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Extended time-dependent mild-slope and wave-action equations for wave–bottom and wave–current interactions

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Extended mild-slope (MS) and wave-action equations (WAEs) are derived by taking into account high-order derivatives of the bottom profile and the depth-averaged current that were previously neglected. As a first step for this derivation, a time-dependent MS-type equation in the presence of ambient currents that consists of these high-order components is constructed. This mild-slope equation is used as a basis to form a wave-action balance equation that retains high-order refraction and diffraction terms of varying depths and currents. The derivation accurately accounts for the effects of the currents on the Doppler shift. This results in an 'effective' intrinsic frequency and wavenumber that differ from the ones of wave ray theory. Finally, the new WAE is derived for the phase-averaged frequency-direction spectrum in order to allow its use in stochastic wave-forecasting models.

Keywords: water waves; wave–current interactions; mild-slope equations; wave-action equations

1. Introduction

The flow of surface water waves is a time-varying three-dimensional flow problem. In various scenarios, it is plausible to assume that the flow is governed by the Laplace equation together with a free surface and an impermeable bottom. The three-dimensional problem is complicated to model, and its solution requires significant computational effort. Therefore, in many applications, one dimension is reduced in order to simplify the model.

A commonly employed approach for this reduction yields the frequency domain mild-slope (MS)-type equations, which can be applied to linear time-harmonic problems. The origin of this discipline is the mild-slope equation (MSE) based on the works of Eckart (1952), Biesel (1952), Svendsen (1967) and Berkhoff (1972) for time-harmonic waves with no ambient flow. The MSE and other MS-type models achieve this reduction by assuming a vertical profile and averaging the governing equation over the depth, which results in the elimination of the vertical coordinate. The same approach was applied by Booij (1981) (later corrected by

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Kirby 1984) for waves in the presence of depth-averaged ambient currents in the time-dependent case. Kostense *et al.* (1988) constructed a numerical model for this equation, presenting an iteration method for solving the dependency of intrinsic frequency on the amplitude of the solution.

Many works continued the MSE's derivation (e.g. Massel 1993; Porter & Staziker 1995; Porter 2003; Kim & Bai 2004; Porter & Porter 2006; Hsu *et al.* 2006*b*; Toledo & Agnon 2010, 2011). Specifically, the extension of this model to hold for steep slopes and curvatures was addressed. For the time-harmonic case with no ambient currents, the higher order bottom components $\nabla^2 h$ and $|\nabla h|^2$, which were originally neglected, were shown to be of significance by Chamberlain & Porter (1995) (see Klopman & Dingemans 2010 for the significance also in a Boussinesq-like approach). In addition, retaining these terms allows for an improved accuracy in modelling the dominating class I Bragg resonance (see Agnon 1999). Nevertheless, these components were not fully taken into account in the time-dependent MSE in the presence of currents. Other components that refer to changes of the ambient current or combinations of the bottom slope and the current were neglected as well. These components are needed in order to yield a more accurate model for areas with strong currents and rapidly changing ones such as near-shore regions and inlets. In order to further increase the accuracy of the MSE models in these cases, a set of evanescent modes may as well be added (e.g. Massel 1993; Belibassakis 2007; Belibassakis *et al.* 2008 for the cases without or with ambient currents, respectively).

Another simplification for this flow problem is the balance wave-action equation (WAE; e.g. Komen *et al.* 1994; Young 1999). This equation is a stochastic (phase-averaged) time-dependent frequency domain (or wavenumber domain) equation that models the action flow, which is classically defined as the energy divided by the wave's intrinsic frequency. It does not present the chaotic behaviour of wave-forecasting problems as deterministic models are used (see Annenkov & Shrira 2001), and leaps over the Nyquist–Shannon limitation, which enables a solution for large domains.

The classical WAE in the presence of currents was derived using a variational approach by Bretherton & Garrett (1968). Willebrand (1975) has extended the wave-propagation part of this formulation by nonlinearity (i.e. the wave rays are dependent on each other), which is harder in its application. The interest in extensions of the WAE has grown recently, as spectral wave models are coupled to circulations models in order to take into account wave-induced currents and effects of the currents on the waves (e.g. Ardhuin *et al.* 2005, 2008; Mellor 2008). Furthermore, these wave-forecasting models are extended towards the coasts, where steep slopes and strong currents are present (e.g. Uchiyama *et al.* 2010).

Bretherton & Garrett's WAE was re-derived using the time-dependent MSE by Jonsson (1981) and Kirby (1984). This relation consisted of neglecting the low-order refraction–diffraction component of the MS-type equation. Liu (1990) used the same method without neglecting this component, which resulted in an improved WAE. In this work, Liu accurately handled the changes in the Doppler shift of the waves resulting from the improved formulation. Still, his derivation was not extended to the frequency-direction spectrum that should allow its use in wave-forecasting models. In addition, higher order bottom components, which in this time were thought to be insignificant, were neglected together with the higher order current components.

Applications of improved frequency-direction spectrum WAEs without an ambient flow were presented by Mase (2001) and Holthuijsen *et al.* (2003) using the parabolic approximation of Berkhoff's time-harmonic MSE and the same MSE without this approximation, respectively. Both presented good improvements in their numerical results for steady problems. Still, because of the use of a time-harmonic equation, their models are less appropriate for the time-dependent case. Furthermore, higher order bottom and current components were not taken into account in their derivations.

The case of waves in the presence of ambient currents was discussed in the appendix of Holthuijsen *et al.* (2003). In their derivation, the wave properties, which were calculated using Berkhoff's time-harmonic MSE with no currents, were substituted into Bretherton & Garrett's equation. This results in an inconsistent model, because Holthuijsen *et al.*'s improved wave properties were not derived for waves in the presence of currents. A further advancement in this field was done by Hsu *et al.* (2006*a*), where the time-harmonic extended MSE in the presence of currents was used as the basic function. In this work, the higher bottom changes were taken into account, but not the higher order terms related to the current. Both works did not take into account the changes in the Doppler shift as in Liu (1990). Furthermore, starting from a time-harmonic equation is appropriate only for steady wave problems.

The aim of the present study is to construct in terms of linear wave theory an extended MSE and WAE that have an improved behaviour for rapid spatial bottom changes as well as ambient current changes both in time and space. The paper is constructed as follows: a derivation of an extended MSE for time-dependent waves with high-order components in the presence of an ambient current is given in §2; the new MS-type equation is used to construct an extended deterministic WAE in §3, and its stochastic frequency-direction spectrum version is derived in §4. Numerical results are presented in §5, and the work is summarized and discussed in §6.

2. The extended mild-slope equation for wave–current interactions

(*a*) *The formulation of the Hamiltonian under vertical profile assumptions*

In this section, the time-dependent MSE of Booij (1981) and Kirby (1984) is extended to contain higher order bottom profile and ambient current terms. The derivation in this section is given from first principles using a variational approach in the same manner as Dingemans (1997, §3.2). The Hamiltonian describing the irrotational flow of an incompressible inviscid fluid with a free surface is

$$\mathcal{H} = \iint H \,\mathrm{d}x\,\mathrm{d}y = \iint (V + T)\mathrm{d}x\,\mathrm{d}y, \tag{2.1}$$

$$V = \frac{1}{2}\rho g \eta^2 \tag{2.2}$$

and

$$T = \frac{1}{2}\rho \int_{-h}^{\eta} [|\nabla \Phi|^2 + (\Phi_z)^2]\mathrm{d}z. \tag{2.3}$$

Here, V is the potential energy density; T is the kinetic energy density; $\Phi(x, y, z, t)$ is the wave velocity potential; $\eta(x, y, t)$ is the free-surface displacement; $h(x, y)$ is the bottom profile; ρ is the density; ∇ is the horizontal gradient operator; and the subscripts denote partial derivatives. The vertical coordinate z of the Cartesian coordinate system $Oxyz$ directs upward with the Oxy-plane located on the still-water surface. The evolution equations for the free-surface elevation $\eta(x, t)$ and the velocity potential at the free surface, $\varphi(x, y, t) = \Phi(x, y, \eta, t)$, are given by Hamilton's canonical equations,

$$\rho \frac{\partial \eta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \varphi} \tag{2.4}$$

and

$$\rho \frac{\partial \varphi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \eta}. \tag{2.5}$$

It is important to note that equations (2.1)–(2.5) describe potential motions only. For water waves, potentiality is a reasonable approximation, which is widely used. Ambient currents, on the other hand, are restricted by this formulation, as realistic currents usually consist of a small vorticity component. Nevertheless, consistent rigorous analysis of such an idealized model would allow us to gauge the importance of effects currently not taken into account in the models described in the literature. Demonstration of the importance of currently neglected terms for this particular model would imply the necessity of incorporating them into all models. Moreover, application of the results to vertical currents might also prove to be plausible (see discussion in §6).

In order to write separate equations for the oscillatory flow and the mean flow, the free-surface elevation and the velocity potential are divided into two parts (see Chu & Mei 1970)—a current and a wave part. The mean current is assumed to be varying slowly on wavelength scale, denoted by μx and μy, and the oscillatory part is restricted to small wave steepnesses (ε). This can be written in the following manner:

$$\eta(\mathbf{x}, t) = \gamma \eta_0(\mu x, \mu y, \mu t) + \varepsilon \eta_1(x, y, t) + O(\varepsilon^2) \tag{2.6}$$

and

$$\Phi(\mathbf{x}, z, t) = \gamma \mu^{-1} \phi_0(\mu x, \mu y, \mu t) + \varepsilon f(z, x, y) \phi_1(x, y, t) + O(\varepsilon^2). \tag{2.7}$$

Here, the component that is denoted by the subscript '0' is dominated by the ambient current, and the component that is denoted by the subscript '1' describes the oscillatory part of the motion. In equation (2.6), the assumption of a two-dimensional mean current flow is incorporated, whereas in equation (2.7), a known vertical profile is assumed for the wave component of Φ. The latter is a common assumption of the MS-type equations, where the vertical profile's dependence on the horizontal coordinates is weak through functions such as the wavenumber k and the bottom profile h. Three scaling parameters are presented in equations (2.6) and (2.7): Stokes's wave steepness parameter, $\varepsilon = O(ka)$, where a is the wave amplitude and the rate of change of the depth over the non-dimensional depth $\mu = O(\nabla h / kh)$. These parameters are assumed to be small ($\varepsilon \ll 1$, $\mu \ll 1$), whereas the ambient current and its surface elevation are allowed to be of any order γ.

Current velocities are usually smaller or comparable with wave orbital velocities, but these two flow types have a fundamental difference. Within the typical time order of surface waves, the ambient current can be regarded as a steady flow. Therefore, the decoupling of the two types of flows are not related to different orders of magnitudes but rather to different Fourier components (e.g. the decoupling of Voronovich 1976). The ambient flow is taken as a mean component, whereas the linear wave is taken as an oscillating component. The ordering using the steepness small parameter ε relates only to the oscillating part of the derivation and not to the ambient current. The assumption of a small steepness is used in this derivation for restricting the formulation for linear waves and neglecting the nonlinear wave interactions, hence the ambient current velocity can be of any magnitude with respect to the wave orbital velocities. This allows us to omit γ for simplicity. Nevertheless, the current formulation of surface water waves does not hold the physics of waves propagating over a hydraulic jump, which occurs in the transition between subcritical and supercritical flow types. Still, in oceanic flows, the ambient currents are subcritical, so for these types of problems, there is no need for modelling the transitional flow regions, and the flow hereinafter is assumed to be subcritical.

Following equations (2.6) and (2.7), Hamilton's canonical equations (2.4) and (2.5) remain valid for the two parts (see Dingemans 1997, §3.2),

$$\rho\frac{\partial\eta_j}{\partial t}=\frac{\delta\mathcal{H}}{\delta\phi_j},\quad j=1,2, \tag{2.8}$$

and

$$\rho\frac{\partial\phi_j}{\partial t}=-\frac{\delta\mathcal{H}}{\delta\eta_j},\quad j=1,2. \tag{2.9}$$

Notice that in equation (2.9), the vertical profile f should be normalized to 1 at the free surface.

Substituting equations (2.6) and (2.7) into equations (2.2) and (2.3) yields

$$V=\tfrac{1}{2}\rho g(\eta_0^2+2\varepsilon\eta_0\eta_1+\varepsilon^2\eta_1^2) \tag{2.10}$$

and

$$T=\frac{1}{2}\rho\int_{-h}^{\eta_0+\varepsilon\eta_1}[|\nabla\phi_0|^2+2\varepsilon\nabla\phi_0\cdot\nabla(f\phi_1)+\varepsilon^2|\nabla(f\phi_1)|^2+\varepsilon^2 f_z^2\phi_1^2]\mathrm{d}z. \tag{2.11}$$

Applying Taylor's series around η_0 to equations (2.10) and (2.11) up to the order of $O(\varepsilon^2)$ and substituting the result into equation (2.1) allows the Hamiltonian density within a linear flow work-frame to be written in the following manner:

$$H=H_0+\varepsilon H_1+\varepsilon^2 H_2, \tag{2.12}$$

$$H_0=\frac{1}{2}\rho[g\eta_0^2+(h+\eta_0)|\nabla\phi_0|^2], \tag{2.13}$$

$$H_1=\rho\left[g\eta_0\eta_1+\frac{1}{2}|\nabla\phi_0|^2\eta_1+(\nabla\phi_0\cdot\nabla\phi_1)\int_{-h}^{\eta_0}f\,\mathrm{d}z+\phi_1\nabla\phi_0\cdot\int_{-h}^{\eta_0}\nabla f\,\mathrm{d}z\right] \tag{2.14}$$

$$\text{and}\quad H_2 = \frac{1}{2}\rho\left[g\eta_1^2 + |\nabla\phi_1|^2\int_{-h}^{\eta_0} f^2 \mathrm{d}z + \phi_1^2\int_{-h}^{\eta_0} f_z^2\,\mathrm{d}z + \phi_1^2\int_{-h}^{\eta_0}|\nabla f|^2\mathrm{d}z \right.$$
$$\left. + 2\phi_1\nabla\phi_1\cdot\int_{-h}^{\eta_0} f\nabla f\,\mathrm{d}z + 2\eta_1(\nabla\phi_0\cdot\nabla\phi_1)f|_{z=\eta_0} + 2\eta_1\phi_1\nabla\phi_0\cdot\nabla f|_{z=\eta_0}\right]. \tag{2.15}$$

The separation of the Hamiltonian density given in equations (2.12)–(2.15) enables the inspection of each of its parts separately. As only linear instances of η_1 and ϕ_1 occur in equation (2.14), the horizontal integration of H_1 0s (see Kirby 1984 for the Lagrangian formulation and Dingemans 1997, §3.2 for the Hamiltonian one). Hence, H_1 will be neglected from this stage.

The vertical profile for the oscillatory part is taken as one of the analytical solution for linear waves propagating over a flat bottom,

$$f(z,k,h,\eta_0) = \frac{\cos[k(h+z)]}{\cos[k(h+\eta_0)]}, \tag{2.16}$$

with

$$\sigma^2 = gk\tanh k(h+\eta_0), \tag{2.17}$$

$$\omega = \sigma + \mathbf{k}\cdot\mathbf{U} \tag{2.18}$$

and

$$\mathbf{U} \equiv \nabla\phi_0. \tag{2.19}$$

Here, $\mathbf{k}$ is the wavenumber vector $(k_1,k_2)^{\mathrm{T}}$, and the wavenumber k is given by $|\mathbf{k}|$. ω is the absolute wave frequency for harmonic waves, and σ is the intrinsic frequency, which relates to ω through the Doppler shift (2.18), and allows the calculation of the wavenumber k through the linear dispersion relation (2.17). In addition, the vertical profile given in equation (2.16) is normalized to 1 on $z=\eta_0$. Note that the definition given in equation (2.16), which follows Dingemans (1997, §3.2) takes into account both the bottom profile and the current-induced surface elevation. It retains its accuracy also for current-induced surface elevation of $O(1)$.

Applying equation (2.16) to equation (2.15) yields

$$H_2 = \tfrac{1}{2}\rho[g\eta_1^2 + q_1|\nabla\phi_1|^2 + q_2\phi_1^2 + q_3\phi_1^2 + 2\phi_1\nabla\phi_1\cdot\mathbf{q}_4$$
$$+ 2\eta_1(\mathbf{U}\cdot\nabla\phi_1) + 2\eta_1\phi_1\mathbf{U}\cdot\mathbf{q}_5], \tag{2.20}$$

$$q_1 = \int_{-h}^{\eta_0} f^2\,\mathrm{d}z = \frac{1}{g}cc_{\mathrm{g}}, \tag{2.21}$$

$$q_2 = \int_{-h}^{\eta_0} f_z^2\,\mathrm{d}z = \frac{1}{g}(\sigma^2 - k^2cc_{\mathrm{g}}), \tag{2.22}$$

$$q_3 = \int_{-h}^{\eta_0}|\nabla f|^2\,\mathrm{d}z,\quad \mathbf{q}_4 = \int_{-h}^{\eta_0} f\nabla f\,\mathrm{d}z \tag{2.23}$$

and

$$\mathbf{q}_5 = \nabla f|_{z=\eta_0} = -\mu\frac{\sigma^2}{g}\nabla\eta_0. \tag{2.24}$$

Here, c and c_g are the phase and group velocities, respectively. Both are calculated using the total water depth $\bar{h} = (h + \eta_0)$. Coefficients q_1, q_2, q_3, $\mathbf{q}_4$ and $\mathbf{q}_5$ are given in more detail in appendix A. Equations (2.8) and (2.9) can now be used together with equations (2.1), (2.12), (2.13) and (2.20) to construct two sets of canonical equations describing the coupled ambient current flow and the oscillatory wave component.

(b) The canonical equations for the ambient current

The equations governing the motion of the ambient flow can be formulated by writing the canonical equations (2.8) and (2.9) for $j = 1$. Upon neglecting terms of order $O(\varepsilon^2)$, we arrive at the usual shallow water equations in potential formulation,

$$\frac{\partial \eta_0}{\partial t} + \mu \nabla \cdot [(\mu^{-1} h + \eta_0) \nabla \phi_0] = 0 \tag{2.25}$$

and

$$\frac{\partial \phi_0}{\partial t} + \frac{1}{2} |\nabla \phi_0|^2 + g\eta_0 = 0. \tag{2.26}$$

Equation (2.25) is the vertically integrated continuity equation for the ambient current, and the gradient of equation (2.26) yields two momentum equations for the current. In order to account for wave-induced currents, which arise from linear wave motion, terms of $O(\varepsilon^2)$ should be retained. In this case, the canonical equations take the form

$$\frac{\partial \eta_0}{\partial t} + \nabla \cdot [(h + \eta_0) \nabla \phi_0] = \varepsilon^2 \nabla \cdot (\eta_1 \nabla \phi_1 + \eta_1 \phi_1 \mathbf{q}_5) \tag{2.27}$$

and

$$\begin{aligned} \frac{\partial \phi_0}{\partial t} + \frac{1}{2} |\nabla \phi_0|^2 + g\eta_0 = -\frac{1}{2} \varepsilon^2 [p_1 |\nabla \phi_1|^2 + p_2 \phi_1^2 + p_3 \phi_1^2 - \nabla \cdot (\mathbf{p}_4 \phi_1^2) \\ + 2\phi_1 \nabla \phi_1 \cdot \mathbf{p}_5 - 2\nabla \cdot (\phi_1 P_6 \nabla \phi_1)], \end{aligned} \tag{2.28}$$

where

$$p_1 = \frac{\partial q_1}{\partial \eta_0}, \quad p_2 = \frac{\partial q_2}{\partial \eta_0}, \quad p_3 = \frac{\partial q_3}{\partial \eta_0}, \tag{2.29}$$

$$\mathbf{p}_4 = \frac{\partial q_3}{\partial \nabla \eta_0}, \quad \mathbf{p}_5 = \frac{\partial \mathbf{q}_4}{\partial \eta_0} \quad \text{and} \quad P_6 = \frac{\partial \mathbf{q}_4}{\partial \nabla \eta_0}, \tag{2.30}$$

and coefficients q_1, q_2, q_3, $\mathbf{q}_4$ and $\mathbf{q}_5$ are given in appendix A.

(c) The canonical equations for the oscillatory component

The equations governing the motion of the wave component are formulated by writing the canonical equations (2.8) and (2.9) for $j = 2$,

$$\frac{\partial \eta_1}{\partial t} = (q_2 + q_3)\phi_1 + \nabla \phi_1 \cdot \mathbf{q}_4 + \eta_1 \mathbf{U} \cdot \mathbf{q}_5 - \nabla \cdot [q_1 \nabla \phi_1 + \mathbf{U} \eta_1 + \phi_1 \mathbf{q}_4] \tag{2.31}$$

and

$$\frac{\partial \phi_1}{\partial t} = -g\eta_1 - \mathbf{U} \cdot \nabla \phi_1 - \phi_1 \mathbf{U} \cdot \mathbf{q}_5. \tag{2.32}$$

Equations (2.31) and (2.32) can be rewritten as

$$\frac{D\eta_1}{Dt} = (q_2 + q_3)\phi_1 + \nabla\phi_1 \cdot \mathbf{q}_4 + \eta_1 \mathbf{U} \cdot \mathbf{q}_5 - \nabla \cdot [q_1 \nabla\phi_1 + \phi_1 \mathbf{q}_4] + \eta_1 \nabla \cdot \mathbf{U} \tag{2.33}$$

and

$$\frac{D\phi_1}{Dt} = -g\eta_1 - \phi_1 \mathbf{U} \cdot \mathbf{q}_5, \tag{2.34}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla. \tag{2.35}$$

The aim of this section is to formulate a single equation that governs the flow of the wave component. This is done as follows: a material derivative is applied to equation (2.34), the term $D\eta_1/Dt$ is substituted using equation (2.33), and instances of η_1 are eliminated using the original form of equation (2.34). This results in the following equation:

$$\begin{aligned}\frac{D^2\phi_1}{Dt^2} + (\nabla \cdot \mathbf{U})\frac{D\phi_1}{Dt} - \nabla \cdot (gq_1 \nabla\phi_1) + \Bigg[g(q_2 + q_3 - \nabla \cdot \mathbf{q}_4) \\ + \frac{D(\mathbf{U} \cdot \mathbf{q}_5)}{Dt} + (\mathbf{U} \cdot \mathbf{q}_5 - \nabla \cdot \mathbf{U})\mathbf{U} \cdot \mathbf{q}_5 \Bigg] \phi_1 = 0,\end{aligned} \tag{2.36}$$

which after substitution of equations (2.21), (2.22) and (2.24) yields

$$\frac{D^2\phi_1}{Dt^2} + (\nabla \cdot \mathbf{U})\frac{D\phi_1}{Dt} - \nabla \cdot (cc_{\mathrm{g}} \nabla\phi_1) + [\sigma^2 - k^2 cc_{\mathrm{g}} + R]\phi = 0 \tag{2.37}$$

with

$$R = g(q_3 - \nabla \cdot \mathbf{q}_4) - \frac{\mu^2\sigma^2}{g}\left(\frac{D(\mathbf{U} \cdot \nabla\eta_0)}{Dt} + (\mathbf{U} \cdot \nabla\eta_0)^2 - (\nabla \cdot \mathbf{U})\mathbf{U} \cdot \nabla\eta_0\right). \tag{2.38}$$

Equation (2.37), including the function R, is the contribution of this section. Neglecting R leads to the equation of Kirby (1984).

The first term of equation (2.38) can be written in the following form:

$$\begin{aligned}g(q_3 - \nabla \cdot \mathbf{q}_4) = r_1 \nabla^2 h + r_2 |\nabla h|^2 + r_3 \nabla h \cdot \nabla\eta_0 + r_4 \nabla h \cdot \nabla\sigma + r_5 |\nabla\sigma|^2 \\ + r_6 |\nabla\eta_0|^2 + r_7 \nabla\eta_0 \cdot \nabla\sigma + r_8 \nabla^2\sigma + r_9 \nabla^2\eta_0,\end{aligned} \tag{2.39}$$

where $r_i, i = 1, \ldots, 9$, are coefficients that depend on the wavenumber k and on the total water depth $\bar{h} = (h + \eta_0)$ and may also depend on the current velocity (figures 1 and 2). Setting the current to 0 while retaining r_1 and r_2, together with the assumption of a time-harmonic flow, reduces equation (2.37) to the modified

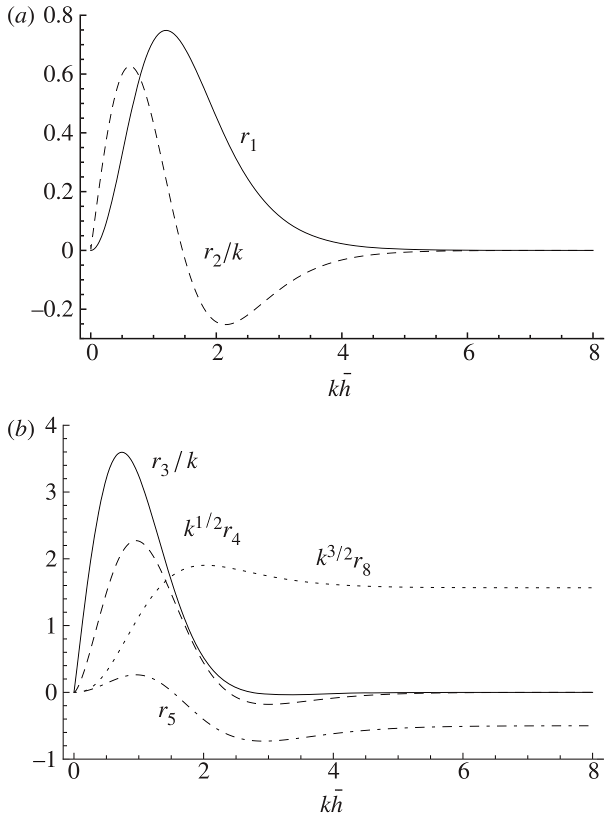

Figure 1. The high-order coefficients with respect to the total water depth $k\bar{h} = k(h + \eta_0)$. (*a*) r_1/k (solid) and r_2 (dashed). (*b*) r_3/k (solid), $\sqrt{k}r_4$ (dashed), r_5 (dotted-dashed) and $k^{3/2}r_8$ (dotted).

MSE by Chamberlain & Porter (1995), as well as to the MSE by Massel (1993), when the evanescent modes are neglected. When an ambient current is present in the equation, these coefficients improve the one-equation of Belibassakis *et al.* (2008) by taking into account the changes in the surface level owing to the ambient current and the changes of the intrinsic frequency owing to changes in the Doppler shifts.

Note that coefficients $r_1, \ldots, r_4$ are reduced to 0 in deep water. This is not surprising as these coefficients relate directly to the bottom slope and curvature, which should have an effect only in shallow to intermediate water depths. Coefficients $r_5, \ldots, r_9$ and the second part of equation (2.38) are significant in any water depth and should allow a better prediction of the effects of the ambient current on the waves.

3. The extended wave-action equation

In this section, a deterministic WAE will be derived from the extended time-dependent MSE (2.37). This will be done while taking into account changes in the intrinsic frequency and the wavenumber rather than using the more simplistic

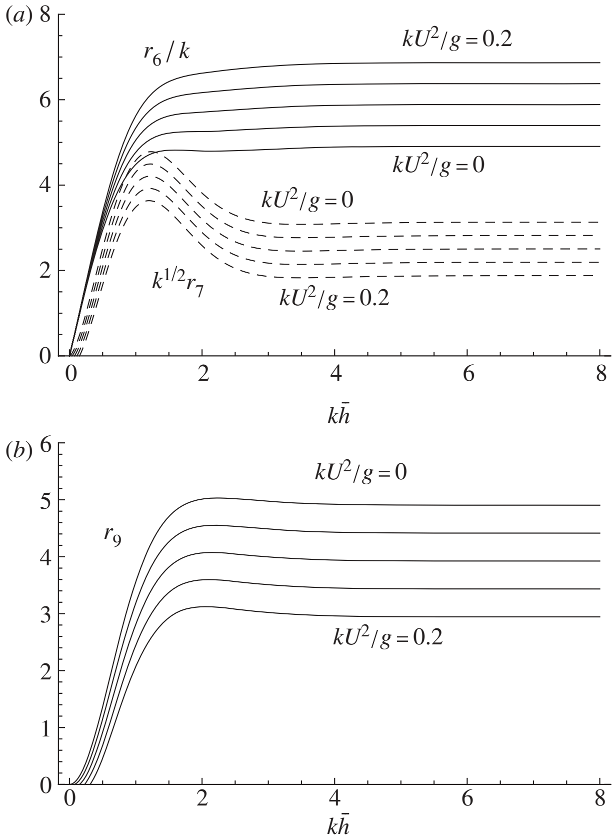

Figure 2. High-order coefficients with respect to the total water depth $k\bar{h} = k(h + \eta_0)$. (*a*) r_6/k (solid) and $\sqrt{k}r_7$ (dashed). (*b*) r_9 (solid). The coefficients are plotted for the non-dimensional values $kU^2/g = 0, 0.05, 0.1, 0.15, 0.2$.

ray theory definitions. This approach, which was given by Kostense *et al.* (1988) (see also a short discussion given in Dingemans 1997, §3.2), extends the one that was used by Jonsson (1981), Kirby (1984), Dingemans (1985), Mase (2001), Holthuijsen *et al.* (2003) and Hsu *et al.* (2006*a*). Here, it is followed with a minor correction.

Let us apply a wave-like structure to the velocity potential

$$\phi_1 = \mathrm{Re}\left\{B(\mu x, \mu y, \mu t)\exp\left[\frac{\mathrm{i}}{\mu}S(\mu x, \mu y, \mu t)\right]\right\}. \tag{3.1}$$

Here, both B and S are real functions, which correspond to the velocity potential amplitude and phase, respectively. For simplicity, from this point, the subscript of the wave component of the velocity potential will be omitted (i.e. $\phi_1 \to \phi$).

Substituting equation (3.1) into equation (2.37) and multiplying by $-B\mathrm{e}^{-\mathrm{i}\mu^{-1}S}$ yields, after some manipulations,

$$\begin{aligned}&-\left\{\mu^2\frac{D^2B}{Dt^2}+\mu(\nabla\cdot\mathbf{U})\frac{DB}{Dt}-\mu\nabla(cc_\mathrm{g})\cdot\nabla B-\mu^2cc_\mathrm{g}\nabla^2B\right.\\&\left.+\left[\sigma^2-\left(\frac{DS}{Dt}\right)^2+(|\nabla S|^2-k^2)cc_\mathrm{g}+R\right]B\right\}B\\&+\mathrm{i}\mu\frac{\partial}{\partial t}\left[\left(-\frac{DS}{Dt}\right)B^2\right]+\mathrm{i}\mu\nabla\cdot\left\{\left[cc_\mathrm{g}\nabla S+\left(-\frac{DS}{Dt}\right)\mathbf{U}\right]B^2\right\}=0.\end{aligned}\tag{3.2}$$

In order to satisfy equation (3.2), both the real and the imaginary parts should be 0 independently. Solving the real part for $|\nabla S|^2$ yields the following eikonal equation:

$$\begin{aligned}K^2=|\nabla S|^2=k^2+\frac{1}{cc_\mathrm{g}}(\Sigma^2-\sigma^2-R)-\frac{1}{Bcc_\mathrm{g}}\\\times\left[\mu^2\frac{D^2B}{Dt^2}+\mu(\nabla\cdot\mathbf{U})\frac{DB}{Dt}-\mu\nabla(cc_\mathrm{g})\cdot\nabla B-\mu^2cc_\mathrm{g}\nabla^2B\right],\end{aligned}\tag{3.3}$$

whereas the imaginary part yields the following transport equation:

$$\frac{\partial}{\partial t}(\Sigma B^2)+\nabla\cdot[(cc_\mathrm{g}\mathbf{K}+\Sigma\mathbf{U})B^2]=0.\tag{3.4}$$

Here, an ‘effective’ wavenumber vector was defined as

$$\mathbf{K}=\nabla S,\tag{3.5}$$

and by considering monochromatic progressive waves, i.e.

$$\frac{\partial S}{\partial t}=-\omega,\tag{3.6}$$

an ‘effective’ intrinsic frequency is given by

$$\Sigma=-\frac{DS}{Dt}=\omega-\mathbf{K}\cdot\mathbf{U}.\tag{3.7}$$

Applying new wave-action and wave-action velocity definitions,

$$N=\Sigma B^2\tag{3.8}$$

and

$$\mathbf{V}_\mathrm{g}=cc_\mathrm{g}\frac{\mathbf{K}}{\Sigma},\tag{3.9}$$

allows equation (3.4) to be written in a simplified manner as

$$\frac{\partial N}{\partial t}+\nabla\cdot\{[\mathbf{V}_\mathrm{g}+\mathbf{U}]N\}=0.\tag{3.10}$$

Equation (3.10), together with equations (3.3) and (3.5)–(3.9), forms the deterministic extended WAE (EWAE) model in the xy-space for the velocity potential amplitude. Note that equation (3.10) can be used as well as a stochastic phase-averaged equation because it contains no information on the phasing. A description of this model in terms of the surface elevation amplitude is given in appendix B.

Let us inspect the relations between the EWAE model and former representations. By neglecting both low- and high-order refraction/diffraction terms of the EWAE, i.e. setting $\mathbf{K}=\mathbf{k}$ (and therefore also $\Sigma=\sigma$), the extended model reduces to the one of Bretherton & Garrett (1968). This means that all diffraction effects are neglected, and only lower order refraction effects are retained for very slow bottom–current variations. The case of no currents, where the higher order term R is also neglected, reduces the EWAE model for steady problems to the one of Holthuijsen *et al.* (2003). This case is also similar to the model of Mase (2001). Mase also used Berkhoff's MSE for the derivation of the wavenumber. For the transport equation, a parabolic approximation to the MSE was applied, assuming that the waves were propagating mainly in a prescribed direction.

In the presence of currents, the derivation given in the appendix of Holthuijsen *et al.* (2003) substitutes the new wave velocity into the transport equation of Bretherton & Garrett (1968) with no rigorous derivation. This results in an inconsistent model because Holthuijsen *et al.*'s effective wavenumber does not depend on currents. This issue was resolved by Hsu *et al.* (2006*a*) with the addition of two high-order terms relating to the bottom curvature and slope (r_1 and r_2, respectively). Still, in both cases, the starting equation was a time-harmonic one (i.e. accurate only for steady problems), and did not contain the Doppler effect of the effective wavenumber and wave direction (i.e. setting $\Sigma=\sigma$, even though $\mathbf{K}\neq\mathbf{k}$). In addition, both of these models did not use a vertical profile that depends on the surface elevation changes owing to the ambient current and neglected other high-order terms that relate to the ambient current.

4. The extended wave-action equation in a frequency-direction spectrum

In this section, the EWAE derived in §3 will be transformed to fit stochastic phase-averaged wave models with a frequency-direction spectrum. In order to do so, the action advection velocities should be given, as well in the additional Σ and θ coordinates representing the frequency-shifting owing to the Doppler effect and the turning rate of the wave component. The wave-action definition $N(x,y,t)$ will be substituted by a sum of wave actions $\mathcal{N}(\theta,\Sigma,x,y,t)$ for each frequency and direction.

Writing equation (3.10) for a single frequency-direction component $\mathcal{N}$ reads

$$\frac{\partial\mathcal{N}}{\partial t}+\frac{\partial}{\partial x}(C_x\mathcal{N})+\frac{\partial}{\partial y}(C_y\mathcal{N}) \\ +\frac{\partial C_\theta}{\partial\theta}\mathcal{N}+\frac{\partial C_\Sigma}{\partial\Sigma}\mathcal{N}+\frac{\partial\mathcal{N}}{\partial\theta}C_\theta+\frac{\partial\mathcal{N}}{\partial\Sigma}C_\Sigma=0, \tag{4.1}$$

where

$$[C_x, C_y, C_\theta, C_\Sigma] = \left[\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}\theta}{\mathrm{d}t}, \frac{\mathrm{d}\Sigma}{\mathrm{d}t}\right]. \tag{4.2}$$

Equation (4.1) is the action balance equation on a four-dimensional infinitesimal volume in time. Its last two terms represent the chain rule derivatives of θ and Σ with respect to time resulting from the first term of equation (3.10), whereas the two previous terms represent the flux balance of the four-dimensional infinitesimal volume in the θ and Σ directions. A more detailed derivation can be found in Rasmussen (1998), appendix E.

After a minor simplification, the WAE for the frequency-direction spectrum takes the form

$$\frac{\partial \mathcal{N}}{\partial t} + \frac{\partial}{\partial x}(C_x \mathcal{N}) + \frac{\partial}{\partial y}(C_y \mathcal{N}) + \frac{\partial}{\partial \theta}(C_\theta \mathcal{N}) + \frac{\partial}{\partial \Sigma}(C_\Sigma \mathcal{N}) = 0, \tag{4.3}$$

where the wave action $\mathcal{N}$ is classically defined as $\langle A^2\rangle/\sigma$, with $\langle\cdots\rangle$ denoting the ensemble average operator; Σ classically collapses to σ and θ is the angle of the wave propagation in the counter-clockwise direction starting from the x-axis (e.g. Komen *et al.* 1994). In our case, the phase-averaged wave action is defined by averaging equation (B 9) in appendix B as

$$\mathcal{N} = \frac{\langle A^2\rangle}{\Sigma}. \tag{4.4}$$

This wave-action definition resembles the one of Bretherton & Garrett (1968), but with the more accurate 'effective intrinsic frequency' (3.7). These wave-action definitions, which relate to the ratio between the energy and the intrinsic frequency, are valid for linear waves. For a problem without a small wave steepness assumption, a different wave-action definition should result.

The horizontal advection speeds can be easily written from equation (3.10) as

$$C_x = V_\mathrm{g}\cos\theta + U_1 \quad \text{and} \quad C_y = V_\mathrm{g}\sin\theta + U_2, \tag{4.5}$$

where U_1 and U_2 denote the x- and y-components of the ambient current vector $\mathbf{U}$, respectively; $V_\mathrm{g} = |\mathbf{V}_\mathrm{g}|$ and θ defines the direction of $\mathbf{K}$ as seen in figure 3.

The relation between the local coordinates along the wave crest and the wave direction to the x and y Cartesian coordinates (figure 3) are given as

$$x = n\cos\theta - m\sin\theta \tag{4.6}$$

and

$$y = n\sin\theta + m\cos\theta, \tag{4.7}$$

which yield their derivative relations as

$$\frac{\partial}{\partial n} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \tag{4.8}$$

and

$$\frac{\partial}{\partial m} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}. \tag{4.9}$$

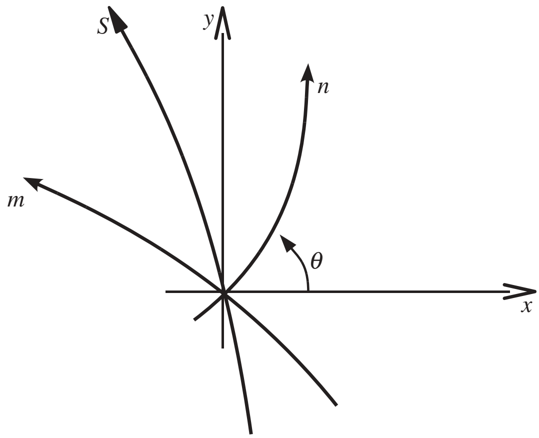

Figure 3. A diagram for the curved coordinate system: m is the coordinate along the wave crest, n is the coordinate normal to the wave crest and s is the coordinate along the streamline of the ambient currents.

The turning ratio owing to the bottom and current changes is given as (e.g. Holthuijsen 2007)

$$C_\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial\theta}{\partial n}\frac{\partial n}{\partial t} - \frac{\partial U_n}{\partial m}. \tag{4.10}$$

Here, U_n is the projection of the current vector on the normal to the wave crest given by the effective wave direction as

$$U_n = \mathbf{U}\cdot\frac{\mathbf{K}}{K}, \tag{4.11}$$

and m is the coordinate in the direction of the wave crest. The $\hat{z}$-component of the curl of equation (3.5), together with definitions (4.8) and (4.9), yields the following relation:

$$\frac{\partial\theta}{\partial n} = \frac{1}{K}\frac{\partial K}{\partial m}. \tag{4.12}$$

Substituting equation (4.12) and the propagation velocity of the wave front with equation (4.10) gives the turning speed as

$$C_\theta = \frac{V_\mathrm{g}}{K}\frac{\partial K}{\partial m} - \frac{\partial U_n}{\partial m}. \tag{4.13}$$

Note that C_θ given in equation (4.13) resembles its classical definition, but uses the new properties of the wave-action speed, wavenumber and wave direction.

The definition of the propagation speed in the effective frequency space owing to the Doppler shifting is given as

$$C_\Sigma = \frac{\mathrm{d}\Sigma}{\mathrm{d}t} = \frac{\partial\Sigma}{\partial t} + \frac{\partial\Sigma}{\partial n}\frac{\partial n}{\partial t} + \frac{\partial\Sigma}{\partial s}\frac{\partial s}{\partial t}. \tag{4.14}$$

Here, s is the direction of the stream line of the ambient current $\mathbf{U}$. Taking the time derivative of equation (3.5) and adding it to the horizontal gradient of equation (3.6) yields the conservation of crests relation for the effective wavenumber,

$$\frac{\partial \mathbf{K}}{\partial t} + \nabla \omega = 0. \tag{4.15}$$

The definition of Σ given in equation (3.7) is applied to equation (4.14), while the derivatives of ω are substituted using the relation (4.15). This yields, after some manipulations, the propagation speed in the frequency space owing to the Doppler shifting as

$$C_\Sigma = -\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s} + V_{\mathrm{g}}\frac{\partial}{\partial n}\right)(KU_n) - (\mathbf{V}_{\mathrm{g}} + \mathbf{U}) \cdot \frac{\partial \mathbf{K}}{\partial t}, \tag{4.16}$$

where

$$\frac{\partial}{\partial s} = \frac{U_1}{U}\frac{\partial}{\partial x} + \frac{U_2}{U}\frac{\partial}{\partial y} \tag{4.17}$$

and

$$U = |\mathbf{U}|. \tag{4.18}$$

The propagation speed C_Σ given in equation (4.16) is different to the classical C_σ, as in the derivation of the latter, σ depends only on k and h through the dispersion relation and this dependency is used to simplify the relation. The effective intrinsic frequency Σ relates to σ through equations (3.7) and (2.18) in the following manner:

$$\Sigma = \sigma + (\mathbf{k} - \mathbf{K}) \cdot \mathbf{U}. \tag{4.19}$$

In addition to the dependency on k and h, equation (4.19) gives a more complex relation between Σ and other wave properties through the effective wavenumber definition (3.5).

5. Numerical results

In this section, numerical calculations are presented in order to check the significance of the high-order component R. The calculation inspects the class I reflection coefficients for various monochromatic waves propagating over an undulating bottom. For the numerical integration, the NDSolve function of the Mathematica 8 software was used.

The bottom profile is taken as in the experiment of Magne *et al.* (2005),

$$h(x) = \begin{cases} 1.9 - \frac{4}{15}(x-5), & 5 \le x < 8, \\ 1.5 - 0.4\cos\left(\dfrac{2\pi}{2.5}(x-8)\right), & 8 \le x < 18, \\ 1.1 + \frac{4}{15}(x-18), & 18 \le x < 21, \\ 1.9, & \text{elsewhere}, \end{cases}$$

as well as the mean ambient current in the flat regions, which is given as $U = 0.4\,\mathrm{m\,s^{-1}}$.

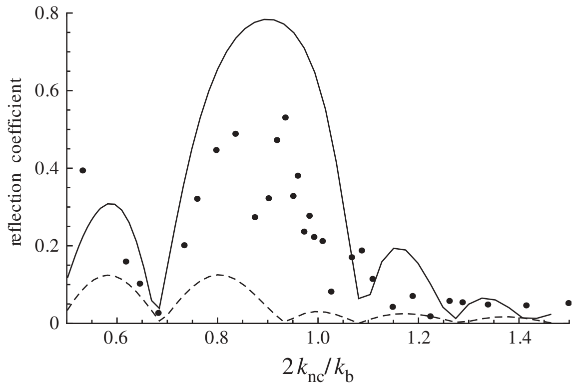

Figure 4. Class I Bragg reflection coefficients in the presence of a mean ambient current ($U = 0.4\,\mathrm{m\,s^{-1}}$). The bottom wavenumber k_b is $2\pi/2.5\,\mathrm{m}$ and k_{nc} is the wavenumber on a mean bottom level ($h = 1.5\,\mathrm{m}$) with no current. Solid line represents the present deterministic model—equations (3.10) and (3.3); dashed line represents Kirby (1984) (i.e. equations (3.10) and (3.3) with $R = 0$); solid circles denote experimental results of Magne *et al.* (2005).

Equation (3.10), together with equation (3.3), was solved and compared with the equation by Kirby (1984) (i.e. neglecting R) and with the experimental results of Magne *et al.* (2005). The results of the above reflection coefficients are presented in figure 4 with respect to $2k_{nc}/k_b$. Here, $k_b = 2\pi/2.5\,\mathrm{m}$, and k_{nc} is the wavenumber of the mean bottom level ($h = 1.5\,\mathrm{m}$) without currents. It can be easily seen from figure 4 that the higher order coefficient R can have a very significant contribution. This implies that its neglection in former models cannot be justified in many surface wave-propagation problems in the presence of ambient currents.

6. Summary and discussion

In the present work, extended MSEs and WAEs were presented. These equations retain high-order terms for changes of the bottom profiles and ambient currents. The extended time-dependent MSEs were derived from first principles, and were the basis for the construction of the WAEs. Even though it is not mentioned in previous works, the MS type of equations is less accurate for wave reflections in the presence of an ambient current. The reason behind this is that owing to the directionality of the ambient current, the assumed vertical profile is a good approximation for a single propagation direction because the wavenumber's magnitude depends on the wave's direction with respect to the current's direction. Therefore, when reflection occurs, the assumed profile can be accurate only for the incident wave component, as the reflected wave is characterized by a different wavenumber. The difference between the wavenumbers of an incident wave and a reflecting wave in the opposite direction are the largest possible, as they are the result of double the effect of the Doppler shift. Therefore, it is less advisable to use it for reflection of strong currents. When the waves propagate mostly in a single direction (which does not need to be known *a priori*), the Doppler shift differences are small, and the same vertical profile can be a plausible approximation for the whole wave field. This, of course, also applies to the case where there is no ambient current because, in this case, the wavenumber is not related to the wave direction at all.

In the next step, a travelling wave form was applied to the new MS-type equation. This resulted in an extended deterministic WAE. The derivation consisted of the Doppler shift in an accurate manner that had taken into account the corrected wave direction and speed. Furthermore, it yielded a different wave-action definition for the wave-action transport equation. This equation can be solved using the iterative method of Kostense *et al.* (1988), which should result in a wavenumber direction that is a good approximation to the actual wave field in each location. In addition, a stochastic (phase-averaged) WAE was derived for a frequency-direction spectrum in order to present a more precise transport equation for wave-forecasting models.

Finally, numerical calculations were presented in order to reassure the significance of the high-order components. The calculation inspected the class I reflection coefficients for various monochromatic waves propagating over an undulating bottom in comparison with wave tank experiments. The results reassured the significance of the high-order terms for the reflection coefficients owing to the class I Bragg resonance in the presence of an ambient current. As any bottom profile can be represented by a sum of sinusoidal components, these high-order terms are expected to be of significance for various shoaling scenarios.

This work was derived under the assumption of potential flow. For water waves, this is a common and a reasonable approximation. However, it may not apply to many realistic flow scenarios of the ambient current, as they are rotational in nature. Nevertheless, wave-propagation models that include this assumption have shown to be also applicable for rotational ambient currents (e.g. Liu 1983; Kostense *et al.* 1988; Chen *et al.* 2005). This interesting point, which was previously mostly overlooked, deserves some explanation. As was discussed in §3, the extended WAE derived in this paper can be reduced to the WAE of Bretherton & Garrett (1968). Therefore, these wave-propagation models are valid for rotational ambient currents in the case of slowly varying wavetrains. In addition, within the wave-propagation equations, rotational terms of the ambient current were taken into account, and irrotationality of the ambient current was not imposed. It can be taken into account that the vortical part of realistic ambient currents is usually a small parameter and that the contribution of high-order terms can be of a major significance (as shown in figure 4). This implies that the improvement of the model of Bretherton & Garrett (1968) is expected also for rotational ambient currents, even if the equation does not consider the full contribution of the rotationality. A rigorous derivation of extended MSEs and WAEs in the presence of rotational flows is, of course, still required in order to have a consistent model that accounts for high-order bottom–current effects with vortical ambient flow components.

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Appendix A. Coefficients of high-order current and bottom changes

The gradient of the wavenumber k in the presence of an ambient current depends both on the bottom profile and on the current. It can be described by taking the gradient of the dispersion relation (2.17) together with the Doppler shift (2.18). This yields the relation

$$\nabla k = c_{\mathrm{g}}^{-1}\nabla\sigma - \frac{2k^2}{2k\bar{h} + \sinh 2k\bar{h}}\nabla\bar{h}. \tag{A 1}$$

Equation (A 1), together with equation (2.16), can be used to construct the coefficients given in equations (2.21)–(2.24). After some simplifications, these coefficients take the form

$$q_1 = \int_{-h}^{\eta_0} f^2\,\mathrm{d}z = \frac{\tanh k\bar{h} + k\bar{h}\,\mathrm{sech}^2\,k\bar{h}}{2k} = \frac{1}{g}cc_{\mathrm{g}}, \tag{A 2}$$

$$q_2 = \int_{-h}^{\eta_0} f_z^2\,\mathrm{d}z = \frac{1}{2}k(\tanh k\bar{h} - k\bar{h}\,\mathrm{sech}^2 k\bar{h}) = \frac{1}{g}(\sigma^2 - k^2 cc_{\mathrm{g}}), \tag{A 3}$$

$$\begin{aligned}
q_3 = \int_{-h}^{\eta_0} |\nabla f|^2\,\mathrm{d}z &= \frac{k\,\mathrm{sech}^2 k\bar{h}|\nabla h|^2}{12(2k\bar{h} + \sinh(2k\bar{h}))^2} \\
&\times \{3\sinh 4k\bar{h} - 4k\bar{h}[k\bar{h}(2k\bar{h} + 3\sinh 2k\bar{h}) + 3]\} \\
&+ \frac{\tanh k\bar{h}|\nabla\sigma|^2}{3gk^2(\tanh k\bar{h} + k\bar{h}\,\mathrm{sech}^2 k\bar{h})^2} \\
&\times [2(k\bar{h})^3(\cosh 2k\bar{h} - 2)\mathrm{sech}^4 k\bar{h} + 3\tanh k\bar{h} - 3k\bar{h}\,\mathrm{sech}^2 k\bar{h}] \\
&+ \frac{\sigma\mathrm{sech}^3 k\bar{h}\nabla\eta_0\cdot\nabla\sigma}{12gk(\tanh k\bar{h} + k\bar{h}\,\mathrm{sech}^2 k\bar{h})^2} \times \{3\sinh 3k\bar{h} - 7\sinh k\bar{h} \\
&+ 4k\bar{h}[3\ \mathrm{sech}k\bar{h} + k\bar{h}(6\sinh k\bar{h} + 2k\bar{h}\,\mathrm{sech}k\bar{h} - 9\tanh k\bar{h}\,\mathrm{sech}k\bar{h})]\} \\
&+ \frac{k\mathrm{sech}^6 k\bar{h}|\nabla\eta_0|^2}{192(\tanh k\bar{h} + k\bar{h}\,\mathrm{sech}^2 k\bar{h})^2} \times \{63\sinh 2k\bar{h} - 24\sinh 4k\bar{h} + 3\sinh 6k\bar{h} \\
&+ 4k\bar{h}[4k\bar{h}(3\sinh 2k\bar{h} - 2k\bar{h}) - 36\cosh 2k\bar{h} + 9\cosh 4k\bar{h} + 15]\} \\
&+ \frac{\sigma\mathrm{sech}^5 k\bar{h}\nabla h\cdot\nabla\sigma}{12gk(\tanh k\bar{h} + k\bar{h}\,\mathrm{sech}^2 k\bar{h})^2} \times \{-3(\sinh k\bar{h} + \sinh 3k\bar{h}) \\
&+ 4k\bar{h}[3\cosh^3 k\bar{h} + k\bar{h}(3\sinh^3 k\bar{h} - k\bar{h}\cosh k\bar{h})]\} \\
&+ \frac{k\nabla h\cdot\nabla\eta_0\mathrm{sech}^6 k\bar{h}}{48(\tanh k\bar{h} + k\bar{h}\,\mathrm{sech}^2 k\bar{h})^2} \times \{24\sinh 2k\bar{h} - 3\sinh 4k\bar{h} \\
&+ 2k\bar{h}[2k\bar{h}(2k\bar{h} + 3\sinh 2k\bar{h}) + 3(1 - 6\cosh 2k\bar{h} + \cosh 4k\bar{h})]\},
\end{aligned} \tag{A 4}$$

$$\mathbf{q}_4 = \int_{-h}^{\eta_0} f\nabla f \, \mathrm{d}z = \left(\frac{1}{4}\mathrm{sech}^2(k\bar{h}) - \frac{k\bar{h}}{2k\bar{h} + \sinh(2k\bar{h})}\right)\nabla h$$
$$- \frac{\sinh 2k\bar{h} + 2k\bar{h}(2k\bar{h}\tanh k\bar{h} - 1)}{2gk^2(2k\bar{h} + \sinh 2k\bar{h})}\sigma\nabla\sigma$$
$$+ \left(-\frac{k\bar{h}}{2k\bar{h} + \sinh(2k\bar{h})} + \frac{3}{4}\mathrm{sech}^2(k\bar{h}) - \frac{1}{2}\right)\nabla\eta_0 \tag{A 5}$$

and $$\mathbf{q}_5 = \nabla f|_{z=\eta_0} = -k\tanh k\bar{h}\nabla\eta_0 = -\frac{\sigma^2}{g}\nabla\eta_0. \tag{A 6}$$

Appendix B. A surface elevation amplitude formulation of the extended wave-action equation

Equations (3.3), (3.8) and (3.10) formulate the extended wave action in terms of the velocity potential amplitude. Let us also describe this model in terms of the surface elevation amplitude. In order to do so, the relation between the two amplitudes should be derived. The same structure that was assumed for the velocity potential in equation (3.1) can be applied to the surface elevation,

$$\eta_1 = \mathrm{Re}\left\{A(\mu x, \mu y, \mu t)\exp\left\{\frac{\mathrm{i}}{\mu}[S(\mu x, \mu y, \mu t) + \gamma(\mu x, \mu y, \mu t)]\right\}\right\}. \tag{B 1}$$

Here, both A and γ are real functions, which correspond to the surface elevation amplitude and the relative phase between the velocity potential and the surface elevation.

Substituting equations (3.1) and (B 1) in equation (2.34) yields a complex equation

$$\mu\frac{DB}{Dt} + \mathrm{i}\frac{DS}{Dt}B = -gA\mathrm{e}^{\mathrm{i}\gamma} + B\frac{\sigma^2}{g}\mu\nabla\eta_0\cdot\mathbf{U}, \tag{B 2}$$

which can be written as the following two real equations:

$$gA\sin\gamma = \Sigma B \tag{B 3}$$

and

$$gA\cos\gamma = B\frac{\sigma^2}{g}\mu\nabla\eta_0\cdot\mathbf{U} - \mu\frac{DB}{Dt}. \tag{B 4}$$

The relative phase γ can be eliminated by squaring and adding equations (B 3) and (B 4), which results in a relation between the two amplitudes,

$$g^2A^2 = \Sigma^2B^2 + \mu^2\left(B\frac{\sigma^2}{g}\nabla\eta_0\cdot\mathbf{U} - \frac{DB}{Dt}\right)^2. \tag{B 5}$$

Note that the second term on the right-hand side (r.h.s.) of equation (B 5) is of $O(\mu^2)$ i.e.

$$g^2A^2 = \Sigma^2B^2 + O(\mu^2). \tag{B 6}$$

After applying a square root and Taylor series, while taking into account that the two amplitudes are positive, a simple definition of B can be given as

$$B = \frac{g}{\Sigma} A + O(\mu^2). \tag{B 7}$$

Equation (3.2) can now be rewritten in terms of the surface elevation amplitude while neglecting terms of $O(\mu^3)$ by using the simplified equations (B 6) and (B 7). The outcomes are the following eikonal and wave-action definitions:

$$\begin{aligned} K^2 = |\nabla S|^2 = k^2 + \frac{1}{cc_{\rm g}}(\Sigma^2 - \sigma^2 - R) - \frac{\Sigma}{gAcc_{\rm g}} \left[g\mu^2 \frac{D^2}{Dt^2}\left(\frac{A}{\Sigma}\right) \right. \\ \left. + \, g\mu(\nabla \cdot \mathbf{U}) \frac{D}{Dt}\left(\frac{A}{\Sigma}\right) - g\mu \nabla(cc_{\rm g}) \cdot \nabla \left(\frac{A}{\Sigma}\right) - g\mu^2 cc_{\rm g} \nabla^2 \left(\frac{A}{\Sigma}\right) \right] \end{aligned} \tag{B 8}$$

and

$$N = \frac{A^2}{\Sigma}. \tag{B 9}$$

Equation (3.10), together with equation (B 8), the wave action (B 9) and the velocity (3.9), forms the deterministic EWAE model in the xy-space for the surface elevation amplitude. Note that equation (3.10) was divided by the constant g^2, which should be applied to its r.h.s. in the case of a non-zero forcing term.

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