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Taylor-Görtler Vortices Expected in the Air Flow on Sea Surface Waves—II*

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Abstract: The instability of Taylor-Görtler vortices which are expected in the air flow on water waves was studied in part I, under the assumption that the curvature around the crest or the trough of water waves, where the instability was expected to take place first, was constant, namely that the characteristics of the vortices were affected little by the local change of the curvature along the direction of the progress of water waves (the direction of x-axis) However, the curvature actually varies from positive to negative, or vice versa. In order to study this effect, the instability of Taylor-Görtler vortices is examined with respect to the range of the part of a constant curvature, in the model in which the curvature is positive constant near the trough and negative constant near the crest, and zero in the intermediate regions, respectively. It is shown that as the region of the constant curvature becomes narrower, the instability tends to weaken. For the same example with part I, namely, when the wind of 12.2 m s^{-1} is blowing over swells of 15 m in wavelength, if the range of constant curvature near the trough is taken as a quarter of one wave length, the critical wave height becomes 0.96 m instead of 0.50 m, and conversely, the wave length and the height of center of the vortex become 11.9 m and 2.1 m instead of 24 m and 3.7 m, respectively.

Further, using the energy equations, quantitative estimates are performed of the intensity of the vortices which develop when the wave height of the swell is 1.05 m in the above described example, and also of the influence of the vortices upon the wind profile when the equilibrium state is reached. When the vortices are generated and grow to attain to an equilibrium state interacting with the mean flow, the maximum x-component of velocity in the vortices is about 1.04 m s^{-1}. Consequently, the wind profile undergoes a considerable distortion from the logarithmic one near the level of 2 m height. This distorted wind profile has a form similar to those sometimes observed above the sea surface.

1. Introduction

In part I, the possibility of the generation of Taylor-Görtler vortices in the air flow above water waves was proposed. The conditions for the instability were examined, under the assumptions that the propagating water waves were of a two-dimensional sinusoidal form with infinitely long crests, and the profile of turbulent air flow was logarithmic.

There we discussed the question of where on the wave the instability would occur fastest. So, we considered only around the crest or the trough, where vortices were expected to be generated first, and then assumed that the curvature was constant. The Görtler parameter, G_r, became to be a wave steepness, δ_w, since the sinusoidal form and the turbulent flow were used as the water waves and the air flow, respectively.

The relation between the wind velocity and the wave phase velocity was expressed as the level of matched layer, y_c, where the two velocities are equal to each other. Accordingly, the characteristics of the instability of vortices were determined only by two factors of the wave number of vortices and the level of matched layer. The numerical calculations indicated the possibility of its generation on the trough of water waves in a state of strong wind. The following two points, however, were to be investigated as mentioned in part I:

(1) It was necessary to consider the effect of the change of curvature along the x-axis on the characteristics of instability in the vicinity

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of the trough, since there is the reverse of the sign of curvature between the trough and the crest of water waves.
(2) The vertical scale of the critical minimum vortices were comparable with the extent of constant local curvature. So, it was unreasonable to neglect the terms depending on the extent, comparing with terms of the variation along the vertical axis.

Both of these problems raise the question of how much the characteristics of vortices vary with the distribution of the curvature in the vicinity of the trough. In order to study this point, we have assumed the distribution of the curvature as such that it is positive constant near the trough and negative constant near the crest, and zero in the intermediate region, and see how the instability changes by the region of the constant curvature. This kind of approach was adopted by TOBAK (1971), in his study of the local applicability of the Görtler theory. He specified that the wall, as a boundary, had a small concave curvature over only a limited extent in the direction of flow, and it is plane in the other region. The terms depending on x-axis could not be neglected in such a specification. So, he had the idea that the variable x was eliminated by the introduction of Fourier transforms with respect to the x-axis, assuming that the perturbations would die out both far upstream and far downstream. By a suitable approximation, he succeeded in deriving a simple correction to the Görtler critical factor for the instability, and brought into evidence the dependence of the factor on the extent of wall curvature.

By using his idea, we may investigate the generation of these vortices above water waves in case of strong wind. It became clear in part I that the vortex instability occurred only around the trough in this case. It will be reasonable to assume that the vortices would decay sufficiently rapidly both upstream and downstream from the trough, (BC) and (B′C′), as indicated in Fig. 1. Consequently, our problem may be made coincide with Tobak's study, about the product of the velocity component of vortices, u_i'', and the constant curvature around the trough, κ.

Next, by using the energy equation, we may perform the quantitative estimates of the intensity of vortices, and of the influence of the vortices upon the profile of the mean wind flow for the equilibrium state, which could not be done from the linear theory of part I. As STUART (1958) and LANDAU-LIFSHITZ (see MONIN and YAGLOM, 1971) attempted to extend a linear theory to a non-linear theory, we introduce the following assumption: even though the generated vortices grow over the range of infinitesimal amplitude (the critical state) into that of finite amplitude (the super-critical state), the basic flow merely intensifies the amplitude without changing appreciably the distribution of vortices. Using this assumption, the profile and the intensity of vortices may be determined from the energy equation on the basis of the linear theory in part I.

In our description below, the same notations for variables will be used as those in part I.

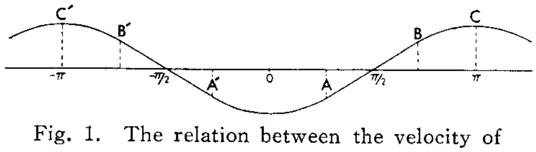

Fig. 1. The relation between the velocity of vortices, u_i'', and the curvature of water waves, κ, under the condition of strong wind.

Sections on water wave.

	(AA′)	(AB) and (A′B′)	(BC) and (B′C′)
Vortices' velocity, u_i''	Non-zero	Small non-zero	Zero
Curvature, κ	Positive const.	Zero	Negative const.
The product, $\kappa u_i''$	Non-zero	Zero	Zero

2. Change of curvature of water wave surface

By means of the method mentioned in section 1, we will consider only the conditions of strong wind, in which the vortices are expected to be generated on the trough of water waves. Of course, the conditions of gentle wind can also be treated similarly.

Let us introduce the curvilinear co-ordinate system as shown in Fig. 2, and the equations for a secondary flow, following the method of part I, except for the point that the terms depending on the x-axis are not neglected in this paper. In Fig. 2, $(-a, a)$ denotes the range of the constant curvature near the trough, the U and C_w the basic flow and the phase velocity

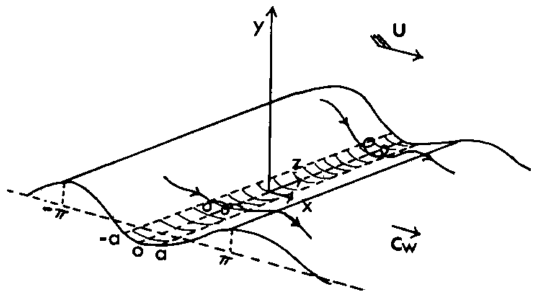

Fig. 2. Schematic picture of the form of vortices and the range of constant curvature, $(-a, a)$, on water waves and the curvilinear co-ordinates, for the condition of strong wind.

of water waves, respectively. Since it was indicated in part I that the characteristics of vortices were almost independent of the Reynolds number, we assume for convenience that the Reynolds number is constant of

$$R_e=\frac{U_0}{k_w\nu_T}=\frac{1}{0.16}=6.25 \qquad (2.1)$$

In this case, the eddy kinematic viscosity, ν_T, becomes from Eq. (3.13) in part I,

$$\nu_T=\frac{\kappa_a u_*}{k_w}=\frac{0.16\,U_0}{k_w} \qquad (2.2)$$

where κ_a indicates von Kármán constant, u_* the friction velocity, k_w the wave number of water waves and U_0 the reference value of wind velocity. This assumption does not affect the conclusion of this paper. Also the exchange coefficient, A, becomes

$$A=\mu+\rho\frac{\overline{(-u_i'u_j')}}{\dfrac{dU}{d\eta}}\approx\rho\frac{\overline{(-u_i'u_j')}}{\dfrac{dU}{d\eta}}$$

A dimensionless form of A is defined as

$$M=\frac{A}{\rho\nu_T} \qquad (2.3)$$

By using (2.2) and (2.3), we can obtain $M=y$ easily, and assume that the curvature along the x-axis is constant, although this is not exact at the points B′, A′, A and B in Fig. 1. Reducing the equation of motion to a dimensionless form with the same reference values as those in part I yields

$$\begin{aligned}
&\frac{\partial u''}{\partial t}+U\frac{\partial u''}{\partial x}+\frac{dU}{dy}v''=-\frac{\partial p''}{\partial x}\\
&\quad+\frac{M}{Re}\left(\nabla^2u''-2\kappa\frac{\partial v''}{\partial x}\right)+\frac{1}{Re}\frac{dM}{dy}\left(\frac{\partial u''}{\partial y}+\frac{\partial v''}{\partial x}\right)\\
&\frac{\partial v''}{\partial t}+U\frac{\partial v''}{\partial x}+2\kappa Uu''=-\frac{\partial p''}{\partial y}\\
&\quad+\frac{M}{Re}\left(\nabla^2v''+2\kappa\frac{\partial u''}{\partial x}\right)+\frac{2}{Re}\frac{dM}{dy}\left(\frac{\partial v''}{\partial y}\right)\\
&\frac{\partial w''}{\partial t}+U\frac{\partial w''}{\partial x}=-\frac{\partial p''}{\partial z}\\
&\quad+\frac{M}{Re}(\nabla^2w'')+\frac{1}{Re}\frac{dM}{dy}\left(\frac{\partial v''}{\partial z}+\frac{\partial w''}{\partial y}\right)\\
&\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}=0 \qquad (2.4)
\end{aligned}$$

with $\kappa=\pi\delta_w{}^*e^{-y}\cos x$, where κ and $\delta_w{}^*$ denote the curvature and the steepness of water waves, respectively. The eigen value in this problem is indicated by the asterisk to distinguish it from δ_w of part I. The dimensionless forms, u'', v'', w'', p'' and U are the x-, the y-, and the z-component of velocity, pressure of vortices, and the x-component of velocity of mean flow, respectively. These do not include the Reynolds number for convenience, being different from equations in part I. The forms o vortices are assumed as in the case of part I as

$$\left.\begin{aligned}
u''&=e^{\gamma t}\cos kz\,u(x,y)\\
v''&=e^{\gamma t}\cos kz\,v(x,y)\\
w''&=e^{\gamma t}\sin kz\,w(x,y)\\
p''&=e^{\gamma t}\cos kz\,p(x,y)
\end{aligned}\right\} \qquad (2.5)$$

The boundary conditions are

$$u(x,y),\ v(x,y),\ w(x,y)\ \text{and}\ \frac{\partial u(x,y)}{\partial x}\to 0\ \text{as}\ |x|\to\pi \qquad (2.6)$$

Now, the variable x in (2.4) may be eliminated by the introduction of the Fourier transforms:

$$\tilde{u}(y;\omega)=\int_{-\pi}^{\pi}u(x,y)e^{-i\omega x}dx,$$

$$u(x,y)=\frac{1}{2\pi}\sum_{\omega_1=-\infty}^{+\infty}\tilde{u}(y;\omega_1)e^{-i\omega_1 x} \qquad (2.7)$$

Similarly, $\tilde{v}(y;\omega)$, $\tilde{w}(y;\omega)$ and $\tilde{p}(y;\omega)$ are defined. We substitute Eq. (2.5) in (2.4), and obtain, using the neutral condition, $\gamma=0$, together with the relation indicated in Fig. 1,

$$U_y\tilde{v}+2i\frac{M\omega}{Re}\kappa\int_{-a}^{a}ve^{-i\omega x}dx$$
$$=-i\omega\tilde{p}+\frac{M}{Re}(\tilde{u}_{yy}-\alpha^2\tilde{u})+M_y(\tilde{u}_y+i\omega\tilde{v})$$
$$2U\kappa\int_{-a}^{a}ue^{-i\omega x}dx-2i\frac{M\omega}{Re}\kappa\int_{-a}^{a}ue^{-i\omega x}dx$$
$$=-\tilde{p}_y+\frac{M}{Re}(\tilde{v}_{yy}-\alpha^2\tilde{v})+2\frac{M_y}{Re}\tilde{v}_y$$
$$0=k\tilde{p}+\frac{M}{Re}(\tilde{w}_{yy}-\alpha^2\tilde{w})+\frac{M_y}{Re}(\tilde{w}_y-k\tilde{v})$$
$$i\omega\tilde{u}+\tilde{v}_y+k\tilde{w}=0 \qquad (2.8)$$

where $\alpha^2=\alpha^2(y;\,\omega)=k^2+\omega^2+i\omega\dfrac{Re}{M}U$ and the suffix y denoting differentials with respect to y. The substitutions

$$\tilde{u}_1=\frac{\lambda^2}{k^2}\left(\tilde{u}-i\frac{\omega}{\lambda^2}\tilde{v}_y\right)$$
$$\tilde{v}\to\frac{1}{Re}\tilde{v} \qquad (2.9)$$

and the successive elimination of $\tilde{p}$ and $\tilde{w}$ yield a pair of equations involving $\tilde{u}_1$, $\tilde{v}$ and integrals of u and v, omitting a negligibly small terms in U_{yy} and $\alpha^2{}_y$:

$$M\{\tilde{u}_{1yy}(y;\omega)-\alpha^2\tilde{u}_1(y;\omega)\}+M_y\tilde{u}_{1y}(y;\omega)$$
$$=U\tilde{v}(y;\omega)+I_1(y;\omega)$$
$$M\{\tilde{v}_{yyyy}(y;\omega)-(\lambda^2+\alpha^2)\tilde{v}_{yy}(y;\omega)$$
$$+\alpha^2\lambda^2\tilde{v}(y;\omega)\}+2M_y\{\tilde{v}_{yyy}(y;\omega)$$
$$-k'^2\tilde{v}_y(y;\omega)\}=I_2(y;\omega) \qquad (2.10)$$

where

$$I_1(y;\omega)=2iM\omega\kappa\int_{-a}^{a}v(x,y)e^{-i\omega x}dx$$
$$=2i\frac{M\omega}{\pi}\kappa\left(a+2\tan^{-1}\frac{\sin a}{1-\cos a}\right)\tilde{v}(y;0)$$
$$I_2(y;\omega)=-2\lambda^2Re^2U\kappa\int_{-a}^{a}u(x,y)e^{-i\omega x}dx$$
$$+2\omega\kappa\int_{-a}^{a}e^{-i\omega x}\{i\lambda^2ReMu+\omega M(v_y-v)$$
$$+\omega M_y v\}dx=\frac{2\kappa}{\pi}\Big[k^2Re(-ReU$$
$$+i\omega M)\tilde{u}_1(y;0)\Big\{\Big(a+2\tan^{-1}\frac{\sin a}{1-\cos a}$$
$$-\pi\frac{\sinh k(\pi-a)}{\sinh k\pi}\cos a\omega\Big)$$
$$+\frac{\pi\omega}{k}\frac{\cosh k(\pi-a)}{\sinh k\pi}\sin a\omega\Big\}$$
$$+i(-ReU+i\omega M)\tilde{v}_y(y;0)$$
$$\times\Big\{\omega\Big(a+2\tan^{-1}\frac{\sin a}{1-\cos a}$$
$$-\pi\frac{\sinh k(\pi-a)}{\sinh k\pi}\cos a\omega\Big)$$
$$-\pi k\frac{\cosh k(\pi-a)}{\sinh k\pi}\sin a\omega\Big\}$$
$$+\omega^2\{M\tilde{v}_y(y;0)+(M_y-M)\tilde{v}(y;0)\}$$
$$\times\left(a+2\tan^{-1}\frac{\sin a}{1-\cos a}\right)\Big]$$
$$\lambda^2=k^2+\omega^2$$
$$\alpha^2=k^2+\omega^2+i\frac{\omega Re}{M}U$$
$$k'^2=k^2+\frac{\omega}{2}\left(3\omega+i\frac{Re}{M}U\right) \qquad (2.11)$$

From the substitutions of (2.9), the dimensionless forms become the same as those of part I. To obtain the similar solution to those obtained in part I, we must put $\omega=0$ in Eqs. (2.10) and (2.11). It yields a pair of homogeneous differential eqations for $\tilde{u}_1(y;0)$ and $\tilde{v}(y;0)$ with $I_1\to0$, $k'^2\to k^2$ and $\tilde{u}_1\to\tilde{u}$:

$$M\{\tilde{u}_{yy}(y;0)-k^2\tilde{u}(y;0)\}+M_y\tilde{u}_y(y;0)=U_y\tilde{v}(y;0)$$
$$M\{\tilde{v}_{yyyy}(y;0)-2k^2\tilde{v}_{yy}(y;0)+k^4\tilde{v}(y;0)\}$$
$$+2M_y\{\tilde{v}_{yyy}(y;0)-k^2\tilde{v}_y(y;0)\}$$
$$=-2\pi\delta^*{}_{wc}e^{-y}\Big\{\frac{1}{\pi}\Big(a+2\tan^{-1}\frac{\sin a}{1-\cos a}$$
$$-\pi\frac{\sinh k(\pi-a)}{\sinh k\pi}\Big)\Big\}Uk^2Re^2\tilde{u}(y;0) \qquad (2.12)$$

The implication is that when ω in (2.11) and (2.12) is put equal to zero in solving the eigen value problem, the integrands of I_1 and I_2 are significantly modified by the variations in $\tilde{u}_1(y;\omega)$ and $\tilde{v}(y;\omega)$ only in the vicinity of $\omega_1=0$.

Putting $a=\pi$ and $\delta^*{}_{wc}\to\delta_{wc}$ in (2.12)

$$M\{\tilde{u}_{yy}(y;0)-k^2\tilde{u}(y;0)\}+M_y\tilde{u}_y(y;0)=U_y\tilde{v}(y;0)$$
$$M\{\tilde{v}_{yyyy}(y;0)-2k^2\tilde{v}_{yy}(y;0)+k^4\tilde{v}(y;0)\}$$
$$+2M_y\{\tilde{v}_{yyy}(y;0)-k^2\tilde{v}_y(y;0)\}$$
$$=-2\pi\,\delta_{wc}e^{-y}Uk^2Re^2\tilde{u}(y;0) \qquad (2.13)$$

The perturbation Eqs. (2.13) coincide with those of part I. Hence, the solutions in part I mean the condition of which the curvature of water waves has a constant curvature everywhere over one wavelength. From comparing (2.12) with (2.13), we may see that these equations are formally identical, by the following relation:

$$\delta^*_{wc} = f(k, a)\delta_{wc} \qquad (2.14)$$

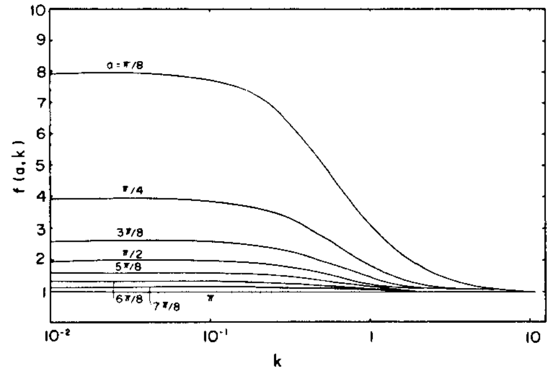

Fig. 3. Correction factor, $f(a, k)$, given by Eq. (2.15), for several values of the extent of constant curvature, a.

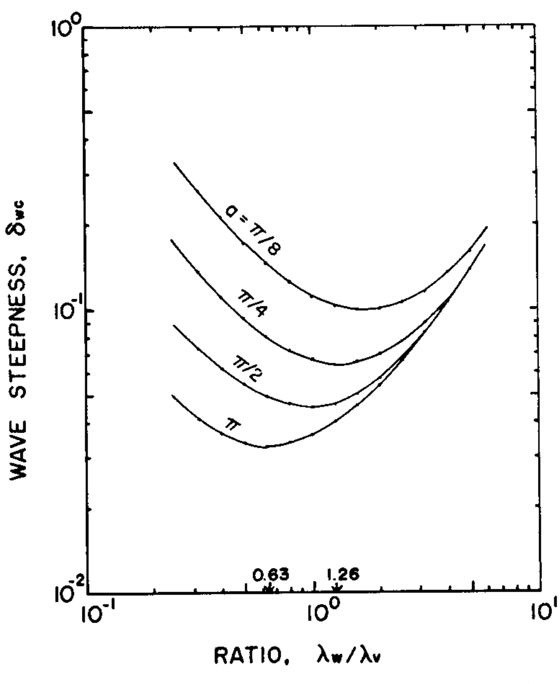

Fig. 4. Critical wave steepness, δ^*_{wc}, for several values of the extent of constant curvature, a, for the same condition of $\lambda_w = 15$ m and $u^* = 0.6\ \mathrm{m\,s^{-1}}$, as the case of E in part I.

where

$$f(k, a) = \frac{\pi}{a + 2\tan^{-1}\dfrac{\sin a}{1-\cos a} - \pi\dfrac{\sinh k(\pi - a)}{\sinh k\pi}} \qquad (2.15)$$

Eq. (2.14) shows how the critical eigen value, δ^*_{wc}, is modified by the extent of constant curvature, a. So, we have only to correct the value of δ_{wc} obtained in part I using the correction factor (2.15), without changing the solution in part I substantially. The distribution of correction factor, $f(k, a)$, inceases rapidly as a and k decrease. The results are illustrated in Fig. 3. So, from (2.14), the δ^*_{wc} is modified significantly only when a and k are small. It is indicated from these results that a larger value of the critical wave steepness than that obtained in part I is required to initiate instability, in the same manner as Tobak's result.

The example of curve E in part I is shown in Fig. 4, using the correction factor (2.15)

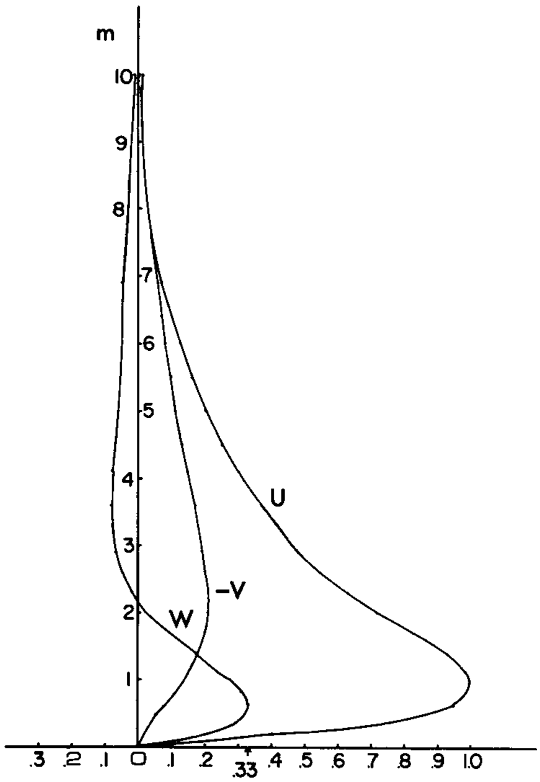

Fig. 5. The profiles of disturbances in the critical minimum state, under the same condition as Fig. 4. The ratio of components may be calculated, but the absolute magnitude may not, on account of the linear theory.

with several values of a^*. For this case, the critical minimum wave numbers, $k_{cm}=0.631$, and 1.26 indicate the extents, $a=\pi$ and $\pi/4$, respectively. We can obtain the results from Fig. 4 that $k_{cm}=(\lambda_w/\lambda_v)_{cm}\approx 1.26$ and $\delta^*_{wcm}\approx 0.064$ for $a=\pi/4$, indicating the followings. When a wind of about 12.2 m s^{-1} at 10 m level is blowing over water waves of the wave length of 15 m, the vortices are generated in the critical state of the wave height of about 0.96 m, or $\delta^*_{wcm}=0.064$. The critical wave length and the height of the center of the vortices are about 11.9 m and 2.1 m, respectively. The values are about 2 and 1/2 times those of part I, respectively. The profile is shown in Fig. 5. The value of correction factor, $f(k, a)$, is also a constant value of about 1.59, since the value of k_{cm} is constant as indicated in section 1. So, δ^*_{wcm} is corrected to about 1.59 times that of $a=\pi$.

From the above-described results, we may consider that better solutions than those of part I have been obtained, although it is not an exact answer to the questions discussed in section 1.

Lastly, we calculate the special case of the above example with zero phase velocity where the water wave surface is fixed. This case may be treated in a similar way as the above case. Putting the phase velocity of water wave, C_w, zero in (3.11) of part I,

$$U=\frac{U_1}{U_0}=\ln\frac{y}{z_0} \qquad (2.16)$$

with $y_c=z_0=8\times 10^{-3}\,k_w u_*^2$, where U_1 denotes the wind velocity.

From Eq. (2.16), the non-dimensional critical level, y_c, becomes to be equal to the non-dimensional roughness length, z_0. There is no significantly different point between the cases of non-zero phase velocity and zero phase velocity, except for the point that the latter has a lower matched level, y_c, than that of the former. Hence we may calculate the instability of this case, using the neutral eigen value of Fig. 7 in part I and the correction factor (2.15), and may anticipate that in this case it is more unstable. The calculation indicates that for the wave height of swells of about 0.46 m, the vortices may be generated with the wave length of about 9.46 m. So, we may conclude that the phase velocity in the direction of the wind velocity, C_w, act as the stabilizing factor, being due to the increase of the value of the critical level, y_c.

* "Curve D" on p. 251 of part I should all be replaced by "Curve E".

3. Estimation of intensity of the vortices by energy equations

It is believed from a theory of instability that the instability is not expected to occur until the eigen value of applied disturbance attains the critical minimum value. As mentioned in section 1, we may calculate the intensity of vortices and the influence upon the mean flow in the vicinity of the critical minimum state. The following approximation is obtained from calculation of part I and the discussion in section 2. The disturbance equation for the condition of the constant curvature may be applicable to conditions around the critical minimum state of the present problem, by introducing the corrected eigen value, δ^*_w. So, the basic equations of the air flow, (2.4), can be simplified, using this approximation, as the following.

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=y\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+(\pi Re^2\delta_w^*)e^{-y}u^2 =-\frac{\partial p}{\partial y}+y\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+2\frac{\partial v}{\partial y}$$

$$\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} =-\frac{\partial p}{\partial z}+y\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\partial w}{\partial y}$$

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (3.1)$$

Now let

$$\begin{aligned} u&=U(t, y)+u''(t, y, z)\\ v&=\qquad\quad\; v''(t, y, z)\\ p&=P(t, y)+p''(t, y, z) \end{aligned} \qquad (3.2)$$

where u'', v'' and p'' denote the components of finite disturbances with zero mean value. Eqs. (3.2) differ from the formulation of section 3 in part I, with respect to the following points: (a) the mean flow, $U(t, y)$, is a function both of t and y, and (b) the amplitude of disturbances is finite, in which non-linear terms can not be neglected. So, the substitution of Eq. (3.2) in (3.1) yields, with rearranging the non-linear terms in it using the equation of continuity, the following equations of motion of total flow:

$$\begin{aligned}
&\frac{\partial}{\partial t}(U+u'')+v''\frac{\partial U}{\partial y}+\frac{\partial}{\partial y}(u''v'')+\frac{\partial}{\partial z}(u''w'') \\
&\quad=\frac{\partial}{\partial y}\left(y\frac{\partial U}{\partial y}\right)+y\nabla^2 u''+\frac{\partial u''}{\partial y} \\
&\frac{\partial v''}{\partial t}+\frac{\partial}{\partial y}(v''^2)+\frac{\partial}{\partial z}(v''w'')+\pi Re^2\delta^*_w(U+u'')^2 \\
&\quad=-\frac{\partial}{\partial y}(P+p'')+y\nabla^2 v''+2\frac{\partial v''}{\partial y} \\
&\frac{\partial w''}{\partial t}+\frac{\partial}{\partial y}(v''w'')+\frac{\partial}{\partial z}(w''^2) \\
&\quad=-\frac{\partial p''}{\partial y}+y\nabla^2 w''+\frac{\partial w''}{\partial y} \\
&\qquad\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}=0
\end{aligned} \tag{3.3}$$

A case of undisturbed flow is specified by $u''=v''=w''=0$, and $U=U(y)$ is independent of t. So, the flow becomes from the first Eq. in (3.3)

$$\frac{d}{dy}\left(y\frac{dU}{dy}\right)=0$$

From the above equation, it is indicated that a mean flow has the following logarithmic profile for the case of non-disturbed flow.

$$U(y)=\ln\frac{y}{y_c} \tag{3.4}$$

Taking the average of (3.3) with respect to the z-axis yields the equations of mean flow for the case of disturbed flow:

$$\begin{aligned}
&\frac{\partial U}{\partial t}+\frac{\partial}{\partial y}(\overline{u''v''})=\frac{\partial}{\partial y}\left(y\frac{\partial U}{\partial y}\right) \\
&\quad\frac{\partial}{\partial y}(\overline{v''^2})+\pi Re^2\delta_w^* e^{-y}(U^2+\overline{u''^2}) \\
&\qquad=-\frac{\partial P}{\partial y}
\end{aligned} \tag{3.5}$$

with $\frac{\partial}{\partial z}(\overline{u''v''})=\frac{\partial}{\partial z}(\overline{v''w''})=\frac{\partial}{\partial z}(\overline{w''^2})=0$. The bar denotes an average over the z-axis. In a state of equilibrium, we may obtain the following stationary equations:

$$\left.\begin{aligned}
&\frac{\partial U}{\partial t}=0, \text{ and } U(t, y)\to U_e(y) \\
&\frac{d}{dy}(\overline{u''v''})=\frac{d}{dy}\left(y\frac{dU_e}{dy}\right)
\end{aligned}\right\} \tag{3.6}$$

The boundary conditions are

$$U_e=\ln\frac{y}{y_c} \quad \text{at } y\to z_0 \text{ and at } y=y_\infty \tag{3.7}$$

where the suffix e refers to an equilibrium state. We use y_∞ as the height where the disturbances die out. By the approximation of $\int_{z_0}^{y_\infty}\left(-\frac{u''v''}{y}\right)\times dy\approx\int_0^\infty\left(-\frac{u''v''}{y}\right)dy$ and $\int_{z_0}^{y}\left(-\frac{u''v''}{y}\right)dy\approx\int_0^y\left(-\frac{u''v''}{y}\right)dy$, the solution of (3.6) for the conditions (3.7) is,

$$\begin{aligned}
U_e(y)=\ln\frac{y}{y_c}-\Bigg\{&\int_0^y\left(-\frac{u''v''}{y}\right)dy \\
&-\left(\frac{\ln\frac{y}{z_0}}{\ln\frac{y_\infty}{z_0}}\right)\int_0^\infty\left(-\frac{u''v''}{y}\right)dy\Bigg\}
\end{aligned} \tag{3.8}$$

where y_∞ is a kind of parameter. Moreover, putting $y_\infty\to\infty$ in (3.8) yields

$$U_{e\infty}(y)=\ln\frac{y}{y_c}-\int_0^y\left(-\frac{u''v''}{y}\right)dy \tag{3.9}$$

These equations show how the distribution of mean flow is affected by the Reynolds stress due to the disturbances. If the amplitude is finite, the second-order terms can not be neglected, and the distribution of the mean flow is not logarithmic in the equilibrium state. The solution of (3.9) is also obtained with the first boundary condition in (3.7) and following boundary condition,

$$\frac{dU_e}{dy}\to\frac{1}{y} \quad \text{as } y\to\infty$$

indicating that the gradient of $U_e(y)$ coincides the logarithmic profile at sufficiently high levels.

The unknown function in (3.8) is only the intensity of disturbance which is a function of time, since the forms of disturbance are the same as those calculated in part I. Let us obtain this factor, $A(t)$, by using the energy equation. Before it, we require the perturbation equations of vortices. These are afforded by the total flow (3.3) and the mean flow (3.5)

$$\frac{\partial u''}{\partial t}+v''\frac{\partial U}{\partial y}+X_1=y\nabla^2u''+\frac{\partial u''}{\partial y}$$
$$\frac{\partial v''}{\partial t}+\pi Re^2\delta_w^*Uu''+X_2$$
$$=-\frac{\partial p''}{\partial y}+y\nabla^2v''+2\frac{\partial v''}{\partial y}$$
$$\frac{\partial w''}{\partial t}+X_3=-\frac{\partial p''}{\partial z}+y\nabla^2w''+\frac{\partial w''}{\partial y}$$
$$\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}=0 \tag{3.10}$$

where

$$X_1=\frac{\partial}{\partial y}(u''v''-\overline{u''v''})+\frac{\partial}{\partial z}(u''w'')$$
$$X_2=\frac{\partial}{\partial y}(v''^2-\overline{v''^2})+\frac{\partial}{\partial z}(v''w'')$$
$$X_3=\frac{\partial}{\partial y}(v''w''-\overline{v''w''})+\frac{\partial}{\partial z}(w''^2)$$

If we integrate the disturbance equations to obtain the energy balance relation, we have, without approximation,

$$\frac{\partial}{\partial t}\iint\frac{1}{2}(u''^2+v''^2+w''^2)dy\,dz$$
$$=\iint(-u''v'')\left(\frac{dU}{dy}+2\pi Re^2\delta_w^*e^{-y}U\right)dydz$$
$$-\iint y(\xi''^2+\eta''^2+\zeta''^2)dy\,dz \tag{3.11}$$

where

$$\xi''=\frac{\partial w''}{\partial y}-\frac{\partial v''}{\partial z},\ \eta''=\frac{\partial u''}{\partial z},\ \text{and}\ \zeta''=\frac{\partial u''}{\partial y}$$

where the integrands are evaluated over a volume bounded by $(0, \infty)$ in the direction of y-axis and by on wavelength in the direction of z-axis, and the quadratic terms, X_i, of (3.10) integrate out. The net rate of increase of disturbance energy is equal to the difference between the integral of the product of the Reynolds stress and the flow shear, which represents the rate of transfer of kinetic energy from the mean flow to the disturbance, and the rate of dissipation of kinetic energy due to eddy viscosity. Substituting Eq. (3.8) in (3.11), we may obtain the energy balance equation composed of the velocities of the disturbed flow only. From the above, profiles of the disturbance due to the vortices may be expressed by the following forms:

$$\left.\begin{aligned}u''(t,y,z)&=A(t)u(y)\cos kz\\ v''(t,y,z)&=A(t)v(y)\cos kz\\ w''(t,y,z)&=A(t)w(y)\sin kz\end{aligned}\right\} \tag{3.12}$$

where $A(t)$ denotes the amplitude of disturbances. Substituting Eqs. (3.12) and (3.11) yields

$$\frac{d|A|^2}{dt}=2kc_i|A|^2-2k\alpha|A|^4 \tag{3.13}$$

where

$$2kc_i=(\gamma_2-\gamma_4+\delta_w^*\gamma_3)/\gamma_1$$
$$\text{and}\ 2k\alpha=(\gamma_5+\delta_w^*\gamma_6)/\gamma_1$$
$$\gamma_1=\int_0^\infty\frac{1}{2}(u^2+v^2+w^2)dy,\quad \gamma_2=\int_0^\infty\frac{(-uv)}{y}dy$$
$$\gamma_3=2\pi Re^2\int_0^\infty(-uv)\,e^{-y}\ln\frac{y}{y_c}dy$$
$$\gamma_4=\int_0^\infty y\left\{\left(\frac{dw}{dy}+kv\right)^2+k^2u^2+\left(\frac{du}{dy}\right)^2\right\}dy$$
$$\gamma_5=\frac{1}{2}\int_0^\infty\frac{(-uv)}{y}\left\{(-uv)-\frac{\int_0^\infty\left(\frac{(-uv)}{y_1}\right)dy_1}{\ln\frac{y_\infty}{z_0}}\right\}dy$$
$$\gamma_6=\pi Re^2\int_0^\infty(-uv)e^{-y}\left\{\int_0^y\left(-\frac{uv}{y_1}\right)dy_1\right.$$
$$\left.-\frac{\ln\frac{y}{z_0}}{\ln\frac{y_\infty}{z_0}}\int_0^\infty\left(-\frac{uv}{y_1}\right)dy_1\right\}dy \tag{3.14}$$

The differential Eq. (3.13) has the same form with (2.15) in Stuart's paper. The $A(t)$ has a real value, since the instability of Taylor-Görtler vortices is believed to have no oscillating solutions. The solution of $A(t)$ depends on its magnitude, as follows.

For infinitesimal disturbances, putting $|A(t)|^4 \ll |A(t)|^2$ in (3.13) yields

$$\frac{d|A|^2}{dt}=2kc_i|A|^2 \tag{3,15}$$

Eq. (3.15) agrees with the linear equation. In a critical state, Eqs. (3.14) and (3.15) yield

$$c_i=0 \text{ and } \delta^*_{wc}=\frac{\gamma_4-\gamma_2}{\gamma_3} \quad (3.16)$$

The critical eigen value (3.16) is calculated according to the energy equation. Although it is a different method from part I, the obtained value 0.0639 is in close agreement with that of 0.0633, for the minimum critical state for the example in part I.

For finite disturbances, the general solution of (3.13) is of the form

$$|A|^2=\frac{Bc_i\, e^{2kc_it}}{1+B\alpha\, e^{2kc_it}} \quad (3.17)$$

$$|A|^2\rightarrow|A|_e^2=\frac{c_i}{\alpha} \quad \text{as } t\rightarrow\infty \quad (3.18)$$

where B is an arbitrary real constant. As shown in Fig. 6, the amplitude (3.17) is the result of linear theory as $t\rightarrow-\infty$ and non-linear theory as $t\rightarrow\infty$. The amplitude factor, $|A|$, converges to $|A|_e$ in a equilibrium state as $t\rightarrow\infty$. In a super-critical state, $c_i>0$, indicated by $\alpha>0$ from (3.14), the equilibrium state has the amplitude (3.18).

Eqs. (3.14), (3.16) and (3.18) yield

$$|A|_e^2=\frac{\gamma_3}{\gamma_5+\delta_w{}^*\gamma_6}(\delta^*_w-\delta^*_{wc}) \quad \text{for } \delta^*_w>\delta^*_{wc} \quad (3.19)$$

From (3.19), $|A|_e^2$ is proportional to the value $(\delta^*_w-\delta^*_{wc})$. This equation may not be applicable to the range of $\delta^*_w\gg\delta^*_{wc}$, since it is possible that the higher harmonics of the basic mode of the disturbance may be generated in this range. It may be expected that the method used above will be valid for some range of δ^*_w above the critical value, δ^*_{wc}.

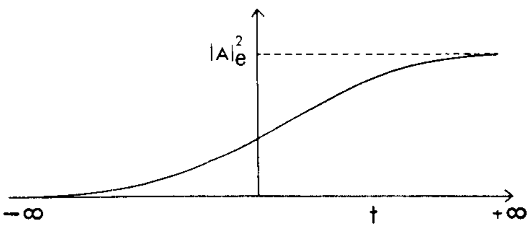

Fig. 6. Amplitude growth of vortices in super-critical case.

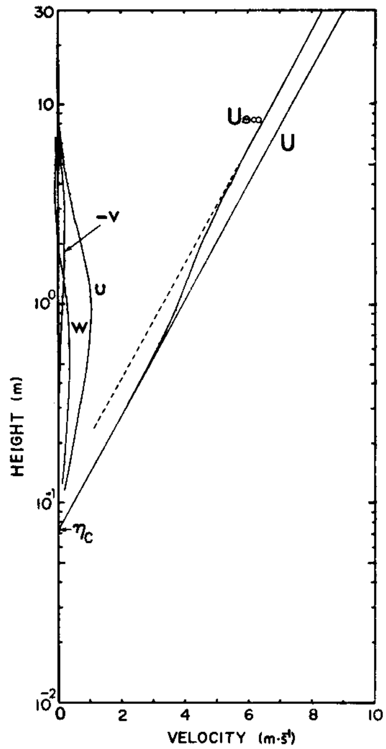

Fig. 7. The distribution of the disturbance flows $(u, -v, w)$ and the mean flow, $U_{e\infty}(y)$, in the super-critical minimum equilibrium state of $\delta_w=0.07$, under the same conditions as Fig. 5. The height where the disturbances die out, η_∞, is taken as ∞ (or $y_\infty\rightarrow\infty$). In a mean flow, the logarithmic profiles of the heavy line and the broken line indicate the cases of infinitesimal amplitude and finite amplitude of disturbance, respectively.

Let us consider the example in section 2 again, putting $\delta_w=0.07$ and $y_\infty\rightarrow\infty$. From the result in section 2, it is expected that the vortices with the wave steepness of 0.07 grow to attain the super-critical equilibrium state, interacting with the mean flow. In this state, the amplitude of vortices and the distribution of the mean flow are calculated by Eqs. (3.19) and (3.9), respectively, as indicated in Fig. 7. From the results, the following two points should be noted. Firstly, about the distribution of the vortices, the disturbed flow has the profile which varies sinusoidally with the amplitude of about 1.04 m s^{-1} at the center of about 3.80 m s^{-1}, in the direction of z-axis, around a height of 1 m. Secondly, the vertical distribution of the mean flow, which was

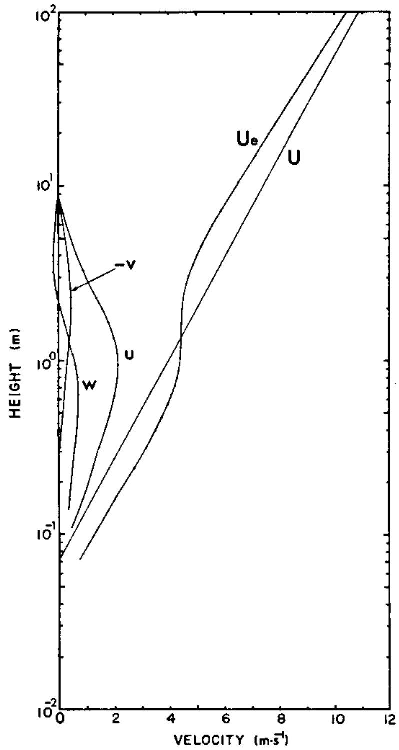

Fig. 8. Same with Fig. 7, but for the condition $\eta_\infty = 700$ m.

logarithmic in the condition of the absence of the vortices, has a bending or a kind of kink near the wave surface and becomes logarithmic at heights much larger than the wave height.

4. Discussion

By the use of the boundary conditions (3.7), together with the condition of $y_\infty = \infty$, we have obtained the solution of $U_{e\infty}(y)$, as shown in Fig. 7. These boundary conditions mean that the gradient of the wind speed at higher levels does not change by the existence of the considering vortices. The original logarithmic wind profile $U(y)$ has shifted to the left in the figure, or to the side of weaker wind speed, in higher levels. This does not indicate that the wind speed at higher levels is weakened by the occurrence of the Taylor-Görtler vortice, but may be interpreted as follows. When the Taylor-Görtler vortices exist, the same conditions of the wind profile in lower levels correspond to weaker wind velocities at higher levels. In this example of the calculation, the disturbance velocity of the order of $1\ \mathrm{m\,s^{-1}}$ appears at 1-m level, and the same conditions of the wind correspond to about 10 % lower wind speed at 10-m level. The same thing will be said for the water surface wind stress, which may be determined by the wind conditions at the lower levels, and expressed by, say, z_0. It should be noticed that there is a significant bending in the vertical distribution of the mean wind velocity, or the wind profile, between the levels of from 0.5 m to 3 m. There is a possibility that a kink of the wind profile, which has been sometimes reported in the observation above the water waves, corresponds this bending of wind profile caused by the existence of the Taylor-Görtler vortices.

For the sake of comparison, a calculation by use of the other boundary conditions has been performed. Namely, the Eq. (3.8) for the condition $\eta_\infty = 700$ m, instead of the condition $y_\infty = \infty$, yields the distribution of vortices and the mean flow as indicated in Fig. 8. It seems that there is a noticeable difference in the distributions between Fig. 7 and Fig. 8. In the latter case, the gradients of the wind speed at the higher and the lower levels are larger, and the disturbance velocities are much greater. In this calculation, however, a little different value of z_0 is used for the convenience of the calculation. But it does not produce a significant matter for our present consideration to see a tendency in the change in the disturbance velocity and in the wind profile, since the change in the gradient of the wind speed at the lower levels will cause a change in z_0, of which we have no method of prediction as z_0 is used for the lower boundary. Also, the selection of the value of y_∞ is arbitrary, and the situation varies large according to the value of y_∞. Consequently, it is considered that the former boundary conditions, yielding the result of Fig. 7, are more appropriate.

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海洋波浪上に発生すると予想される Taylor-Görtler 渦について—II

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要旨: part I において，波浪上の対数風速分布に対して，その波面上で Taylor-Görtler 渦による不安定性がもっとも早く生ずる谷および峯を選び，その付近で波面の曲率が一定という仮定，すなわち，波面の曲率の波の進行方向（x軸方向）の変化による影響が小さいという仮定のもとに，この不安定性の特性を評価した．

しかし，現実にはx軸方向に曲率が正負に変化している．その影響をみるために，曲率が，波面の谷または峯の近傍で，それぞれ正または負の定数，その前後で0として曲率の存在する範囲に関して，そこに生ずる Taylor-Görtler 渦の不安定性を吟味した．それによると，一定曲率の範囲が小さくなると，渦は起こりにくいことが示された．例えば，part I と同じ例で，波長 15 m のうねりの上に 12.2 m s^{-1} の風が吹くとき，一定曲率の範囲が一波長の 1/4 の場合，渦が発生するに必要な波高が part I の約 0.44 m から約 0.96 m へと大きくなり，その時の渦の波長と軸の高さは，それぞれ，約 24 m と約 3.7 m から約 11.9 m と約 2.1 mへと小さくなった．

次に，エネルギー方程式から，渦の強さと渦による平均風速の分布への影響を求めた．上の例で，波高だけを 1.05 m に高めた場合，渦ができて平衡状態に達したとき，渦のx軸方向の速度成分の分布の最大値は，約 1.04 m s^{-1} となり，従って，平均風速は，高さ 2 m 付近で，対数法則から，かなりずれる結果となった．これは，海面上で，しばしば観測されるキンクのある風速分布と類似の形となった．

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