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On the existence of a wave of greatest height and Stokes's conjecture

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It is shown that there exists a solution of Nekrasov's integral equation which corresponds to the existence of a wave of greatest height and of permanent form moving on the surface of an irrotational, infinitely deep flow. It is also shown that this wave is the uniform limit, in a specified sense, of waves of almost extreme form. The question of the validity of Stokes's conjecture is reduced to one of the regularity of the solution of Nekrasov's equation in this limiting case.

1. INTRODUCTION

In this paper we study a nonlinear integral equation

$$\phi(s) = \frac{1}{3}\int_{-\pi}^{\pi} \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k} \frac{\sin \phi(t)}{(1/\mu) + \int_0^t \sin \phi(w)\,\mathrm{d}w}\,\mathrm{d}t \tag{1.1}$$

which Nekrasov (1920) introduced in the course of his investigation of the free surface of a deep, inviscid heavy flow acted on by gravity. Using it he was able to show that such a free surface can take the form of a periodic travelling wave-train of permanent shape which moves with constant velocity c. Here we only give an outline of the derivation of (1.1) from the free surface problem, but the details are to be found in the book by Milne-Thomson (1968) or the monograph (in English) by Nekrasov (1967).

The basic idea is this. Suppose that, on an infinitely deep flow which is at rest at infinite depth, the profile of the free surface is that of a periodic wave of permanent form travelling with velocity c. Then we can use a hodograph transformation to map the region under one period (wavelength λ) onto the unit disk in the complex w-plane. This mapping takes the free surface onto the unit circle, and the point at infinite depth onto the origin of the w-plane. Then Bernoulli's free surface condition implies that if $\phi(s)$ is the slope of the wave profile at the point corresponding to the point $\mathrm{e}^{\mathrm{i}s}$, $s \in [-\pi, \pi]$, on the unit circle, then ϕ satisfies (1.1) with

$$(1/\mu) = 2\pi Q^3/3g\lambda c.$$

Here Q denotes the speed of particles at the wave crest relative to coordinates moving with the wave.

Now this derivation can be reversed and it can be shown that the existence of a solution of (1.1) for given μ corresponds to the existence of a flow with periodic free surface. So it can be argued, as it is done in the works cited above, that (1.1) is equivalent to the existence of periodic waves on the free surface of an infinitely deep flow.

It is evident from the oddness of the kernel in (1.1) that any solution ϕ is odd. This is one of the assumptions made in the derivation of (1.1), but it is justified, as was shown by Levi-Civita (1925), since any periodic free surface of the water-wave problem must be symmetrical about the crest (this is precisely what the oddness of ϕ indicates).

Recently there has been much interest, both theoretical and numerical, in the qualitative features of solutions of the water-wave problem when μ is large. The case $Q = 0$ ($\mu = \infty$) corresponds to the presence of a *stagnation point* at the wave crest, and it was claimed by Stokes (1880) that in this limiting case a wave exists, the so-called *wave of greatest height*. Moreover, for this wave he suggested an argument to show that $\lim_{s\to 0+} \phi(s) = \frac{1}{6}\pi = -\lim_{s\to 0-} \phi(s)$. In other words he proposed that the wave of greatest height is a sharp-crested wave, and not smooth crested, as waves for finite μ are known to be.

Stokes's method is to find a local solution of the free surface problem in a neighbourhood of the *stagnation point* and to show that such a solution exhibits the sharp-crestedness which we mention above. But there has always been some question as to whether this local solution can be matched onto a solution elsewhere in the flow. Recently the nature of the singularity for waves close to the *wave of greatest height* has been investigated theoretically by Grant (1973) and computationally by Schwartz (1974).

Recent numerical investigations (Longuet-Higgins & Cokelet 1976; Longuet-Higgins & Fox 1977; and Cokelet 1977) have examined solutions of (1.1) for large values of μ in the belief that the *wave of greatest height* is the limit of a sequence of such solutions as $\mu \to \infty$.

However, the *wave of greatest height* (i.e. a solution of (1.1) with $\mu = \infty$) has not yet been proved to exist. The Russian mathematician Yu. P. Krasovskii (1961), using a variant of (1.1) proved that, for each β, $0 < \beta < \frac{1}{6}\pi$ there exists a solution of (1.1) for some $\infty > \mu > 3$ which is positive on $]0, \pi[$ with the property that $\sup_{s\in[-\pi,\pi]} |\phi(s)| = \beta$. More recently, Keady & Norbury (1978) have shown that, for each $\mu > 3$ there exists a solution ϕ of (1.1) which is positive on $]0, \pi[$ such that $\sup_{s\in[-\pi,\pi]} |\phi(s)| < \frac{1}{2}\pi$.

In this paper we prove that *there does exist* a solution of the equation

$$\phi(s) = \frac{1}{3}\int_{-\pi}^{\pi} \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k} \frac{\sin\phi(t)}{\int_0^t \sin\phi(w)\,\mathrm{d}w}\,\mathrm{d}t \tag{1.2}$$

which is continuous on $[-\pi, \pi]$ except at 0 where it has a point of discontinuity. Furthermore we shall show that for each $\epsilon > 0$, ϕ is the uniform limit on $[-\pi, -\epsilon] \cup [\epsilon, \pi]$ of a sequence $\{\phi_n\}$ of solutions of (1.1) corresponding to a sequence $\{\mu_n\}, \mu_n \to \infty$.

It is significant in the light of Stokes's claim mentioned earlier that we can prove the following: for each $\epsilon > 0$, $\sup_{s \in [-\epsilon, \epsilon]} |\phi(s)| \geq \frac{1}{6}\pi$.

Unfortunately all this is not enough to decide the question of the local behaviour of the *wave of greatest height* at its crest. The reason is that we have not resolved the nature of the discontinuity of ϕ at 0. If ϕ has a simple jump discontinuity at 0 then we shall show that indeed

$$\lim_{s \to 0+} \phi(s) = \tfrac{1}{6}\pi.$$

If ϕ does not have a jump discontinuity then $\lim_{s \to 0+} \phi(s)$ has no meaning. Thus the question of the shape of the *wave of greatest height* at its crest is reduced to one of whether or not it has an infinite number of ripples in a neighbourhood of its crest and this question is so far unanswered.

Our approach to the existence problem for (1.2) is functional-analytic. Because the operator in (1.2) is non-compact the usual topological fixed point methods do not apply. The result follows from first principles by showing that a sequence of solutions of (1.1) corresponding to a sequence $\{\mu_n\}$, $\mu_n \to \infty$, converges in L^2 to a non-trivial function ϕ. The existence of this sequence $\{\phi_n\}$ is guaranteed by the following theorem (Keady & Norbury 1978).

THEOREM 1.1. *For each* $\mu > 3$ *there exists a solution* ϕ *of* (1.1) *with the following properties*:

(i) ϕ *is continuous on* $[-\pi, \pi]$ *and is odd*;

(ii) $\phi(s) > 0, s \in]0, \pi[, \phi(\pi) = 0$ *and* $|\phi(s)| < \frac{1}{2}\pi, s \in [-\pi, \pi]$.

The weak convergence in L^2 of a sequence $\{\phi_n\}$ of solutions of (1.1) corresponding to a sequence $\{\mu_n\}$, $\mu_n \to \infty$, is then immediate, and our first task is to show that this weak limit is non-trivial. That this weak limit is a solution of (1.2) follows from the *a-priori* estimates of § 4 which ensure that $\{\phi_n\}$ converges strongly in L^2 to its weak limit.

The other properties which we claim for ϕ are proved in § 5 using some rather deep results from the theory of Fourier series (Zygmund 1959).

In § 6 the Stokes's conjecture is discussed and it is shown that if ϕ has a jump discontinuity at 0, then $\lim_{s \to 0+} \phi(s) = \frac{1}{6}\pi$. In § 7, we discuss briefly the implications of hydrodynamic comparison theorems for this problem.

Throughout this paper we restrict our attention to the free surface water-wave problem over an infinitely deep flow. In Keady & Norbury (1978) and Krasovskii (1961) the Nekrasov equations for both the finite and the infinite-depth problems are treated (in the finite-depth problem the kernel of the integral equation is more involved). It is clear that the analysis of this paper can be extended to cover the finite-depth problem as well.

2. Preliminary results about Fourier series

In choosing our notation for Fourier series we have followed Zygmund (1959) even when he disagrees with other standard works, e.g. Hardy & Rogosinski (1962).

As usual, $L^p, p > 1$, is used to denote the space of pth power integrable functions on the closed interval $[-\pi, \pi]$. The space of continuous functions on $[-\pi, \pi]$, endowed with the maximum norm, is denoted by C.

A trigonometric series,

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos ks + b_k \sin ks), \tag{2.1}$$

is said to be the Fourier series of a function $f: [-\pi, \pi] \to R$ if

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \mathrm{d}x;$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, \mathrm{d}x;$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, \mathrm{d}x.$$

If (2.1) is the Fourier series of f we write

$$f \sim a_0 + \sum_{k=1}^{\infty} (a_k \cos ks + b_k \sin ks). \tag{2.2}$$

Only when the series in (2.2) converges for a given $s \in [-\pi, \pi]$ to $f(s)$ will we write

$$f(s) = a_0 + \sum_{k=1}^{\infty} (a_k \cos ks + b_k \sin ks). \tag{2.3}$$

The question of the pointwise convergence of Fourier series is a vexed one and we will return to it in a moment. But first we introduce the notion of the conjugate of a trigonometric series.

The trignometric series

$$\sum_{k=1}^{\infty} (a_k \sin ks - b_k \cos ks) \tag{2.4}$$

is *the conjugate* of the series (2.1). The series (2.4) is not necessarily a Fourier series, but if it is, and if (2.1) is the Fourier series of f, then we write

$$Cf \sim \sum_{k=1}^{\infty} (a_k \sin ks - b_k \cos ks).\dagger \tag{2.5}$$

† We adopt this definition (Zygmund 1959) for the sake of consistency. It is more usual in water-wave problems to follow Hardy & Rogosinski (1962) in taking the negative of (2.4) as the conjugate Fourier series of (2.1). Thus in (5.3) we have a minus sign in front of $3C\phi$, whereas no such minus appears in, say, Krasovskii (1961).

Now we are in a position to collect those theorems on Fourier series which we use in the analysis of the subsequent sections. If $f: [-\pi, \pi] \to R$ we define $\tilde{f}(x)$, when it exists, as follows:

$$\tilde{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\tan\frac{1}{2}t}\,\mathrm{d}t. \tag{2.6}$$

THEOREM 2.1. *If, for* $s \in [-\pi, \pi]$,

$$\int_0^\pi \frac{|f(s+t)+f(s-t)-2f(s)|}{2\tan\frac{1}{2}t}\,\mathrm{d}t \tag{2.7}$$

is finite, then (2.3) *holds.*

If, for $s \in [-\pi, \pi]$,

$$\int_0^\pi \frac{|f(s+t)-f(s-t)|}{2\tan\frac{1}{2}t}\,\mathrm{d}t \tag{2.8}$$

is finite, then (2.4) *converges to* $\tilde{f}(s)$.

Remark. This is thm 6.1, of chapter 2 of Zygmund (1959). Since the conjugate of a Fourier series is not necessarily a Fourier series we cannot assert, under the above hypotheses, that Cf exists.

THEOREM 2.2. *Suppose that f is continuous in the closed interval* $[a, b] \subset [-\pi, \pi]$ *and let*

$$\omega(\delta) = \sup_{\substack{|x-y|\leqslant\delta \\ x,\, y\, \in[a,b]}} |f(x)-f(y)|.$$

Then if $\omega(\delta)/\delta$ *is integrable near* $\delta = 0$ *and if the integrals*

$$\int_0^\pi \frac{|f(a)-f(a-t)|}{t}\,\mathrm{d}t \quad \textit{and} \quad \int_0^\pi \frac{|f(b+t)-f(b)|}{t}\,\mathrm{d}t$$

are both finite, then (2.1) *and* (2.4) *converge uniformly* in $[a, b]$ *to f and $\tilde{f}$ respectively.*

This theorem, which is a uniform version of theorem 2.1, is again to be found in Zygmund (1959), thm 6.8 of chapter 2.

The next theorem is the celebrated theorem of M. Riesz on conjugate Fourier series.

THEOREM 2.3. *If* $f \in L^p$, $1 < p < \infty$, *then* $\tilde{f} \in L^p$, *and there exists a constant* A_p *such that*

$$\int_{-\pi}^{\pi} |\tilde{f}(x)|^p\,\mathrm{d}x \leqslant A_p \int_{-\pi}^{\pi} |f(x)|^p\,\mathrm{d}x.$$

Moreover, $\tilde{f} = Cf$, *i.e. the Fourier series of $\tilde{f}$ is the conjugate of the Fourier series of f.*

This result is thm 2.4 of chapter 7 of Zygmund (1959) and the next result is thm 2.11 of the same chapter.

THEOREM 2.4. (i) *If* $|f| \leqslant 1$, *then*

$$\int_{-\pi}^{\pi} \exp(\lambda|\tilde{f}(x)|)\,\mathrm{d}x \leqslant 4\pi/\cos\lambda$$

for $0 \leqslant \lambda < \frac{1}{2}\pi$.

(ii) *If f is continuous on* $[-\pi, \pi]$ *then*

$$\int_{-\pi}^{\pi} \exp(\lambda|\tilde{f}(x)|)\,\mathrm{d}x$$

is finite for all $\lambda > 0$.

Finally we introduce some special trignometric series which are central in the analysis to follow. Throughout we shall use the notation

$$\boldsymbol{k}(s,t) = \sum_{k=1}^{\infty} \frac{1}{\pi} \frac{\sin ks \sin kt}{k}, \ (s,t) \in [-\pi, \pi] \times [-\pi, \pi].$$

$\boldsymbol{k}$ has some important properties, the most important of which, for our purposes is that

$$\mathrm{K} : \boldsymbol{k}(s,t) = \frac{1}{4\pi} \ln \left\{\frac{1-\cos(s+t)}{1-\cos(s-t)}\right\},$$

which is non-negative on $[0, \pi] \times [0, \pi]$.

3. *A priori* ESTIMATES

In this section we will prove some pointwise estimates about a solution ϕ of (1.1) which depend on the parameter μ. It will be our task in the next section to show that these estimates hold uniformly for a sequence of solutions of (1.1) corresponding to a sequence μ_n tending to ∞.

THEOREM 3.1. *If ϕ is a non-trivial solution of* (1.1) *for some* $\mu > 3$, *and* $0 \leqslant \phi(s) < \frac{1}{2}\pi$ *for all* $s \in [0, \pi]$, *then there exists* $\beta > 0$ *such that for* $s \in [0, \pi]$,

$$\phi(s) \geqslant \beta \sin s.$$

Proof.

$$\phi(s) = \frac{2}{3}\int_0^{\pi} \boldsymbol{k}(s,t) \left\{\frac{\sin\phi(t)}{\frac{1}{\mu} + \int_0^t \sin\phi(w)\,\mathrm{d}w}\right\} \mathrm{d}t$$

$$\geqslant M \int_0^{\pi} \boldsymbol{k}(s,t)\,\phi(t)\,\mathrm{d}t, \quad \text{where} \quad M = \frac{4}{3\pi(\frac{1}{3}+\pi)} \quad \text{for} \quad s \in [0,\pi].$$

Now if

$$\phi \sim \sum_{k=1}^{\infty} a_k \sin ks,$$

then

$$\phi(s) = \sum_{k=1}^{\infty} a_k \sin ks,$$

and so for $s \in [0, \pi]$,

$$\phi(s) \geqslant \frac{M}{2} \sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right) \sin ks.$$

For $m > 1$ and $s \in [0, \pi]$,

$$\begin{aligned}\phi(s) &\geqslant \left(\frac{M}{2}\right)^m \sum_{k=1}^{\infty} \left(\frac{a_k}{k^m}\right) \sin ks \\ &\geqslant \left(\frac{M}{2}\right)^m \left\{a_1 \sin s - \sum_{k=2}^{\infty} \left|\frac{a_k}{k^m} \sin ks\right|\right\} \\ &\geqslant \left\{\frac{M}{2}\right\}^m \left\{a_1 - \sum_{k=2}^{\infty} \frac{k|a_k|}{k^m}\right\} \sin s.\end{aligned}$$

Since $\phi > 0$ on a set of positive measure in $[0, \pi]$, $a_1 > 0$, and so we can choose m sufficiently large that the term in chain brackets is positive, and the proof is complete.

THEOREM 3.2. *If ϕ is a non-trivial solution of* (1.1) *for some $\mu > 3$ and $0 \leqslant \phi(s) < \frac{1}{2}\pi$ for all $s \in [0, \pi]$, then*

$$\alpha = \sup_{t \in [0, \pi]} \left\{\frac{1}{\mu} + \int_0^t \sin \phi(w)\, \mathrm{d}w\right\} = \frac{1}{\mu} + \int_0^{\pi} \sin \phi(w)\, \mathrm{d}w \geqslant \frac{2}{3\pi}.$$

Proof. Let ϕ be such a solution of (1.1) and let

$$\chi(t) = \frac{1}{\mu} + \int_0^t \sin \phi(w)\, \mathrm{d}w.$$

By the previous theorem there exists $\beta > 0$ such that $\phi(s) \geqslant \beta \sin s$, $s \in [0, \pi]$.

Let
$$\beta = \sup\{\beta' > 0 : \phi(s) \geqslant \beta' \sin s, s \in [0, \pi]\}.$$

Then

$$\begin{aligned}\frac{\beta}{2} \sin s &= \int_0^{\pi} k(s,t)\, \beta \sin t\, \mathrm{d}t \leqslant \int_0^{\pi} k(s,t)\, \phi(t)\, \mathrm{d}t \leqslant \frac{\pi}{2} \int_0^{\pi} k(s,t) \left\{\frac{\sin \phi(t)}{\chi(t)}\right\} \chi(t)\, \mathrm{d}t \\ &\leqslant \alpha \frac{\pi}{2} \int_0^{\pi} k(s,t) \left\{\frac{\sin \phi(t)}{(1/\mu) + \int_0^t \sin \phi(w)\, \mathrm{d}w}\right\} \mathrm{d}t \\ &= \tfrac{3}{4} \alpha \pi \phi(s).\end{aligned}$$

Since β was chosen to be maximal we may conclude that

$$\alpha \geqslant \frac{2}{3\pi}$$

and the proof of the theorem is complete.

4. ASYMPTOTIC ESTIMATES AS $\mu \to \infty$

According to theorem 1.1, for each $\mu > 3$ there exists a solution, ϕ, of (1.1) corresponding to that value of μ such that ϕ is odd on $[-\pi, \pi]$ and $0 \leqslant \phi(s) < \frac{1}{2}\pi$ for all $s \in [0, \pi]$. Let $\{\mu_n\}$ be an unbounded increasing sequence of real numbers and let $\{\phi_n\}$ denote the corresponding sequence of solutions of (1.1) as above. Since $\{\phi_n\}$

is bounded in L^2 it has a subsequence $\{\phi_{n(j)}\}$ which is such that both $\{\phi_{n(j)}\}$ and $\{\sin\phi_{n(j)}\}$ are weakly convergent in L^2 as $j\to\infty$.

From now on we shall use $\{\phi_n\}$ instead of $\{\phi_{n(j)}\}$ to denote this subsequence, and $\{\mu_n\}$ to denote the corresponding sequence of real numbers.

We shall suppose that $\phi_n \rightharpoonup \phi$, and that $\sin\phi_n \rightharpoonup \sigma$, as $n\to\infty$, where $\rightharpoonup$ denotes weak convergence in L^2. It follows easily that both ϕ and σ are non-negative almost everywhere on $[0,\pi]$.

In addition we shall adopt the following notation:

$$\text{for each}\quad t\in[-\pi,\pi],\quad \psi_n(t) = \ln\left\{\frac{1}{\mu_n}+\int_0^t \sin\phi_n(w)\,\mathrm{d}w\right\}.$$

Since ϕ_n is odd ψ_n is even on $[-\pi,\pi]$.

THEOREM 4.1. *The weak limit ϕ of ϕ_n is non-trivial. Indeed*

$$\frac{2}{\pi}\int_0^\pi \phi(s)\sin s\,\mathrm{d}s = a > 0. \tag{4.1}$$

Proof. Since $\sin\phi_n \rightharpoonup \sigma$ in L^2 as $n\to\infty$,

$$\frac{1}{\mu_n}+\int_0^\pi \sin\phi_n(w)\,\mathrm{d}w \to \int_0^\pi \sigma(w)\,\mathrm{d}w.$$

Hence, by theorem 3.2,

$$\int_0^\pi \sigma(w)\,\mathrm{d}w \geqslant 2/3\pi. \tag{4.2}$$

But

$$\frac{2}{\pi}\int_0^\pi \phi(s)\sin s\,\mathrm{d}s = \lim_{n\to\infty}\frac{2}{\pi}\int_0^\pi \phi_n(s)\sin s\,\mathrm{d}s$$
$$\geqslant \lim_{n\to\infty}\frac{2}{\pi}\int_0^\pi \sin\phi_n(s)\sin s\,\mathrm{d}s = \frac{2}{\pi}\int_0^\pi \sigma(s)\sin s\,\mathrm{d}s > 0,$$

since σ is a non-negative element of L^2, and (4.2) holds. This completes the proof of the theorem.

We are now in a position to prove an asymptotic version of theorem 3.1.

THEOREM 4.2. *Let $\{\phi_n\}$ and $\{\mu_n\}$ be as before. Then there exists $\beta > 0$ (independent of n) such that*

$$\phi_n(s) \geqslant \beta\sin s,$$

for each $s\in[0,\pi]$.

Proof. For each n let

$$\phi_n \sim \sum_{k=1}^{\infty} a_k^{(n)}\sin ks,$$

and so

$$\phi_n(s) = \sum_{k=1}^{\infty} a_k^{(n)}\sin ks,\quad s\in[0,\pi].$$

It then follows, as in the proof of theorem 3.1, that for $s \in [0, \pi]$,

$$\phi_n(s) \geqslant \left\{\frac{M}{2}\right\}^m \left\{a_1^{(n)} - \sum_{k=2}^{\infty} \left|\frac{a_k^{(n)}}{k^{m-1}}\right|\right\} \sin s,$$

where $M = 4/\pi(1+3\pi)$, and $m \geqslant 1$. Since $\phi_n \rightharpoonup \phi$ as $n \to \infty$,

$$a_1^{(n)} = \frac{2}{\pi}\int_0^{\pi} \phi_n(s) \sin s \, \mathrm{d}s \to \frac{2}{\pi}\int_0^{\pi} \phi(s) \sin s \, \mathrm{d}s = a > 0,$$

by theorem 4.1. Hence there exists $N > 0$ such that for all $n \geqslant N$

$$a_1^{(n)} \geqslant \tfrac{3}{4}a.$$

We can choose m independently of n so large that

$$\sum_{k=2}^{\infty} \left|\frac{a_k^{(n)}}{k^{m-1}}\right| \leqslant \tfrac{1}{4}a.$$

So for this value of m, and $n \geqslant N$,

$$\phi_n(s) \geqslant \{\tfrac{1}{2}M\}^m \tfrac{1}{2}a \sin s.$$

By theorem 3.1, for each $n \leqslant N$ there exists $\beta_n > 0$ such that

$$\phi_n(s) \geqslant \beta_n \sin s.$$

We put $\beta = \min\{(\tfrac{1}{2}M)^m \tfrac{1}{2}a, \beta_1, \beta_2, \ldots, \beta_N\}$ to complete the proof of the theorem.

5. The wave of greatest height: its existence and its properties

This section begins with a proof of the existence of a wave of extreme form. In other words we prove that there exists a solution of the Nekrasov equation in the limiting case when $1/\mu = 0$. Once the existence result is in hand we consider what can be said about the properties of this limiting wave. We prove that the wave is *not* smooth-crested (as waves for finite μ are known to be), and that it is the limit in an intuitively desirable sense of waves of almost extreme form.

The first result below is of independent interest since it establishes the link between (1.1) and the version of the Nekrasov equation which Krasovskii (1961) used in his treatment of the water-wave problem.

Theorem 5.1. *If ϕ is a solution of (1.1) for some $\mu > 3$ and*

$$\psi(t) = \ln\left\{\frac{1}{\mu} + \int_0^t \sin\phi(w)\,\mathrm{d}w\right\}, \quad t \in [-\pi, \pi]$$

then

$$-3\phi = C\psi, \tag{5.1}$$

$$3C\phi = \psi - a_0, \tag{5.2}$$

and
$$\phi(s) = \frac{\nu}{3}\int_{-\pi}^{\pi} k(s,t)\exp(-3C\phi)\sin\phi(t)\,dt, \tag{5.3}$$

where
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(t)\,dt, \quad \text{and} \quad \nu = \exp(-a_0).$$

Remark. We shall make no explicit use of the connection between (1.1) and (5.3) in what follows. It is worth noting however that (5.3) is actually equivalent to (1.1) (ϕ satisfies (1.1) for some $\mu > 3$ if and only if ϕ satisfies (5.3) for some $\nu > 0$).

Proof. If ϕ satisfies (1.1) then Fubini's theorem implies that
$$\phi \sim \sum_{k=1}^{\infty} a_k \sin ks,$$
where
$$a_k = \frac{1}{3\pi}\int_{-\pi}^{\pi}\frac{\sin kt}{k}\left\{\frac{\sin\phi(t)}{\dfrac{1}{\mu}+\displaystyle\int_0^t \sin\phi(w)\,dw}\right\}dt$$
$$= \frac{-1}{3\pi}\int_{-\pi}^{\pi}\cos kt\psi(t)\,dt.$$
Hence $-3\phi = C\psi$, and $3C\phi = \psi - a_0$. Since
$$\left\{\frac{1}{\mu}+\int_0^t \sin\phi(w)\,dw\right\}^{-1} = \exp(-\psi(t)),$$
(5.3) follows immediately from (1.1) by substitution. This completes the proof of the theorem.

THEOREM 5.2. *Let $\{\phi_n\}$, $\{\psi_n\}$ and $\{\mu_n\}$ be as in the previous section,* i.e. *$\phi_n \rightharpoonup \phi$, $\sin\phi_n \rightharpoonup \sigma$ in L^2 as $n\to\infty$, and $-3\phi_n = C\psi_n$ (by theorem 5.1).*

Then $\phi_n\to\phi$, $\sin\phi_n\to\sigma = \sin\phi$ and $\psi_n\to\psi$ in L^2 as $n\to\infty$, where
$$\psi(t) = \ln\left\{\int_0^t \sin\phi(w)\,dw\right\}, \quad t\in[-\pi,\pi].$$

Proof. Since $\sin\phi_n \rightharpoonup \sigma$ in L^2 as $n\to\infty$, it follows that for each $t\in[-\pi,\pi]$, $\psi_n(t)\to\psi(t)$ where $\psi(t) = \ln\left\{\int_0^t \sigma(w)\,dw\right\}$,

By theorem 4.2, for each $t\in[0,\pi]$,
$$t \geqslant \int_0^t \sin\phi_n(w)\,dw \geqslant \int_0^t \sin(\beta\sin w)\,dw \geqslant \frac{2}{\pi}\int_0^t \beta\sin w\,dw = \frac{2\beta}{\pi}(1-\cos t).$$
Hence, for each $t\in[0,\pi]$,
$$|\psi_n(t)| \leqslant \max\{|\ln(\tfrac{1}{3}+t)|,\ \ |\ln 2\beta(1-\cos t)/\pi|\}.$$
Now the Dominated Convergence Theorem implies that $\psi_n\to\psi$ in L^2 as $n\to\infty$. By theorem 5.1, $C\psi_n = -3\phi_n$, and so theorem 2.3 implies that $\phi_n\to\phi$ in L^2 as $n\to\infty$. It remains only to show that $\sigma = \sin\phi$.

Since $\phi_n \to \phi$ in L^2 as $n \to \infty$, there exists a subsequence $\{\phi_{n(j)}\}$ such that $\phi_{n(j)}(t) \to \phi(t)$ almost everywhere as $n \to \infty$. Hence, by the Dominated Convergence Theorem, $\sin \phi_{n(j)} \to \sin \phi$ in L^2 as $n \to \infty$. So $\sigma = \sin \phi$, and the proof of the theorem is complete.

THEOREM 5.3. *Let ϕ be the L^2 limit of $\{\phi_n\}$ as in the previous theorem. Then*

(i)
$$\phi \sim \sum_{k=1}^{\infty} -\frac{1}{3\pi}\left\{\int_{-\pi}^{\pi} \cos kt \left\{\ln \int_0^t \sin \phi(w)\, dw\right\} dt\right\} \sin ks; \tag{5.4}$$

(ii) *the series in* (5.4) *above is uniformly convergent on* $[\epsilon, \pi]$ *for each* $\epsilon > 0$. *Hence* ϕ *is continuous on* $[-\pi, 0[\cup]0, \pi]$;

(iii) ϕ *is discontinuous at* 0;

(iv) $\sup_{s\in[-\epsilon,\epsilon]} |\phi(s)| \geqslant \frac{1}{6}\pi$, *for each* $\epsilon > 0$;

(v) $\phi(s) = \dfrac{1}{3}\displaystyle\int_{-\pi}^{\pi} \boldsymbol{k}(s,t) \left\{\frac{\sin \phi(t)}{\int_0^t \sin \phi(w)\, dw}\right\} dt$ *a.e on* $[-\pi, \pi]$.

Proof. (i) This follows from theorems 5.2 and 2.3 by taking L^2 limits in the expression $-3\phi_n = C\psi_n$.

(ii) Since $f(t) = \ln \int_0^t \sin \phi(w)\, dw$, $t \in [-\pi, \pi]$,

is a continuously differentiable function on $[-\pi, 0[\cup]0, \pi]$ and is integrable on $[-\pi, \pi]$ all the hypotheses of theorem 2.2 are satisfied when $a = \epsilon$, $b = \pi$. The result is now immediate from theorem 2.2.

(iii) By (5.4) and theorem 2.3

$$3C\phi = \ln \int_0^t \sin \phi(w)\, dw - a_0,$$

where
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left\{\int_0^t \sin \phi(w)\, dw\right\} dt.$$

So
$$\exp(-3C\phi) = \left\{\int_0^t \sin \phi(w)\, dw\right\}^{-1} \exp a_0.$$

Hence, according to theorem 2.4 (ii), ϕ is not a continuous function on $[-\pi, \pi]$. It can only be discontinuous at 0, and the proof of (iii) is complete.

(iv) Suppose that, on the contrary, there exists $\epsilon > 0$ with the property that

$$\sup_{s\in[-\epsilon,\epsilon]\backslash\{0\}} |\phi(s)| = \alpha < \tfrac{1}{6}\pi$$

Now put
$$\phi_1(s) = \begin{cases} \phi(s), & |s| \geqslant \epsilon, \\ [\phi(\epsilon)/\epsilon]\, s, & |s| \leqslant \epsilon, \end{cases}$$

and put
$$\phi_2(s) = \phi(s) - \phi_1(s), \quad s \in [-\pi, \pi]\backslash\{0\}.$$

Then $\phi_1(s)$ is a continuous function on $[-\pi, \pi]$, $|\phi_2(s)| \leqslant \alpha < \frac{1}{6}\pi$ almost everywhere, and $\phi(s) = \phi_1(s) + \phi_2(s)$. Hence $\exp\{-3C\phi\} = \exp\{-3C\phi_1\}\{\exp\{-3C\phi_2\}\}$.

Hölders inequality along with the fact that $\exp\{-3C\phi_1\} \in L^p$ for all $p \geqslant 1$ (theorem 2.4 (ii)) and $\exp\{-3C\phi_2\} \in L^q$, for all $1 \leqslant q < \pi/6\alpha$ (theorem 2.4 (i)) implies that $\exp\{-3C\phi\} \in L^1$. This is a contradiction and so (iv) is established.

(v) Since $\phi_n \to \phi$ in L^2 as $n \to \infty$ we may suppose, without loss that $\phi_n(x) \to \phi(x)$ almost everywhere as $n \to \infty$ (in the next theorem we shall see that much more is true). Hence

$$\frac{\sin \phi_n(x)}{1/\mu_n + \int_0^x \sin \phi_n(w)\, \mathrm{d}w} \to \frac{\sin \phi(x)}{\int_0^x \sin \phi(w)\, \mathrm{d}w}$$

almost everywhere, as $n \to \infty$. Now

$$\phi_n(s) = \frac{2}{3}\int_0^\pi \boldsymbol{k}(s,t) \frac{\sin \phi_n(t)}{(1/\mu_n) + \int_0^t \sin \phi_n(w)\, \mathrm{d}w}\, \mathrm{d}t,$$

$s \in [0, \pi]$, and so, by Fatou's lemma and the positivity of $\boldsymbol{k}$ on $[0, \pi] \times [0, \pi]$,

$$\phi(s) \geqslant \frac{2}{3}\int_0^\pi \boldsymbol{k}(s,t) \frac{\sin \phi(t)}{\int_0^t \sin \phi(w)\, \mathrm{d}w}\, \mathrm{d}t,$$

for almost all $s \in [0, \pi]$. Hence the odd function χ, defined by

$$\chi(s) = \frac{1}{3}\int_{-\pi}^\pi \boldsymbol{k}(s,t) \frac{\sin \phi(t)}{\int_0^t \sin \phi(w)\, \mathrm{d}w}\, \mathrm{d}t, s \in [-\pi, \pi],$$

is square integrable, and by Fubini's theorem

$$\begin{aligned}
\int_{-\pi}^\pi \chi(s) \sin ks\, \mathrm{d}s &= \frac{1}{3}\int_{-\pi}^\pi \int_{-\pi}^\pi \boldsymbol{k}(s,t) \sin ks \frac{\sin \phi(t)}{\int_0^t \sin \phi(w)\, \mathrm{d}w}\, \mathrm{d}t\, \mathrm{d}s, \\
&= \frac{1}{3}\int_{-\pi}^\pi \frac{\sin kt}{k} \frac{\sin \phi(t)}{\int_0^t \sin \phi(w)\, \mathrm{d}w}\, \mathrm{d}t, \\
&= \lim_{\delta \to 0} \frac{2}{3}\int_\delta^\pi \frac{\sin kt}{k} \frac{\sin \phi(t)}{\int_0^t \sin \phi(w)\, \mathrm{d}w}\, \mathrm{d}t, \\
&= \lim_{\delta \to 0}\left\{-\frac{2}{3}\int_\delta^\pi \cos kt \ln\left\{\int_0^t \sin \phi(w)\, \mathrm{d}w\right\} \mathrm{d}t \right. \\
&\qquad \left. - \frac{\sin k\delta}{k} \ln \int_0^\delta \sin \phi(w)\, \mathrm{d}w\right\} \\
&= -\frac{2}{3}\int_0^\pi \cos kt \ln\left\{\int_0^t \sin \phi(w)\, \mathrm{d}w\right\} \mathrm{d}t,
\end{aligned}$$

since $\phi(t) \geqslant \beta \sin t$ for all $t \in [0, \pi]$.

Thus we have shown that ϕ and χ are L^2 functions which have the same Fourier series, and the proof of (v) is complete.

So far we have shown that ϕ is the L^2 limit of a sequence of solutions of Nekrasov's equation which correspond to waves of almost extreme form. In our next theorem we prove that this convergence is uniform in an intuitively desirable sense.

THEOREM 5.4. *Let $\{\phi_n\}$, $\{\mu_n\}$ and ϕ be as before. Then*

$$\phi_n(x) \to \phi(x)$$

uniformly on $[-\pi, -\epsilon] \cup [\epsilon, \pi]$ for each $\epsilon > 0$.

To prove this theorem we need first of all to prove the following lemma, which carries the burden of the proof.

LEMMA 5.5. *Let $\{f_n\}$ be a sequence of real-valued even functions of a real variable of period 2π such that*

$$\int_0^{2\pi} |f_n(x)| \, dx \to 0$$

as $n \to \infty$.

Suppose also that

$$\sup_{n\in\mathbb{N},\, x\in[\delta,\, 2\pi-\delta]} \{|f_n'(x)| \leqslant M_\delta < \infty$$

for each $\delta > 0$ (here $'$ denotes differentiation). Then

$$g_n(x) = -\sum_{k=1}^{\infty} \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f_n(y)\cos ky\right)\sin kx \to 0$$

as $n \to \infty$, uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$ for each $\delta > 0$.

Proof. According to theorem 2.1, for each $x \in [\delta, \pi]$,

$$\begin{aligned}|g_n(x)| &= \frac{1}{\pi}\left|\int_0^{\pi} \frac{f_n(x+t)-f_n(x-t)}{2\tan\frac{1}{2}t}\,dt\right| \\ &\leqslant \frac{1}{\pi}\left\{\left|\int_0^{\sigma}\frac{f_n(x+t)-f_n(x-t)}{t}\,dt\right| + \left|\int_{\sigma}^{\pi}\frac{f_n(x+t)}{2\tan\frac{1}{2}t}\,dt\right| + \left|\int_{\sigma}^{\pi}\frac{f_n(x-t)}{2\tan\frac{1}{2}t}\,dt\right|\right\}.\end{aligned}$$

If $\sigma \leqslant \min\{\frac{1}{2}\delta, \frac{1}{3}\pi\}$ then

$$\begin{aligned}|g_n(x)| &\leqslant \frac{1}{\pi}\left\{\sigma M_{\frac{1}{2}\delta} + \frac{1}{\sigma}\int_0^{\pi}|f_n(x+t)|\,dt + \frac{1}{\sigma}\int_0^{\pi}|f_n(x-t)|\,dt\right\} \\ &\leqslant \frac{1}{\pi}\left\{\sigma M_{\frac{1}{2}\delta} + \frac{2}{\sigma}\int_0^{2\pi}|f_n(y)|\,dy\right\}.\end{aligned}$$

Now given $\delta > 0$, $\epsilon > 0$ we can choose σ sufficiently small that

$$\sigma M_{\frac{1}{2}\delta} \leqslant \tfrac{1}{2}\pi\epsilon.$$

For this value of σ we can find an integer N such that

$$\frac{2}{\sigma\pi}\int_0^{2\pi}|f_n(y)|\,dy \leqslant \frac{\epsilon}{2} \quad \text{for all} \quad n \geqslant N.$$

Hence for all $n \geqslant N$ and $x \in [\delta, \pi]$, $|g_n(x)| \leqslant \epsilon$ and the proof of the lemma is complete.

Proof of theorem 5.4.

We need only apply the lemma with

$$f_n(x) = \frac{1}{3}\left\{\ln\left(\frac{1}{\mu_n} + \int_0^x \sin\phi_n(w)\,\mathrm{d}w\right) - \ln\int_0^x \sin\phi(w)\,\mathrm{d}w\right\}.$$

Clearly $$\int_0^{2\pi} |f_n(x)|\,\mathrm{d}x \to 0$$

by theorem 5.2. Furthermore, for $x \in [\delta, 2\pi - \delta]$,

$$\begin{aligned} |f_n'(x)| &\leqslant \frac{1}{3}\left|\frac{\sin\phi_n(x)}{\int_0^x \sin\phi_n(w)\,\mathrm{d}w}\right| + \frac{1}{3}\left|\frac{\sin\phi(x)}{\int_0^x \sin\phi(w)\,\mathrm{d}w}\right| \\ &\leqslant M_\delta, \end{aligned}$$

since both ϕ_n and ϕ are bounded below on $[0, \pi]$ by $\beta \sin t$. This completes the proof of the theorem.

6. On Stokes's conjecture

In the previous section we have proved the existence of a solution of Nekrasov's equation in the limiting case when $1/\mu = 0$. This solution corresponds to a progressive periodic wave-train which is such that the flow speed at the crest is zero relative to a frame of reference moving with the velocity of the free surface profile.

Recall that the variable $\phi(s)$ in Nekrasov's version of the water-wave problem denotes the *slope* of the wave at a point parameterized by the point $\mathrm{e}^{\mathrm{i}s}$ on the unit circle in the w-plane (see § 1 for further details). The fact that ϕ has a discontinuity at 0 (theorem 5.3 (iii)) means that the wave corresponding to ϕ is not smooth at its crest. Indeed we know that its slope takes quite large values (i.e. close to 30°) at points arbitrarily close to the crest (theorem 5.3 (iv)).

So far the nature of the discontinuity of ϕ at 0 is not understood. But in this section we will prove that the possibilities are still further limited. We will show that, if $\lim_{s \to 0+} \phi(s)$ exists, then its value is $\frac{1}{6}\pi$. Thus Stokes's conjecture will be verified once it is proved that ϕ has a jump discontinuity at 0, and vice-versa.

LEMMA 6.1. *Let χ denote the function given by*

$$\chi(s) = \frac{1}{3\pi}\int_{-\pi}^{\pi} \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k}\frac{1}{t}\,\mathrm{d}t,$$

then $\lim_{s \to 0+} \chi(s) = \frac{1}{6}\pi$.

Proof. If we put
$$\psi(s) = \begin{cases} \frac{1}{6}(\pi - s), & s \in\,]0, \pi], \\ 0, & s = 0, \\ -\frac{1}{6}(\pi + s), & s \in [-\pi, 0[, \end{cases}$$

then
$$\psi(s) = \frac{1}{3}\sum_{k=1}^{\infty}\frac{\sin ks}{k}, \; s\in[-\pi,\pi].$$

For $s\in[-\pi,\pi]\backslash\{0\}$,

$$\chi(s)-\psi(s) = \frac{2}{3\pi}\sum_{k=1}^{\infty}\frac{\sin ks}{k}\int_0^{\pi}\left(\frac{\sin kt}{t}-\tfrac{1}{2}\right)\mathrm{d}t.$$

Now
$$\frac{1}{k}\int_0^{\pi}\left(\frac{\sin kt}{t}-\tfrac{1}{2}\right)\mathrm{d}t = o(1/k^{1+\alpha})$$

for any $\alpha < 1$ (Gradshteyn & Ryzhik 1965, p. 929, art. 8.235), and so $\chi-\psi$ is a continuous function. This completes the proof of the result.

THEOREM 6.2. *If ϕ is as in theorem 5.3 and ϕ has a jump discontinuity at* 0, *then*

$$\lim_{s\to 0+}\phi(s) = \tfrac{1}{6}\pi.$$

Proof. Let $\lim\limits_{s\to 0+}\sin\phi(s) = a$. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{a-\epsilon}{a+\epsilon}\frac{1}{t} \leqslant \frac{\sin\phi(t)}{\displaystyle\int_0^t \sin\phi(w)\,\mathrm{d}w} \leqslant \frac{a+\epsilon}{a-\epsilon}\frac{1}{t}$$

if $0 < t < \delta$.

Hence there exist continuous odd functions $f, g: [-\pi,\pi]\to\mathbb{R}$ such that

$$f(t)+\frac{a-\epsilon}{a+\epsilon}\frac{1}{t} \leqslant \frac{\sin\phi(t)}{\displaystyle\int_0^t \sin\phi(w)\,\mathrm{d}w} \leqslant \frac{a+\epsilon}{a-\epsilon}\frac{1}{t}+g(t)$$

for all $t\in]0,\pi]$. Now $\int_0^{\pi} \boldsymbol{k}(s,t)f(t)\,\mathrm{d}t$ is a continuous odd function of s, as is $\int_0^{\pi} \boldsymbol{k}(s,t)g(t)\,\mathrm{d}t$, since f and g are continuous. Therefore, since we know that $\boldsymbol{k}(s,t)$ is non-negative on $[0,\pi]\times[0,\pi]$, we can assert that for any $\epsilon > 0$,

$$\frac{a-\epsilon}{a+\epsilon}\tfrac{1}{6}\pi \leqslant \lim_{s\to 0+}\frac{2}{3}\int_0^{\pi}\boldsymbol{k}(s,t)\frac{\sin\phi(t)}{\displaystyle\int_0^{\pi}\sin\phi(w)\,\mathrm{d}w}\,\mathrm{d}t \leqslant \frac{a+\epsilon}{a-\epsilon}\tfrac{1}{6}\pi.$$

We conclude then that $\lim\limits_{s\to 0+}\phi(s) = \frac{1}{6}\pi$ and the proof of the theorem is complete.

CONCLUSION

In this paper we have proved that there exists a solution of the Nekrasov equation corresponding to the existence of a periodic water-wave on deep water which has a *stagnation point* at each of its crests (the case $\mu = \infty$ in (1.1)). This solution has been seen to be the uniform limit, at points away from the wave crest, of waves of almost extreme form (i.e. solutions of (1.1) with μ large but finite).

Unfortunately this analysis leaves the important question of the shape of the free-surface of the limiting wave in a neighbourhood of its crest unanswered. However it has been shown that there are only two possibilities.

Either the free surface has a well-defined corner at its crest, in which case the free surface subtends an angle $\frac{1}{6}\pi$ with the horizontal on either side of the *stagnation point*; or else the wave crest is the limit point of a sequence of steep ripples, and there are no well-defined tangents at the crest.

If the latter possibility is not ruled out then it necessarily follows (from theorem 5.4) that solutions of (1.1) become more and more oscillatory as $\mu \to \infty$. In other words, waves of almost extreme form can have arbitrarily large numbers of steep ripples as μ increases.

There is nothing in the numerical evidence so far to suggest that water waves become more oscillatory as $\mu \to \infty$. None the less there is no theoretical argument which excludes the possibility of such poor asymptotic behaviour.

In an effort to understand the qualitative features of irrotational water-waves Keady & Pritchard (1974) suggested (in an appendix) that the Serrin–Lavrientieff comparison theorems might prove useful. Their basic argument was as follows:

Serrin (1952) had shown that, at a point of inflexion in the free surface of an infinitely deep irrotational flow the derivative of the flow speed is positive or negative according to whether the tangent points into or out of the flow. If the free surface satisfies Bernoulli's condition, this rules out the possibility of inflexion points with tangents directed out of the flow.

Keady & Pritchard have very kindly brought to my attention that Serrin's result contains an assumption that the tangent at the point of inflexion cuts the free surface at one point only. Obviously this cannot be guaranteed when the free surface has the form of a periodic wave, and therefore this approach does not produce the required result.

So the question of the behaviour of solutions of (1.1) near the crest as $\mu \to \infty$ is left open. What we have succeeded in showing is that the wave of extreme form is the limit of waves of almost extreme form, and that its qualitative features away from the crest are accurately reflected by the shape of solutions of (1.1) for large but finite μ. Whether Stokes's conjecture is true or not, it has been shown that there are points arbitrarily close to the crest of the limiting wave which are steep; in fact,

$$\limsup_{s \to 0+} \phi(s) \geqslant \tfrac{1}{6}\pi,$$

where ϕ is the solution of (1.1) with $\mu = \infty$.

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References

Cokelet, E. D. 1977 Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans R. Soc. Lond.* A **286**, 183–230.

Gradshteyn, I. S. & Ryzhik, I. M. 1965 *Tables of integrals series and products.* New York and London: Academic Press.

Grant, M. A. 1973 The singularity at the crest of a finite amplitude progressive Stokes wave. *J. Fluid Mech.* **59**, 247–262.

Hardy, G. H. & Rogosinski, W. W. 1962 *Fourier series.* Cambridge University Press.

Keady, G. & Norbury, J. 1978 On the existence theory for irrotational water waves. *Math. Proc. Camb. Phil. Soc.* **83**, 137–157.

Keady, G. & Pritchard, W. G. 1974 Bounds for surface solitary waves. *Proc. Camb. Phil. Soc.* **76**, 345–358.

Krasovskii, Yu. P. 1961 On the theory of steady-state waves of finite amplitude. U.S.S.R. *Comput. Math. & Math. Phys.* **1**, 996–1018.

Levi-Civita, T. 1925 Determination rigoureuse des ordes permanentes d'ampleur finie. *Math. Ann.* **93**, 264–314.

Longuet-Higgins, M. S. & Cokelet, E. D. 1976 The deformation of deep surface waves on water. I. A numerical method of computation. *Proc. R. Soc. Lond.* A **350**, 1–26.

Longuet-Higgins, M. S. & Fox, M. J. H. 1977 Theory of the almost highest wave: The inner solution. *J. Fluid. Mech.* **80**, 721–742.

Milne-Thomson, L. M. 1968 *Theoretical hydrodynamics.* London: Macmillan.

Nekrasov, A. I. 1920 On Stokes waves (in Russian). Izv. Ivan.-Voznesensk. politckh. Inst. pp. 81–91.

Nekrasov, A. I. 1967 The exact theory of steady state waves on the surface of a heavy liquid. *M.R.C. Technical summary report* no. 813. The University of Madison.

Schwartz, L. W. 1974 Computer extension and analytic continuation of Stokes' expansion for gravity waves. *J. Fluid Mech.* **62**, 553–578.

Serrin, J. B. 1952 Uniqueness theorems for two free boundary problems. *Am. J. Math.* **74**, 492–806.

Stokes, G. G. 1880 On the theory of oscillatory waves. Appendix B: Considerations relative to the greatest height of oscillatory irrotational waves which can be propagated without change of form. *Math. Phys. Pap.* **1**, 225–228.

Zygmund, A. 1959 *Trignometric series*, vol. 1. Cambridge University Press.