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A note on breaking waves

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Some simple general properties of wave breaking are deduced from the known behaviour of surface gravity waves in deep water, on the assumption that breaking occurs in association with wave groups. In particular we derive equations for the time interval, τ, between the onset of breaking of successive waves:

$$\tau = \frac{T}{|1-(\boldsymbol{c}\cdot\boldsymbol{c}_{\mathrm{g}})/c^2|},$$

and for the propagation vector $\boldsymbol{c}_{\mathrm{b}}$ (referred to as the 'wave-breaking vector') of the position at which breaking, once initiated, will proceed:

$$\boldsymbol{c}_{\mathrm{b}} = \boldsymbol{c}\left(1-\frac{\boldsymbol{c}\cdot\boldsymbol{c}_{\mathrm{g}}}{c^2}\right)+\boldsymbol{c}_{\mathrm{g}}.$$

Here $\boldsymbol{c}$ is the phase velocity, and $\boldsymbol{c}_{\mathrm{g}}$ the group velocity, of waves of period T. Interfacial waves, internal gravity waves, inertial waves and planetary waves are considered as particular examples. The results apply not only to wave breaking, but to the movement of any property (e.g. fluid acceleration, gradient Richardson number) that is carried through a medium in association with waves. One application is to describe the distribution, in space and time, of regions of turbulent mixing, or transitional phenomena, in the oceans or atmosphere.

1. INTRODUCTION

Our objective is to describe where and when extreme properties, conditions such as the maximum shear or acceleration, are produced by waves in a dispersive medium.

Gravity waves on the ocean surface are probably the best studied examples of dispersive waves, and have the simplicity of propagating on the interface between the sea and the atmosphere rather than in three dimensions as do many other types of waves. The property of surface wave breaking is visible and familiar. We have developed an analysis in terms of this class of waves because what little is known about wave breaking (and, in particular, is substantiated by observations) derives mainly from studies of this particular wave type. We also choose to take examples from fluids, although the theory that is developed will equally well apply to other media.

Surface wave breaking in deep water appears to occur mainly in association with wave groups (Donelan *et al.* 1974; Thorpe & Hall 1983; Longuet-Higgins & Smith 1983; Weissman *et al.* 1984). Deep-water surface gravity waves have the unusual property that their phase velocity, $\boldsymbol{c}$ (the speed of advance of the wave crests in a direction normal to the crest line), is in the same direction as the group velocity, $\boldsymbol{c}_{\mathrm{g}}$. In particular, small-amplitude waves have the property that $\boldsymbol{c} = 2\boldsymbol{c}_{\mathrm{g}}$, and this provides a fairly good description of their propagation characteristics. After the onset of breaking at a certain position in the wave group, with the production of 'white water' or foam and subsurface bubbles, the breaking wave advances through the group at roughly its phase speed, continuing to break for some period of time until it reaches a second position towards the front of the wave group where it ceases to break. The time taken for a wave to advance one wavelength, λ, through the group, to the relative position at which its predecessor had begun to break, is given by

$$\tau = \lambda/|\boldsymbol{c}-\boldsymbol{c}_{\mathrm{g}}|, \tag{1}$$

which is therefore equal to $\lambda/(\frac{1}{2}c)$ or $2T$, where T is the wave period, because $c = \lambda/T$.

The time interval between the successive onset of breaking in the wave group is thus twice the wave period. Such periodicity, derived from an essentially linear wave theory, is indeed observed in the ocean (Donelan *et al.* 1974; Longuet-Higgins 1976). It determines, for example, the frequency at which bursts of underwater sound are produced as a consequence of wave breaking (Farmer & Vagle 1988). The time for which breaking persists, once initiated in a particular wave (i.e. the 'duration' of breaking), determines whether breaking occurs in one wave at a time within a group, or simultaneously in two or more, and whether breaking is found at all fixed positions past which a wave group advances (figure 1) (see Thorpe & Humphries 1980).

There are few reported measurements of the extent of breaking along the crest line of a surface wave. The extent may depend on the angular width of the wave spectrum and on factors relating to the stability of wave trains (see, for example, Su *et al.* 1984). The extent of the breaking along the crest appears, however, to be on average less than the distance over which the wave advances whilst continuing to break (Thorpe & Hall 1983). In shallow water, where the phase and group velocities are nearly equal, breaking is less intermittent and whitecaps may persist on a given wave for many wave periods (see Longuet-Higgins 1988, §4).

Rather little is known of the tendency of other types of waves to form groups, but it seems likely that this is a general property of dispersive waves in natural systems that have a fairly narrow-band spectrum resulting from their mode of generation, their inherent instability, or the tendency for interactions to occur between waves. Transient groups of waves are often found in the stratified ocean and the atmosphere, for example in the inertial internal waves observed in the ocean by Pinkel (1983) or Mied *et al.* (1987), and the planetary-scale waves of Rossby or Kelvin-type observed in the ocean by Freeland *et al.* (1975), or in the atmosphere by Yanai & Maruyama (1966), Wallace & Kousky (1968), and Van Loon *et al.* (1973). Such packets of waves are inherent in many theoretical studies. The transient response of an ocean to a storm (Price 1983), for example, involves inertial gravity waves. The spin-up of an ocean basin (Anderson & Gill 1975) produces horizontally propagating planetary Rossby waves, whereas stratospheric warming has been explained by the presence of pockets of vertically propagating

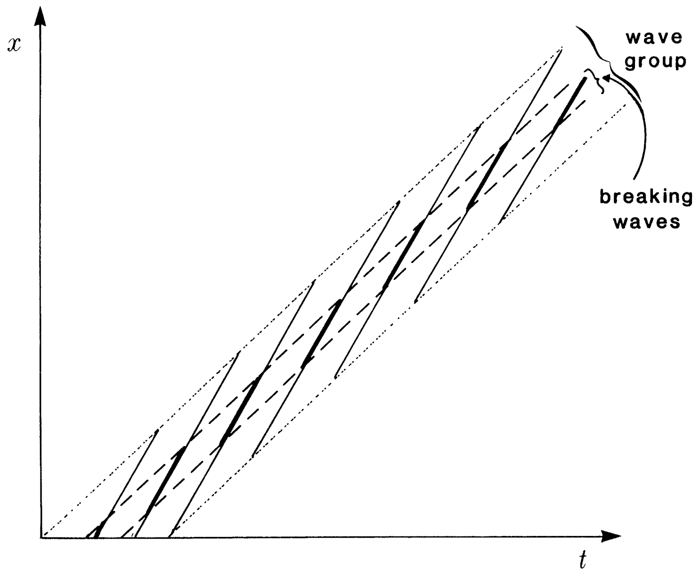

FIGURE 1. Individual surface wave crests, drawn as continuous lines, moving with a wave group, outlined by dotted lines, in the distance, x (measured in direction c_g), and time, t, domain. A region of breaking waves, where the wave crests are drawn as thick lines, moves in the central region of the group within the region bounded by broken lines, which mark the onset or cessation of wave breaking. Because the phase speed, c, of waves in deep water equals twice the group speed, c_g, the slope of the continuous lines marking the wave crests is twice that of the broken or dotted lines marking the group. The horizontal distance between the continuous lines is T, the wave period, and the horizontal distance between the onset of thick lines is the interval between successive wave breaking, $2T$ (see text). The projection of the thick lines onto the t axis is the duration of wave breaking, D. If $D > 2T$ there is at least one wave breaking at all times, t. If $D < T$, then there are positions, x, at which no wave will be breaking as the group propagates past, and the turbulence generated (and bubble clouds and foam) is in discrete patches.

(and breaking) planetary waves (Holton 1972; Edmon *et al.* 1980; McIntyre & Palmer 1985).

We shall consider therefore how the ideas described above, which satisfactorily describe the main fatures of breaking surface waves, can be extended to other wave systems, assuming that they too form groups. It is supposed that the point where a condition, the onset of breaking, is found is advected with the group velocity, but that once reached or exceeded, the local property, 'wave breaking', advances for a while with a point of constant phase in a wave (e.g. the wave crest as illustrated in figure 1), although not necessarily with the phase *velocity* (see later). By describing the condition as that of 'wave breaking', we are assigning a general term to an event in which a critical value of some scalar property of the flow is passed. This might, for example, be a certain acceleration in the surface gravity waves or zero density gradient in waves in a stratified fluid. The theory, however, applies to any scalar property of the wave, and to the periodic exceedance of any value by that property. There is no need to involve a transition of flow state (e.g. from laminar to turbulent flow), dissipation, or a change, perhaps temporary, in the local properties of the medium through which the waves are propagating caused by turbulent mixing, some of the usual consequences of wave breaking. As Donelan *et al.* (1974) point out in the case of surface waves:

> Not only breakers, but other features of the surface elevation, or, we might add, of the velocity field, will tend to occur periodically in the ocean. Thus over any moderate area of the sea surface ... the surface elevation may be expected to repeat itself roughly every two wave periods.

We here identify a 'breaking region' simply as one in which some property is exceeded, and aim to determine both how frequently such regions repeat in more general waves, and the characteristic orientation of such regions. Given this identification of 'breaking', we need not concern ourselves further with the particular characteristics of wave breaking in different wave systems.

Two linear approximations, which the observations show are reasonable for surface waves, are, however, implicit in extending the theory to a more general class of waves. The first is that if 'breaking' produces a change in fluid medium, the changes produced in one wave do not (for example, by changing the density gradient) significantly affect the speed or propagation conditions of waves that subsequently pass through that region of fluid. The second is an approximation about group velocity, namely that it provides a 'working' description of waves even in the generally nonlinear region to which the theory is applied. That it succeeds for surface waves is insufficient evidence that it does for other types of waves! We council caution in applying the results too readily. Further measurements seem appropriate.

With this proviso we consider examples of dispersive plane waves of increasing complexity, interfacial waves in a two-layer fluid, two-dimensional internal gravity waves and pure inertial waves, two-dimensional and eventually three-dimensional, planetary waves. General results are presented in §2.4. A particular application of the work is in describing those regions of fluid in which wave mixing may be produced, and especially in providing estimates of the frequency of the occurrence of extreme properties.

2. Analysis

2.1. *Waves with parallel phase and group velocities: interfacial internal waves and internal waves propagating between horizontal boundaries*

The conclusions that may be drawn about the breaking of these waves are exactly analogous to those of the surface waves decribed in §1. The interval between the successive onset of breaking in groups may be written

$$\begin{aligned} \tau &= \lambda/|\boldsymbol{c}-\boldsymbol{c}_{\mathrm{g}}|, \quad \text{(as in (1))} \\ &= \lambda/c|1-c_{\mathrm{g}}/c|, \\ &= T/|1-c_{\mathrm{g}}/c|, \end{aligned} \tag{2}$$

where $c_{\mathrm{g}} = \partial\sigma/\partial k$ is the group velocity in the direction in which the waves are propagating with wave number, k, and frequency, σ. For interfacial waves the dispersion relation $\sigma = \sigma(k)$ was found by Stokes (1847). For deep layers above and below the interface, $c_g/c = \frac{1}{2}$, and so $\tau = 2T$, as for surface waves. In general for internal gravity waves, $c_g \leqslant c$, even when there is a vertical gradient of the horizontal velocity (see, for example Thorpe 1978, where values of c_g and c are calculated for waves in a shear flow), and τ will be greater than T with waves moving forward through the wave group. (If $c_g > c$, as for capillary gravity waves of length less than about 1.7 cm on water, waves move backwards through the group.) We turn next to the more interesting cases when $\boldsymbol{c}$ and $\boldsymbol{c}_g$ are not parallel.

2.2. *Waves with phase and group velocities at right angles to each other*

2.2.1. *Two-dimensional internal gravity waves; no rotation*

For simplicity we consider next the case of two-dimensional internal gravity waves propagating in an unbounded fluid of constant buoyancy frequency, N. We might imagine, for example, a group of waves that has been generated by oscillating a horizontal cylinder for a short period of time at a frequency, σ less than N, as could be achieved in the apparatus described by Mowbray & Rarity (1967). Such a group is sketched in figure 2. The group may be described by

$$\eta = F(x - t\,\partial\sigma/\partial k,\ z - t\,\partial\sigma/\partial m)\sin(kx + mz - \sigma t) \tag{3}$$

(see Gill 1982, p. 106), where η is a wave amplitude (or density perturbation), x and z and horizontal and vertical coordinates, (k, m) is the wave number vector, F is an amplitude function which defines the envelope of the group and

$$\boldsymbol{c}_g = (\partial\sigma/\partial k, \partial\sigma/\partial m) \tag{4}$$

is the group velocity. For internal gravity waves $\boldsymbol{c}$ and $\boldsymbol{c}_g$ are at right angles (Phillips 1966). The position in the group where a wave first begins to break (say on FF in figure 2*a*) does not move in the direction of $\boldsymbol{c}$ and so the time taken for one wave (e.g. GG in figure 2*a*) to advance to the position in the group where the previous wave had begun to break (i.e. FF) is just the wave period; the time between the onset of breaking in successive waves is thus equal to the wave period.

Suppose now that at time t one wave EE in figure 2*a* is breaking along the crest line over an extent, AB $= l$, which depend on the envelope function, F. (The extent of breaking in the y-direction, akin to breaking along the crest in surface waves, is ignored.) If the group is propagating steadily without change in form, as described in (3), the pattern of waves one period, T, later will be as shown in figure 2*a* but displaced by a distance $\boldsymbol{c}_g T$, i.e. the wave crest FF now lying along the line EE but with its breaking zone displaced by a distance $\boldsymbol{c}_g T$. The breaking zones of successive waves will overlap if $c_g T$ is less than the maximum of the extent, l, of the breaking crest.

Suppose now that at a time $t + \delta t$ the wave EE has advanced to E′E′ (figure 2*b*). The leading edge of the breaking zone A has advanced to D (where AD $= \boldsymbol{c}\,\delta t$) on the wave crest but, because the amplitude function, F, and the critical position for the onset of breaking, have advanced by a distance $\boldsymbol{c}_g\,\delta t$, breaking now extends to A′, where DA′ $= \boldsymbol{c}_g\,\delta t$. Thus the point, A, at which breaking occurs, moves in a direction AA′ with the vector $\boldsymbol{c} + \boldsymbol{c}_g$, $= \boldsymbol{c}_b$, say.

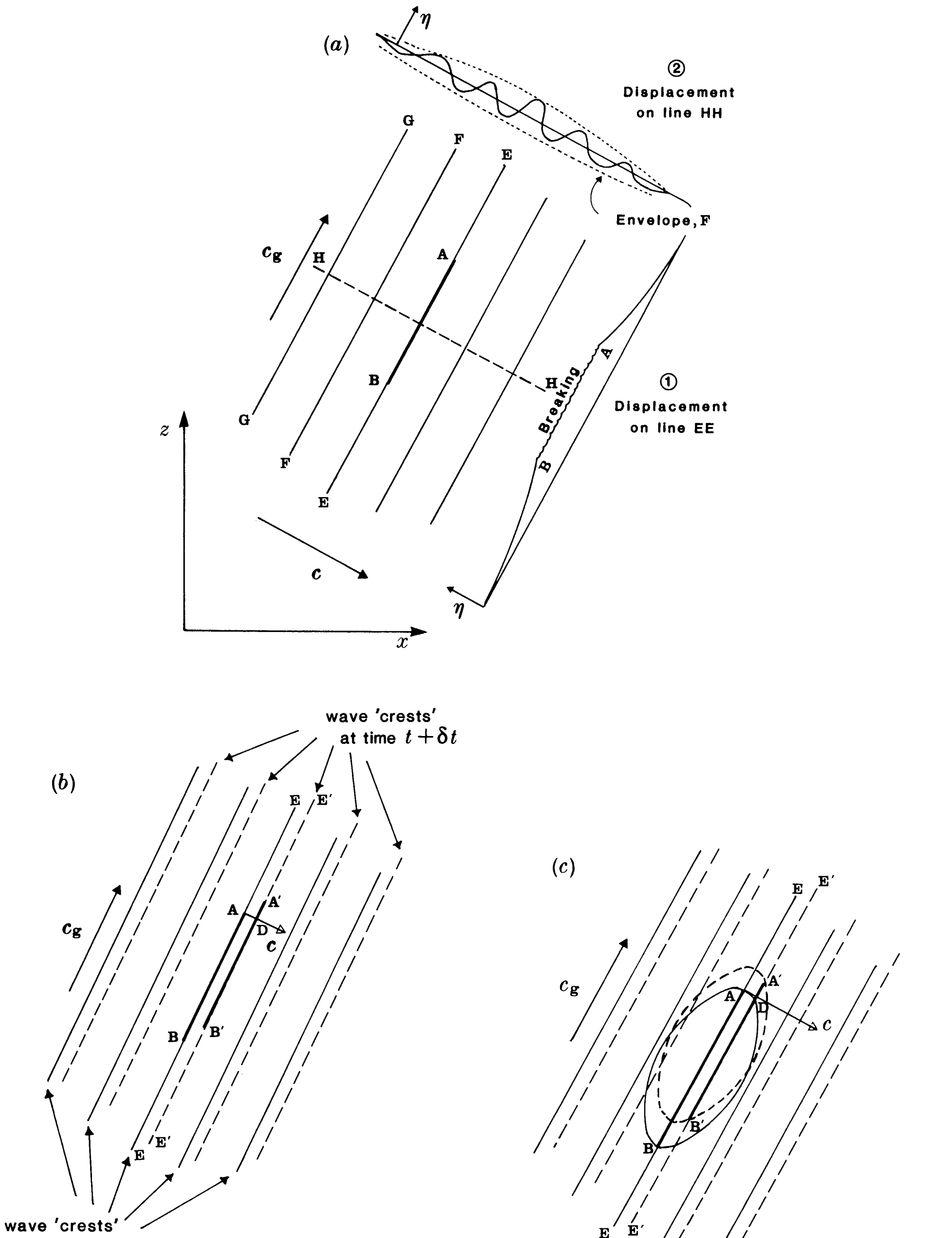

FIGURE 2. A group of waves in which the phase speed, $\boldsymbol{c}$, and group velocity, $\boldsymbol{c}_g$, are at right angles. The lines represent surfaces of constant phase (e.g. wave crests) in the x, z plane (motion independent of y). (a) Section 1 is a diagrammatic representation of wave parameter (e.g. elevation, η), along crest EE (i.e. parallel to $\boldsymbol{c}_g$) whereas section 2 shows the corresponding 'elevation' parallel to vector $\boldsymbol{c}$. (b) Shows the advance of the crests from their position at time t (full lines) to that at $t+\delta t$ (broken lines), whereas (c) shows the movement of a constant contour on the amplitude function (full and broken closed curve) during the same time interval.

This, however, is not a complete description. We might suppose, for example, that breaking occurs, or is suppressed, as wave crest passes a particular contour where the envelope function F has a critical value, F_c (see figure 2*c*). While this contour advects with the group velocity, the point A moves around the contour, so that DA′ is equal to $\boldsymbol{c}_g\,\delta t$ plus the gradient of F at F_c in the direction of $\boldsymbol{c}$, multiplied by $c\,\delta t$, so that A moves with the vector

$$\boldsymbol{c}+\boldsymbol{c}_g\left(1+\frac{c}{c_g}\frac{\partial F}{\partial \zeta}\right),$$

where ζ is the coordinate in the direction of $\boldsymbol{c}$.

For a group for which F is symmetrical about a line parallel to $\boldsymbol{c}$, the mid-point of the breaking crest AB advects at speed $\boldsymbol{c}+\boldsymbol{c}_g$ exactly, and this we refer to as the *wave breaking vector* for internal gravity waves, $\boldsymbol{c}_b$.

Now the dispersion relation for internal gravity waves is

$$\sigma^2 = N^2k^2/(k^2+m^2), \tag{5}$$

and the group velocity (4) is found to be

$$\boldsymbol{c}_g = (\sigma m/kK)\,(m/K, -k/K), \tag{6}$$

where $\boldsymbol{c} = (\sigma/K)(k/K, m/K)$, normal to $\boldsymbol{c}_g$, and $K = (k^2+m^2)^{\frac{1}{2}}$ is the wave number. The wave-breaking vector is therefore

$$\boldsymbol{c}_b = \boldsymbol{c}+\boldsymbol{c}_g = \sigma/k(1,0), \tag{7}$$

which is *horizontal* with a speed σ/k that follows the horizontal movement of the lines of constant phase (see also Phillips 1966, §5.4). The horizontal extent of the breaking wave regions will be $\sigma D/k$, where D is the duration of the breaking event. If $\sigma D/k$ is much greater than the extent of breaking in the $\boldsymbol{c}_g$ direction, the wave-breaking vector defines both the axis of the breaking region and the principal zone of breaking.

2.2.2. *Two-dimensional inertial gravity waves*

Two-dimensional waves in a stratified fluid rotating about a vertical axis with angular velocity f also have phase speeds normal to their group velocities. The interval between the onset of breaking is therefore equal to the wave period. The dispersion relation is

$$\sigma^2 = (N^2k^2+f^2m^2)/(k^2+m^2) \tag{8}$$

and

$$\boldsymbol{c}_g = \frac{km\sigma(N^2-f^2)}{K(N^2k^2+f^2m^2)}\left(\frac{m}{K}, -\frac{k}{K}\right). \tag{9}$$

The wave-breaking vector is found to be

$$\boldsymbol{c}_b = \boldsymbol{c}+\boldsymbol{c}_g = \frac{\sigma\chi}{N^2k^2+f^2m^2}\left(\frac{kN^2}{\chi}, \frac{mf^2}{\chi}\right), \tag{10}$$

where $\chi^2 = k^2N^4+m^2f^4$. For pure inertial waves (when $N = 0$) this vector is vertical, parallel to the axis of rotation. In general, if the inclination of the group velocity to the horizontal is α (as shown in figure 3), then by using (8)

$$\tan\alpha = \frac{k}{m} = \left(\frac{\sigma^2-f^2}{N^2-\sigma^2}\right)^{\frac{1}{2}} \tag{11}$$

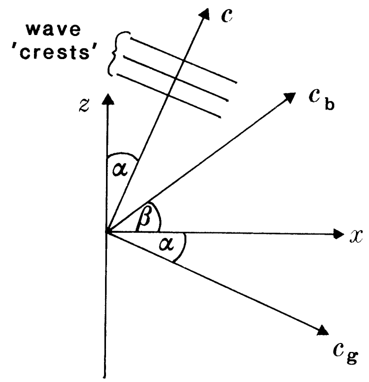

FIGURE 3. The directions of vectors $\boldsymbol{c}$, $\boldsymbol{c}_g$ and $\boldsymbol{c}_b$, for inertial gravity waves (here with $N > f$). The wave breaking vector, $\boldsymbol{c}_b$, is horizontal for pure internal gravity waves and vertical for pure inertial waves.

(σ must be between f and N for waves to exist). The inclination of the wave breaking vector to the horizontal is β where, from (8) and (10),

$$\tan\beta = \frac{mf^2}{kN^2} = \frac{f^2}{N^2}\left(\frac{N^2-\sigma^2}{\sigma^2-f^2}\right)^{\frac{1}{2}}. \tag{12}$$

We find that the angle between $\boldsymbol{c}_b$ and $\boldsymbol{c}_g$ is

$$\alpha+\beta = \arctan\left[\sigma^2 \tan\alpha/(\sigma^2-f^2)\right]. \tag{13}$$

2.3. *Two-dimensional waves with* c *and* $\mathbf{c}_g$ *neither parallel nor at right angles; planetary waves in a homogeneous fluid*

Longuet-Higgins (1965) has shown that the dispersion relation of divergent planetary, or Rossby, waves on a homogeneous fluid of depth h is

$$\sigma(k^2+l^2+f^2/gh)+\beta k = 0, \tag{14}$$

where f is the Coriolis parameter, β is its northwards gradient, and k, l, are the eastward and northwards components of the wavenumber. The locus of the wavenumber (k, l) of waves of given frequency, σ, is a circle with centre, C, at $(-\beta/2\sigma, 0)$ and radius $\surd(\beta^2/4\sigma^2-f^2/gh)$ (see figure 4). The wavenumber component k is always negative so that the phase velocity has always a westward component. The group velocity of a wave with wavenumber denoted by a point A on the circle is parallel to AC. Evidently $\boldsymbol{c}$ and $\boldsymbol{c}_g$ are not parallel, unless $\boldsymbol{c}$ is directly to the west, or at right angles to each other, unless OA is a tangent to the wavenumber circle.

The point at which breaking occurs advances with a wave, but moves along its constant phase surface with the component of $\boldsymbol{c}_g$ lying in the constant phase surface, i.e. with a speed $c_g \sin(\mathrm{CAO})$ or $|\boldsymbol{c}\times\boldsymbol{c}_g|/c$. The *wave-breaking vector* $\boldsymbol{c}_g$ is thus inclined to OA at an angle θ, where $\tan\theta = |\boldsymbol{c}\times\boldsymbol{c}_g|/c^2$. Now

$$\boldsymbol{c}_g = \frac{\sigma}{k(k^2+l^2+f^2/gh)}(l^2-k^2+f^2/gh,\ -2kl), \tag{15}$$

and so $\tan\theta = -l/k = \tan(\mathrm{AOC})$.

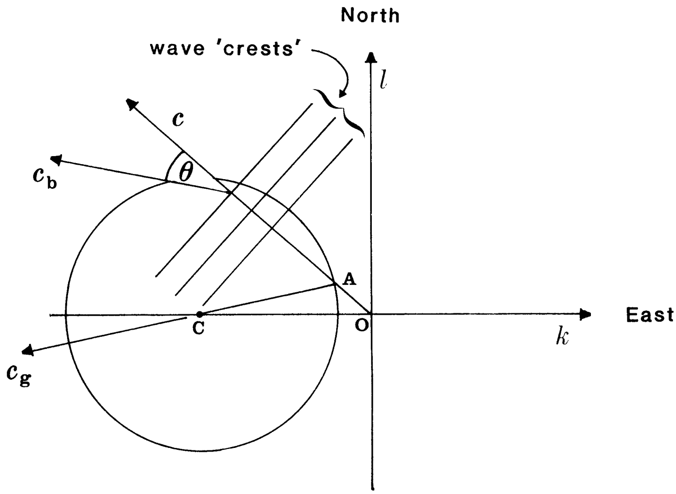

FIGURE 4. The wavenumber diagram for planetary waves in a homogeneous fluid of constant depth. The angle θ is equal to angle AOC so that c_b is westwards (see text).

Hence the wave-breaking vector, the main axis of the breaking region, is westwards, moving at a speed that is the westward component of $\boldsymbol{c}$, i.e. $-\sigma/k$.

The interval between the onset of breaking is $\tau = \lambda/|c - c_g \cos(\pi - \text{CAO})|$, the time taken for a wave crest to travel one wavelength, λ, through the group, or

$$\tau = \frac{\lambda}{|c-(\boldsymbol{c}\cdot\boldsymbol{c}_g)/c|}, \quad = \frac{T}{|1-(\boldsymbol{c}\cdot\boldsymbol{c}_g)/c^2|} \tag{16}$$

because $c = \lambda/T$. Now $1-(\boldsymbol{c}\cdot\boldsymbol{c}_g)/c^2 = 2(k^2+l^2)/(k^2+l^2+f^2gh)$ so that

$$\tau = \tfrac{1}{2}T[1+(f^2/ghK^2)], \tag{17}$$

where $K^2 = k^2+l^2$. (For non-divergent planetary waves, the term f^2/gh in (14)–(17) vanishes and $\tau = \frac{1}{2}T$.) The condition that OA shall be a tangent to the wavenumber circle is $K^2 = f^2/gh$, when from (17), $\tau = T$ and $\boldsymbol{c}$ and $\boldsymbol{c}_g$ are at right angles. Long waves, with wavenumber $K < f/\sqrt{gh}$ have $\tau > T$ whereas short waves, $K > f/\sqrt{gh}$, have $T > \tau > \frac{1}{2}T$.

2.4. *The general case*

Equation (16) derived in 2.3 is a general expression for periodicity of breaking of plane waves. The wave-breaking vector is the sum of two vectors, the vector $\boldsymbol{c}$, which follows the phase surface of the wave, and vector component of $\boldsymbol{c}_g$ in the phase surface, which for the plane waves considered here, is normal to $\boldsymbol{c}$. The latter component is of magnitude $|\boldsymbol{c}\times\boldsymbol{c}_g|/c$ and is in the direction $(\boldsymbol{c}\times\boldsymbol{c}_g)\times\boldsymbol{c}$, equal to $c^2\boldsymbol{c}_g-(\boldsymbol{c}\cdot\boldsymbol{c}_g)\,\boldsymbol{c}$, which, because $|(\boldsymbol{c}\times\boldsymbol{c}_g)\times\boldsymbol{c}| = |\boldsymbol{c}\times\boldsymbol{c}_g|\,c$, has the unit vector $[c^2\boldsymbol{c}_g-(\boldsymbol{c}\cdot\boldsymbol{c}_g)\cdot\boldsymbol{c}]/|\boldsymbol{c}\times\boldsymbol{c}_g|\,c$. The vector is thus $\boldsymbol{c}_g-(\boldsymbol{c}\cdot\boldsymbol{c}_g)\cdot\boldsymbol{c}/c^2$, and the wave-breaking vector is

$$\boldsymbol{c}_b = \boldsymbol{c}[1-(\boldsymbol{c}\cdot\boldsymbol{c}_g)/c^2]+\boldsymbol{c}_g, \tag{18}$$

which is $\boldsymbol{c}$ if $\boldsymbol{c}_g$ and $\boldsymbol{c}$ are parallel, when the breaking region moves with the phase velocity (as in surface and interfacial waves, §2.1), or $\boldsymbol{c}+\dot{\boldsymbol{c}}_g$ when $\boldsymbol{c}_g$ and $\boldsymbol{c}$ are at right angles (as found in §2.2).

2.5. *Planetary waves in a stratified fluid*

We now apply the results to plane planetary waves, which have, on a rotating sphere, a vertical component of motion. The dispersion relation is now

$$\sigma\left(k^2+l^2+\frac{f^2m^2}{N^2}\right)+\beta k = 0 \tag{19}$$

(Gill 1982, p. 523), which describes waves propagating with a westward component ($k<0$), and

$$\boldsymbol{c}_g = \frac{-\sigma}{k(k^2+l^2+\gamma m^2)}[(k^2-l^2-\gamma m^2), 2kl, 2\gamma km] \tag{20}$$

with components to the east, north and upward vertical, where $\gamma = f^2/N^2$ and m is now the vertical wavenumber. We find $(\boldsymbol{c}\cdot\boldsymbol{c}_g)/c^2 = -1$ so that, from (16), $\tau = \frac{1}{2}T$, and from (18)

$$\boldsymbol{c}_b = 2\boldsymbol{c}+\boldsymbol{c}_g = \frac{\beta}{K^2(k^2+l^2+\gamma m^2)^2}[(2k^2+K^2)(k^2-l^2-\gamma m^2)-4k^4,\; 2klm^2(1-\gamma),\; -2mk(k^2+l^2)(1-\gamma)], \tag{21}$$

where K^2 is now $k^2+l^2+m^2$.

If $\gamma<1$, as is usual in the ocean and in the atmospheric stratosphere, the signs of the northward components of $\boldsymbol{c}_g$ and $\boldsymbol{c}_b$ are the same, but the vertical components have opposite signs, the wave-breaking vector component being negative (i.e. downards, in the same vertical direction as the phase is advancing) when the group velocity is upwards ($m<0$).

3. Discussion

Breaking waves are poorly understood. Their dynamics are yet to be thoroughly studied, and although we know some of those processes that give rise to wave breaking (see, for example, Thorpe 1987*b* for a discussion of transitional processes in stratified fluids) we cannot yet quantify the frequency of the phenomena or assess their contribution to heat and momentum transport. The rudimentary analysis that we have discussed is a guideline to further theoretical study and to the observational strategy, which, in the ocean or the atmosphere, may be required to collect further data that might help to identify the causes of mixing. The analysis may also help to interpret existing measurements, as has knowledge of the frequency of onset of surface wave breaking in the study of ambient sound (Farmer & Vagle 1988).

The wave-breaking vector has been defined as the speed and direction in which extreme values or properties of waves, such as their breaking, move in association with waves. For pure internal gravity waves in a uniform density gradient the direction is horizontal (§2.2.1), and the effect of a wave group passing through a stratified fluid is to produce reduced density gradients and perhaps turbulent mixing in layers that are predominantly horizontal, no matter what the wave frequency. This process may contribute to the production of the layered fine-structure found in the ocean, or to the formation of layers when a stratified fluid is stirred by a vertical grid, a process involving internal waves radiated from the near-grid turbulent region (Thorpe 1982). Pure inertial waves have a wave-breaking vector parallel to the axis of rotation (§2.2.2), perhaps associated with the axial vortices observed in grid-stirring experiments in rotating fluids by Hopfinger *et al.* (1982). If so, similar experiments in fluids that are both rotating and uniformly stratified should develop mixing regions or 'traumata' (*vide* McEwan 1973), which are inclined to both the horizontal and vertical directions.

Regions of relatively high temperature variance have been observed in the ocean seasonal thermocline by Marmorino *et al.* (1985) using a towed thermistor chain. The large-scale trends of such patches are not constrained to be along constant density surfaces. This is consistent with the conclusions of §2.2.2 if the patches are produced by breaking inertial gravity waves. Regions of high rates of dissipation of turbulent kinetic energy have also been observed to advect vertically in the thermocline, in associated with vertically propagating inertial waves (Gregg *et al.* 1986). There is some suggestion in these observations (e.g. loc. cit. figure $9a$) that the duration of mixing is of order of 12–24 h.

Groups of internal waves of tidal period are generated at the shelf-break by an interaction with the surface tides, being strongest at Spring tide (see, for example, New 1988). Some of these propagate off-shore into the stratified ocean. For typical values of $N = 10^{-3}\ \mathrm{s}^{-1}$, $f = 1.1\times10^{-4}\ \mathrm{s}^{-1}$ and $\sigma = 1.4\times10^{-4}\ \mathrm{s}^{-1}$ (corresponding to the M_2 tide at 50° N) we find, from (11) that the inclination of the group velocity of the waves to the horizontal is 5.0° and, from (13), the angle between the group velocity and the wave breaking vector is 12.9°. If, as shown in figure 3, the group velocity is downwards as waves propagated away from the relatively shallow shelf-break region into deep water above the continental slope, the wave-breaking vector is inclined upwards at an angle of 7.9°. Similarly in regions of the Biscay Abyssal Plain adjoining the Celtic Sea or Armorican Shelf or off the west slope of the Porcupine Bank, where the M_2 internal tidal motions are observed to be large (Pingree 1988; Thorpe 1987a), one might expect to find regions of small Richardson number where fluid is prone to instability at intervals of one tidal period ($\tau = T$) but, once developed, advance at an angle of a few degrees from the horizontal in a direction different from that of the group or phase velocities. The extent of such regions will depend on the envelope function of the internal wave group, or the duration for which a low value of Richardson number is sustained as a wave advances through the group, as explained in §2.2. As the critical latitude is approached where $\sigma = f$, or for near inertial waves, the wave-breaking vector tends towards the direction $\boldsymbol{c}$, which becomes nearly vertical.

In general, the position at which breaking occurs will move through the fluid with vector $\boldsymbol{c}_\mathrm{b}$ and, in general, this implies a movement in space. In the case of internal gravity waves, however, $\boldsymbol{c}_\mathrm{b}$ is horizontal (equation (5)) and the breaking region will be stationary in space if the fluid is advected by a current equal to, and opposed to, the horizontal component of the phase speed, as in stationary mountain lee waves. Two-dimensional planetary waves (§2.4) may also have stationary breaking regions if opposed by an east-going current. Except in the exceptional case when $N = f$, planetary waves will not have breaking regions that are stationary. If, for example, planetary waves in a stratified atmosphere are propagating in a vertical plane to the west, so that $l = 0$ and $k = fm/N$, then from (20) $\boldsymbol{c}_g$ is vertical and, from (21), the horizontal component of $\boldsymbol{c}_b$ is twice that of $\boldsymbol{c}$.

We emphasize again that, although here labelled as breaking, the property carried by waves and wave groups may be ascribed to any one of the many dynamical properties associated with waves. The persistence of breaking, or the time period during which some condition of motion is exceeded in a single wave, has a fundamental importance in establishing the volume of fluid that is affected and whether it is continuous or patchy (as was shown in figure 1), but is at present largely unknown.

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