeneous approximation neither changes of the absolute frequency nor changes of the relative frequency equal the Doppler shift, but both $\Delta\omega$ and $\Delta\sigma$ are of the same order of magnitude as the Doppler shift (where $\Delta\omega - \Delta\sigma$ equals the Doppler shift). This indicates that both unsteadiness and inhomogeneity of depth and current are relevant for wave–current interactions in the tidal situation considered here.

b. *Wave amplitude*

To determine the wave amplitude a quasi-stationary and a quasi-homogeneous approximation to the action density equation (12) could be considered, analogous to the approach used for the phase parameters in the previous section. However, to allow for a comparison with existing numerical models (e.g., Chen and Wang 1983), the full equation (12) will be solved for all three approximations, whereas the phase parameters ω, σ and k for the three approximations are determined as in the previous section. Consequently a quasi-stationary (quasi-homogeneous) approximation is defined as an approximation in which depth and current are unsteady (inhomogeneous), but in which the change of absolute frequency (wavenumber) is neglected.

Moving with the propagation velocity of the waves, all changes in action density are caused by gradients in the propagation velocity only. Such gradients are caused by variations in current, water level and wavenumber [Eq. (5)], which all vary with phase χ of the tide only. Consequently, the solution to Eq. (12) is stationary in a frame of reference that moves with the velocity c_t along the x-axis (as χ is stationary in this frame of reference). Thus $\partial(\mathbf{c}_w \cdot \mathbf{e}_y)A/\partial y = 0$ and $\partial A/\partial t + c_t \partial A/\partial x = 0$ so that Eq. (12) becomes

$$\frac{\partial}{\partial x}[(c_g \cos\theta + U - c_t)A] = 0. \quad (37)$$

Consequently $(c_{g,0}\cos\theta - c_t)A_0 = (c_g \cos\theta + U - c_t)A$, and using Eq. (34) the relative action density A/A_0 becomes

$$\frac{A}{A_0} = \left[1 - \frac{c^*}{1 - c^* \cos\theta} \frac{\Delta c_g \cos\theta + U}{c_{g,0}}\right]^{-1}. \quad (38)$$

The relative wave amplitude a/a_0 can be obtained from Eqs. (13) and (14):

$$\frac{a}{a_0} = \left[\frac{\sigma}{\sigma_0}\frac{A}{A_0}\right]^{0.5}. \quad (39)$$

Using a truncated Taylor series expansion of Eqs. (38) and (39) for small relative changes $\Delta c_g / c_{g,0}$, $\Delta\sigma/\sigma_0$ and $\Delta A/A_0$, the relative change of wave action and amplitude become

$$\frac{\Delta A}{A_0} = \frac{c^*}{1 - c^* \cos\theta} \frac{\Delta c_g \cos\theta + U}{c_{g,0}} \quad (40)$$

$$\frac{\Delta a}{a_0} = \frac{1}{2}\left[\frac{\Delta\sigma}{\sigma_0} + \frac{\Delta A}{A_0}\right]. \quad (41)$$

As indicated above, Eqs. (40) and (41) hold for all three approximations, as long as the appropriate formulations for $\Delta\sigma/\sigma_0$ and $\Delta c_g / c_{g,0}$ are used. In the following these formulations are first substituted in Eqs. (40) and (41) for all three approximations and then the changes of amplitude for the three approximations are analyzed and intercompared.

1) QUASI-STATIONARY APPROXIMATION

In the quasi-stationary approximation the change of relative frequency $\Delta\sigma_s$ is given by Eq. (24). The change of propagation velocity $\Delta c_{g,s}$ equals αU_p, where α is a function of $k_0 d_0$ only (see appendix A). Substituting these two equations in Eqs. (38) and (39) and using Eq. (3), the relative change of wave amplitude for the quasi-stationary approximation becomes (after some straightforward algebra):

$$\frac{\Delta a_s}{a_0} = \frac{1}{2}\frac{U}{c_{g,0}} G_1 \frac{c^* - G_2}{1 - c^* \cos\theta} \quad (42)$$

where

$$G_1 = 1 + (n_0 + \alpha)\cos^2\theta$$

$$G_2 = n_0 \cos\theta G_1^{-1}.$$

2) QUASI-HOMOGENEOUS APPROXIMATION

In the quasi-homogenous approximation the changes of the relative frequency $\Delta\sigma_h$ are away from the coast usually much smaller than the Doppler shift $k_0 U_p$ [see Eq. (28) and Fig. 2], so that $\Delta\sigma_h \approx 0$. Away from the coast $n \approx n_0$ (Eq. (4) with small relative depth variations $\Delta d/d_0$) so that $\Delta c_{g,h} \approx 0$ [Eq. (3) with $k = k_0$ and $\sigma = \sigma_0$]. Substituting these solutions for $\Delta\sigma_h$ and $\Delta c_{g,h}$ in Eqs. (38) and (39), the relative change of wave amplitude becomes

$$\frac{\Delta a_h}{a_0} = \frac{1}{2}\frac{U}{c_{g,0}}\frac{c^*}{1 - c^* \cos\theta}. \quad (43)$$

3) UNSTEADY INHOMOGENEOUS APPROXIMATION

In the unsteady inhomogeneous approximation the change of relative frequency $\Delta\sigma$ is given by Eq. (35). The change of propagation velocity Δc_g can be expressed in terms of U, $c_{g,0}$, c^* and θ, considering that (i) $\Delta c_g/\Delta c_{g,s} \approx \Delta k/\Delta k_s$ [linearization of Eq. (3) around k_0], (ii) $\Delta c_{g,s} = \alpha U_p$ (appendix A) and (iii) Eq. (36). Substituting such an expression for Δc_g together with Eqs. (35) and (38) in Eq. (39), the relative change of wave amplitude for the unsteady inhomogeneous approximation becomes, after some straightforward algebra:

$$\frac{\Delta a}{a_0} = \frac{1}{2}\frac{U}{c_{g,0}} G_2 \frac{c^*[G_3 - c^* \cos\theta]}{[1 - c^* \cos\theta]^2} \quad (44)$$

where

$$G_3 = [1 + n_0 \cos^2\theta]G_1^{-1}.$$

4) ANALYSIS OF RESULTS

The relative change of amplitude for the case considered here is obviously best estimated by (44) (i.e. by the inhomogeneous and unsteady approximation). This equation shows that the relative change of amplitude $\Delta a/a_0$ is a linear function of relative current velocity U/c_g, which like Doppler shift is easy to estimate for practical shelf sea conditions. The ratio between $\Delta a/a_0$ and U/c_g is a complicated function of the relative depth $k_0 d_0$ and the direction θ. Figure 4 shows that for most relative depths and directions the relative change of amplitude is of the same order of magnitude as the relative current velocity but smaller. Only for waves in extremely shallow water (e.g., $k_0 d_0 < 1$) propagating in approximately the same direction as the tide (e.g. $|\theta| < 30°$), the relative change of amplitude becomes significantly larger than the relative current velocity.

The influence of unsteadiness and inhomogeneity of depth and current (i.e., changes of absolute frequency and wavenumber, respectively) on the change of wave amplitude can be distinguished by intercomparing the changes in amplitude of the three approximations. The change of amplitude in the quasi-stationary and the quasi-homogenous approximations (Δa_s and Δa_h, respectively) are therefore normalized with the change of amplitude of the unsteady inhomogeneous approximation (Δa) in Fig. 5. Since neither $\Delta a_s/\Delta a$ nor $\Delta a_h/\Delta a$ approximately equals 1 [i.e., equals $1 \pm O(10^{-1})$], both the unsteadiness and the inhomogeneity of depth and current are material to the change in amplitude in the situation considered here. Note that the relative importance of unsteadiness and inhomogeneity is independent of the current velocity, i.e., independent of the magnitude of the wave–current interactions, since $\Delta a_s/\Delta a$ and $\Delta a_h/\Delta a$ are functions of $k_0 d_0$ and θ only.

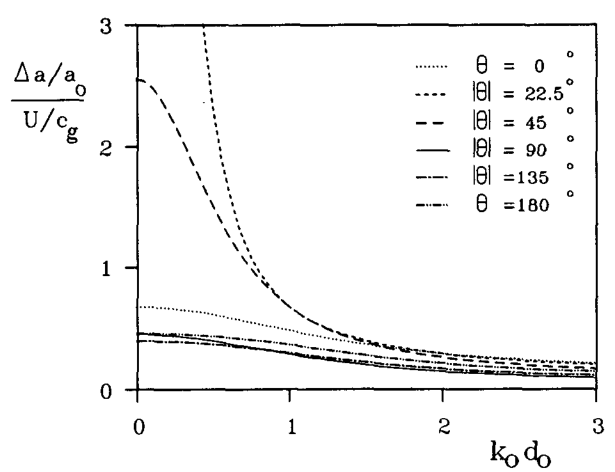

FIG. 4. Relative changes of wave amplitude $\Delta a/a_0$, normalized with the relative current velocity U/c_g as a function of relative depth $k_0 d_0$ for waves traveling on a one-dimensional tide at angle θ.

5. Discussion

As indicated in section 3, depths and currents due to tides in shelf seas are essentially an unsteady and inhomogeneous medium for wind wave propagation. Away from the coast interactions due to current variations are dominant over those due to surface level variations. Changes of the absolute frequency ω for the situation considered here [Eq. (33), Fig. 3] are of the same order of magnitude as the Doppler shift $k_0 U_p$, whereas relative changes of the wave amplitude [Eq. (44), Fig. 4] are of the same order of magnitude as the relative current velocity U/c_g. Considering e.g. a current velocity of 1 m s^{-1}, a water depth of 25 m and a wave period of 7 s (which is fairly realistic for the southern North Sea), relative changes of the order of 10% might be expected for these wave parameters.

Relative errors in changes of wave parameters as induced by the quasi-stationary or quasi-homogeneous approximation are independent of the actual current magnitude, and vary with relative depth $k_0 d_0$ and direction θ only (e.g., see Figs. 3 and 5) for the situation considered here.

In the quasi-stationary approximation the change of absolute frequency due to depth and current unsteadiness is neglected. Consequently [Eq. (33), Fig. 3], errors in the magnitude of the (change of) absolute frequency are of the same order of magnitude as (but not equal to) the Doppler shift. In this approximation

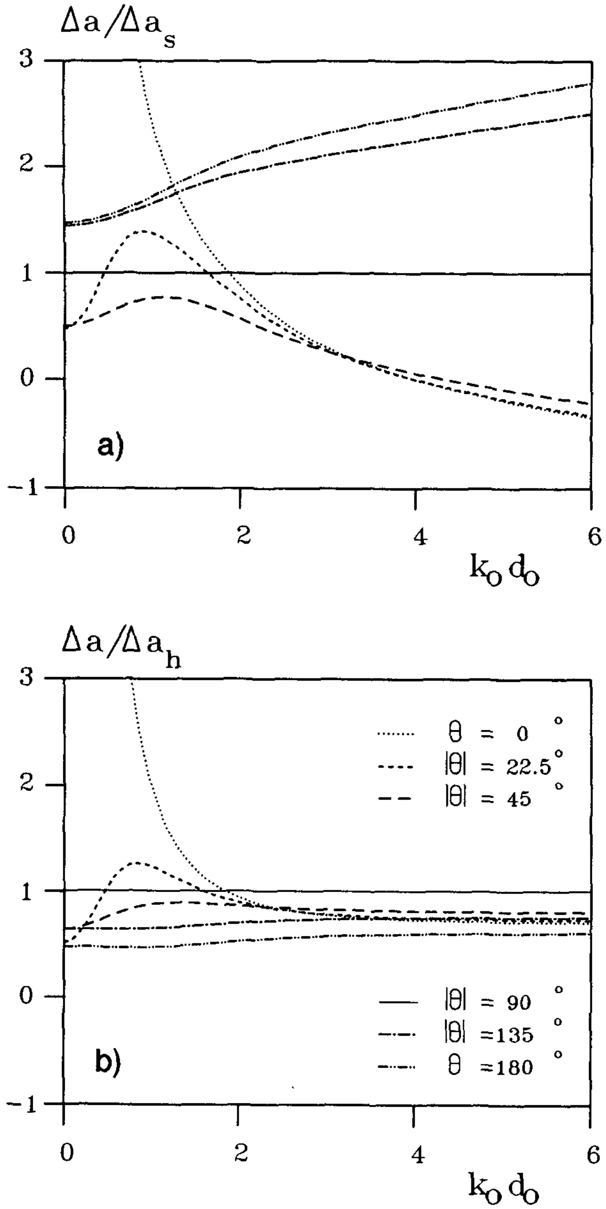

FIG. 5. Normalized changes of wave amplitude as a function of $k_0 d_0$ for waves traveling on a one-dimensional tide at angle θ (a) quasi-stationary approximation ($\Delta a_s/\Delta a$) and (b) quasi-homogeneous approximation ($\Delta a_h/\Delta a$).

changes of relative frequency and wavenumber ($\Delta\sigma$ and Δk) are overestimated by a factor of two or more for waves and a tide traveling in opposite directions ($90° > \theta > 270°$), and even show a wrong sign for waves and a tide traveling in similar directions ($-90° > \theta > 90°$) (see Fig. 3, note that $\Delta\sigma_s/\Delta\sigma = -\Delta\sigma_s/k_0U_p = \Delta k_s/\Delta k$). Finally Fig. 5a shows that the quasi-stationary approximation also introduces large errors in the predicted changes of wave amplitude, including underestimations, overestimations and wrong signs.

In the quasi-homogeneous approximation the change of wavenumber (and away from the coast change of relative frequency) due to depth and current inhomogeneity is neglected. Consequently (Fig. 3) changes of phase parameters are comparable to those of the quasi-stationary approximation, except for extremely deep water, where errors as induced by the quasi-homogeneous approximations vanish. Figure 5b shows that the quasi-homogeneous approximation underestimates the change of amplitude by a factor of two or less for most relative depths $k_0 d_0$ and directions θ. Note that the error in change of amplitude does not vanish for extremely deep water, unlike the error in the phases parameters. For extremely deep water, however, tidal currents are small, so that the total change of amplitude [Eq. (44)] becomes negligible.

Finally it should be noted that the analysis as presented in this study only deals with waves on a freely propagating tide. However, many current fields cannot be described in such a way. Consider, e.g., depth- or river-induced current variations in coastal areas and deep-ocean surface currents such as the Gulf Stream. In those cases the unsteadiness parameter [Eq. (15)] can still be used to determine whether depth and current are quasi-stationary, quasi-homogeneous or unsteady and inhomogeneous. The results of section 4 are only applicable if the length and time scales of depth and current variations are related to some propagation velocity of the depth and current field.

6. Conclusions

The present study shows that tides on the scale of shelf seas such as the (southern) North Sea are an unsteady and inhomogeneous medium for the propagation of (wind) waves. For such conditions the following conclusions can be drawn. 1) The currents are strong enough to induce significant wave-current interactions, manifested in changes of absolute frequency, wavenumber and wave amplitude. 2) The change of absolute frequency, which is typical for unsteady media, is of the same order of magnitude as the Doppler shift, but not equal to the Doppler shift. 3) In the quasi-stationary approximation, which is commonly used in wave propagation models, the absolute frequency of the waves remains constant. In such models errors dominate the predicted current induced changes of wave wavenumber, absolute and relative frequency and wave amplitude.

Acknowledgments. The author acknowledges with pleasure the Dutch Ministry of Public Works and Transportation for supplying Fig. 1 and L. H. Holthuijsen, J. A. Battjes and N. Booij of the Delft University of Technology and R. Sobey of the University of California for their valuable discussion and for reviewing early drafts of this paper.

APPENDIX A

Relation between U_p and $\Delta c_{g,s}$

Considering small perturbations as above and as $U = \Delta U$, α $(=\Delta c_{g,s}/U_p)$ can be written as:

$$\alpha = \frac{\Delta c_g}{U_p} \approx \frac{\partial c_g}{\partial U_p} = \frac{\partial c_g}{\partial k}\frac{\partial k}{\partial U_p} \qquad \text{(A1)}$$

where the suffix s, indicating the stationary approximation, has been dropped. Using equation (3) and (4), $\partial c_g/\partial k$ is written as

$$\frac{\partial c_g}{\partial k} = \frac{\sigma}{k^2}(n^2 - n + n'kd) \qquad \text{(A2)}$$

where n' is the derivative of n with respect to kd. To find an expression for $\partial k/\partial U_p$, the dispersion relation (1) is rewritten as:

$$\Delta\omega = \frac{\partial\sigma}{\partial k}\Delta k + k\Delta U_p + U_p\Delta k. \qquad \text{(A3)}$$

Using a quasi-stationary approximation $\Delta\omega$ equals 0. For small perturbations the current velocity U is (expected to be) small compared to the propagation velocity c_g $(=\partial\sigma/\partial k)$, so that

$$\frac{\partial k}{\partial U_p} \approx \frac{\Delta k}{\Delta U_p} \approx -\frac{k}{c_g} = -\frac{k^2}{n\sigma}. \qquad \text{(A4)}$$

Substitution of (A2) and (A4) in (A1) gives an expression for α, which is a function of kd only:

$$\alpha = \frac{n^2 - n + n'kd}{-n}. \qquad \text{(A5)}$$

REFERENCES

Barber, N. F., 1949: Behaviour of waves on tidal streams. *Proc. Roy. Soc. London,* **A198,** 81–93.

Burrows, R., and T. S. Hedges, 1985: The influence of currents on ocean wave climates. *Coastal Engineering,* **9,** 247–260.

Chen, Y. H., and H. Wang, 1983: Numerical model for nonstationary shallow water wave spectral transformations. *J. Geophys. Res.,* **88**(C14), 9851–9863.

Christoffersen, J. B., 1982: Current depth refraction of dissipative water waves. Institute of Hydrodynamics and Hydraulic Engineering, Technical University of Denmark, Ser. Paper No. 30.

Hedges, T. S., K. Anastasiou and D. Gabriel, 1985: Interaction of random waves and currents. *J. Waterway, Port, Coastal and Ocean Eng.,* ASCE, **111**(2), 275–288.

Longuet-Higgins, M. S., and R. W. Stewart, 1960: Changes in the form of short gravity waves on long waves and tidal currents, *J. Fluid Mech.,* **8,** 565–583.

Mei, C. C., 1983: *The Applied Dynamics of Ocean Surface Waves.* Wiley, 740 pp.

Peregrine, D. H., 1976: Interaction of water waves and currents. *Adv. Appl. Mech.,* **16,** 9–117.

——, and I. G. Jonsson, 1983: Interaction of waves and currents. Miscellaneous Rep. No. 83-6, CERC, U.S. Army Corps of Engineers.

Phillips, O. M., 1977: *The Dynamics of the Upper Ocean.* 2nd ed., Cambridge University Press, 336 pp.

SWIM group, 1985: A shallow water intercomparison of three numerical wave prediction models (SWIM). *Quart. J. Roy. Meteor. Soc.,* **111,** 1087–1112.

Tayfun, M. A., R. A. Dalrymple and C. Y. Yang, 1976: Random wave-current interaction in water of varying depth. *Ocean Eng.,* **3,** 403–420.

Unna, P. J. H., 1941: White horses. *Nature,* **148,** 226–227.

Whitham, G. B., 1974: *Linear and Nonlinear Waves.* Wiley, 636 pp.