0.1 Wave-mean interaction and vorticity transport in nearshore flows.

Emanuele Terrile

0.1.1 Eulerian description of nearshore circulation

Boussinesq model

The standard Boussinesq equations for variable water depth were first derived by Peregrine (1967), who used the depth-averaged velocity as a dependent variable. Extended forms of Boussinesq equations have been derived by Madsen *et al.* (1991) and Nwogu (1993), that used depth varying velocity. Recently a complete set of fully nonlinear Boussinesq-type equations for waves and current over an impermeable bottom has been provided by Chen (2006). The fully non linear Boussinesq type equations introduced by Wei *et al.* (1995) neglect the second-order terms associated with the vertical vorticity. Those terms are small with regard to the swash zone but become significant in the case of wave-induced circulation. The different forms of corrections which we can find in literature (i.e. Chen *et al.* 2001, Gobbi *et al.* 2001, and Hsiao *et al.* 2002), as shown by Kirby (2003), are equivalents. However Chen (2006) has shown that such corrections are incomplete because provide corrected equations which are still only first-order accurate with the respect of the conservation of potential vorticity (hereinafter PV). In the same work he introduced a new set of Boussinesq-type equation which are more complete, leading at the PV conservation. Both continuity and momentum equations are written, as already did by Nwogu (1993), in terms of a reference horizontal velocity vector $\mathbf{u}_\alpha$=(u_α, v_α) at some reference elevation in the fluid layer $z = z_\alpha(x, y) = \alpha h_s(x, y)$. the parameter α is determined by fitting the linear dispersion relation and damping property of the Boussinesq models to the Stokes-type solutions.

The continuity equation is identical to that in Wei et al. (1995) and reads:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \mathbf{M} = O(\mu^4 \sqrt{g h_0}) \tag{0.1.1}$$

where

$$\begin{aligned} \mathbf{M} = \mathbf{M}_\alpha = (h_s + \eta) \Bigg[\mathbf{u}_\alpha + \mu^2 \Bigg\{ \left(z_\alpha - \frac{1}{2}(\eta - h_s) \right) \nabla(\nabla \cdot (h_s \mathbf{u}_\alpha)) \\ + \left(\frac{z_\alpha^2}{2} - \frac{1}{6}(\eta^2 - \eta h_s + h_s^2) \right) \nabla(\nabla \cdot \mathbf{u}_\alpha) \Bigg\} \Bigg], \end{aligned} \tag{0.1.2}$$

h_s is the still water depth and η is the free surface elevation. The associated momentum conservation equation given in Chen (2006) reads:

$$\frac{\partial \mathbf{u}_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + g\nabla\eta + \mu^2(\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3) = O(\mu^4 \frac{g h_0}{l_0}), \tag{0.1.3}$$

where

$$\mathbf{V}_1 = \frac{z_\alpha^2}{2}\nabla(\nabla\cdot\mathbf{u}_{\alpha,t}) + z_\alpha\nabla[\nabla\cdot(h_s\mathbf{u}_{\alpha,t})] - \nabla\left[\frac{\eta^2}{2}\nabla\cdot\mathbf{u}_{\alpha,t} + \eta\nabla\cdot(h_s\mathbf{u}_{\alpha,t})\right], \tag{0.1.4}$$

$$\mathbf{V}_2 = \nabla\left\{(z_\alpha-\eta)(\mathbf{u}_\alpha\cdot\nabla)[\nabla\cdot(h_s\mathbf{u}_\alpha)] + \frac{1}{2}(z_\alpha^2-\eta^2)(\mathbf{u}_\alpha\cdot\nabla)(\nabla\cdot\mathbf{u}_\alpha)\right\}$$
$$+\frac{1}{2}\nabla\{[\nabla\cdot(h_s\mathbf{u}_\alpha)+\eta\nabla\cdot\mathbf{u}_\alpha]^2\}, \tag{0.1.5}$$

$$\mathbf{V}_3 = (V_3^{(x)}, V_3^{(y)}), \tag{0.1.6}$$

in which

$$V_3^{(x)} = -v_\alpha\omega_1 - \omega_0\left\{\left[z_\alpha - \frac{1}{2}(\eta-h_s)\right]\frac{\partial}{\partial y}[\nabla\cdot(h_s\mathbf{u}_\alpha)] + \left[\frac{z_\alpha^2}{2} - \frac{1}{6}(\eta^2-\eta h_s+h_s^2)\right]\frac{\partial}{\partial y}(\nabla\cdot\mathbf{u}_\alpha)\right\}, \tag{0.1.7}$$

$$V_3^{(y)} = u_\alpha\omega_1 + \omega_0\left\{\left[z_\alpha - \frac{1}{2}(\eta-h_s)\right]\frac{\partial}{\partial x}[\nabla\cdot(h_s\mathbf{u}_\alpha)] + \left[\frac{z_\alpha^2}{2} - \frac{1}{6}(\eta^2-\eta h_s+h_s^2)\right]\frac{\partial}{\partial x}(\nabla\cdot\mathbf{u}_\alpha)\right\}. \tag{0.1.8}$$

Here the vertical component of the vorticity at the second order of approximation is given by

$$\omega = \omega_0 + \mu^2\omega_1 + O\left(\mu^4\frac{\sqrt{gh_0}}{l_0}\right), \tag{0.1.9}$$

with

$$\omega_0 = \frac{\partial v_\alpha}{\partial x} - \frac{\partial u_\alpha}{\partial y}; \tag{0.1.10}$$

$$\omega_1 = \frac{\partial z_\alpha}{\partial x}\left\{\frac{\partial}{\partial y}[\nabla\cdot(h_s\mathbf{u}_\alpha)] + z_\alpha\frac{\partial}{\partial y}(\nabla\cdot\mathbf{u}_\alpha)\right\} - \frac{\partial z_\alpha}{\partial y}\left\{\frac{\partial}{\partial x}[\nabla\cdot(h_s\mathbf{u}_\alpha)] + z_\alpha\frac{\partial}{\partial x}(\nabla\cdot\mathbf{u}_\alpha)\right\}. \tag{0.1.11}$$

In comparison with Wei et al. (1995), the only difference is the introduction of the second-order effects of the vertical vorticity, $\mathbf{V}_3$. In comparison with the corrections introduced by Chen et al. (2003), the difference is the extra correction associated with ω_0 in $\mathbf{V}_3$ in the present equations. The implication of the corrections are connected with the conservation property of the PV.

The vorticity equation of such Boussinesq model is obtained taking the curl of the momentum equation (0.1.3) and, after some manipulations, leads to:

$$\frac{\partial\omega}{\partial t} + \mathbf{u}_\alpha\cdot\nabla\omega = -\omega\nabla\cdot\mathbf{u}_\alpha - \mu^2(\mathbf{u}_{\alpha1}\cdot\nabla\omega + \omega\nabla\cdot\mathbf{u}_{\alpha1}) + O\left(\mu^4\frac{gh_0}{l_0^2}\right), \tag{0.1.12}$$

where

$$\mathbf{u}_{\alpha1} = \left(\frac{z_\alpha^2}{2} - \frac{1}{6}(\eta^2-\eta h_s+h_s^2)\right)\nabla(\nabla\cdot\mathbf{u}_\alpha) + \left(z_\alpha - \frac{1}{2}(\eta-h_s)\right)\nabla(\nabla\cdot(h_s\mathbf{u}_\alpha)). \tag{0.1.13}$$

It is noticed that the vertical vorticity ω depends on $\mathbf{u}_\alpha$, even if it is not its direct curl (i.e. $\omega \neq \nabla\times\mathbf{u}_\alpha$ but $\omega_0 = \nabla\times\mathbf{u}_\alpha$), and z_α but is depth uniform. Introducing the depth-averaged horizontal velocity vector, $\bar{\mathbf{u}}$, which can be expressed as

$$\hat{\mathbf{u}} = \mathbf{u}_\alpha + \mu^2\mathbf{u}_{\alpha1} + O\left(\mu^4\sqrt{gh_0}\right), \tag{0.1.14}$$

after some manipulations, inserting (0.1.14) into (0.1.12), we can obtain the following useful vorticity equation (0.1.15):

$$\frac{\partial\omega}{\partial t} + \hat{\mathbf{u}}\cdot\nabla\omega = -\omega\nabla\cdot\hat{\mathbf{u}} + O\left(\mu^4\frac{gh_0}{l_0^2}\right). \tag{0.1.15}$$

Such equation, combined with the continuity equation (0.1.1) and with (0.1.14), leads us to the conservation of the potential vorticity in terms of the transport velocity field $\hat{\mathbf{u}}$ (corresponding to the depth averaged velocity):

$$\frac{D_\wedge}{Dt}\left(\frac{\omega}{h}\right) = O\left(\mu^4 \frac{gh_0}{l_0^3}\right), \tag{0.1.16}$$

$$\text{in which} \qquad \frac{D_\wedge}{Dt} = \frac{\partial}{\partial t} + \hat{\mathbf{u}} \cdot \nabla \qquad \text{and} \qquad h = h_s + \eta, \tag{0.1.17}$$

where h represents the total water depth. The conservation of the PV, obtained in (0.1.16), is consistent with the level of approximation in the Boussinesq model for the pure wave motion, having accuracy up to order $O(\mu^2)$.

To get a better understanding on the physical phenomena, described by such set of equation (0.1.1)-(0.1.3), and to obtain a clearer set of Boussinesq-type equations, we introduce a new definition of the velocity vector, $\mathbf{u}_\omega$, which takes into account the second-order effects of the vertical vorticity. This leads us to achieve a velocity whose curl is exactly the complete vertical vorticity at the second-order, ω, as given in (0.1.12).

The horizontal vector velocity $\mathbf{u}_\omega$ is therefore defined as:

$$\boxed{\mathbf{u}_\omega = \mathbf{u}_\alpha + \mu^2 \mathbf{u}_{12} + O(\mu^4 \sqrt{gh_0})} \tag{0.1.18}$$

$$\text{with} \qquad \mathbf{u}_{12} = z_\alpha \nabla(\nabla \cdot (h_s \mathbf{u}_\alpha)) + \frac{z_\alpha^2}{2} \nabla(\nabla \cdot \mathbf{u}_\alpha) + O(\mu^2 \sqrt{gh_0}), \tag{0.1.19}$$

or, in a better and reasonable way:

$$\boxed{\mathbf{u}_\omega = \hat{\mathbf{u}} + \mu^2 \mathbf{u}_2 + O(\mu^4 \sqrt{gh_0})} \tag{0.1.20}$$

$$\text{with} \qquad \mathbf{u}_2 = \frac{1}{2}(\eta - h_s)\nabla(\nabla \cdot (h_s \mathbf{u}_\alpha)) + \frac{1}{6}(\eta^2 - \eta h_s + h_s^2)\nabla(\nabla \cdot \mathbf{u}_\alpha) + O(\mu^2 \sqrt{gh_0}), \tag{0.1.21}$$

where $\mathbf{u}_2$ represents the difference, at the second-order, of $\mathbf{u}_\omega$ from the depth averaged horizontal velocity vector $\hat{\mathbf{u}}$.

From this definition it follows immediately that $\omega = \nabla \times \mathbf{u}_\omega + O\left(\mu^4 \frac{\sqrt{gh_0}}{l_0}\right)$. Furthermore it should be noted that the dispersion properties of the fully nonlinear Boussinesq model do not change because, however, the velocity $\mathbf{u}_\omega$ dependents always on the reference elevation $z_\alpha(x, y)$. Basing on the velocity $\mathbf{u}_\omega$ we re-write the Boussinesq model previously presented as shown in the following.

The continuity equations (0.1.1) remains formally the same but with:

$$\mathbf{M} = \mathbf{M}^\omega = (h_s + \eta)\left[\mathbf{u}_\omega - \mu^2 \left\{\underbrace{\frac{1}{2}(\eta - h_s)\nabla(\nabla \cdot (h_s \mathbf{u}_\omega)) + \frac{1}{6}(\eta^2 - \eta h_s + h_s^2)\nabla(\nabla \cdot \mathbf{u}_\omega)}_{\mathbf{u}_2}\right\}\right] + O(\mu^4 h_0 \sqrt{gh_0}), \tag{0.1.22}$$

while major changes appears in the associated momentum equations given in the following, with particular regards to the dispersive terms:

$$\boxed{\frac{\partial \mathbf{u}_\omega}{\partial t} + (\mathbf{u}_\omega \cdot \nabla)\mathbf{u}_\omega + g\nabla\eta + \mu^2(\mathbf{V}_{12}^\omega + \mathbf{V}_3^\omega) = O(\mu^4 \frac{gh_0}{l_0})} \tag{0.1.23}$$

where:

$$\mathbf{V}_{12} = -\nabla\left[\frac{\eta^2}{2}\nabla\cdot\mathbf{u}_{\omega,t} + \eta\nabla\cdot(h_s\mathbf{u}_{\omega,t})\right] + \frac{1}{2}\nabla\{[\nabla\cdot(h_s\mathbf{u}_\omega) + \eta\nabla\cdot\mathbf{u}_\omega]^2\}$$
$$+\nabla\left\{(z_\alpha - \eta)(\mathbf{u}_\omega\cdot\nabla)[\nabla\cdot(h_s\mathbf{u}_\omega)] + \frac{1}{2}(z_\alpha^2 - \eta^2)(\mathbf{u}_\omega\cdot\nabla)(\nabla\cdot\mathbf{u}_\omega)\right\}, \tag{0.1.24}$$

$$\mathbf{V}_3 = \left\{\frac{1}{6}(\eta^2 - \eta h_s + h_s^2)\nabla(\nabla\cdot\mathbf{u}_\omega) + \frac{1}{2}(\eta - h_s)\nabla(\nabla\cdot(h_s\mathbf{u}_\omega))\right\}\times(\nabla\times\mathbf{u}_\omega) = \mathbf{u}_2\times(\nabla\times\mathbf{u}_\omega). \tag{0.1.25}$$

Now, the momentum equation (0.1.23) results more clear and, as shown in the following, its curl gives immediately the vorticity equation written on ω. Furthermore the simply curl of the momentum equation (0.1.23), because of $\mathbf{V}_{12}$ give an irrotational dispersive contribute, leads to the following vorticity equation:

$$\boxed{\frac{\partial\omega}{\partial t} + \mathbf{u}_\omega\cdot\nabla\omega = -\omega\nabla\cdot\mathbf{u}_\omega - \mu^2\underbrace{\nabla\times\{\mathbf{u}_2\times(\nabla\times\mathbf{u}_\omega)\}}_{VORTEXFORCE}} + O\left(\mu^4\frac{gh_0}{l_0^2}\right), \tag{0.1.26}$$

where a *VORTEX FORCE* term, coming from the dispersive contribution of $\mathbf{V}_3$, appears. Note that $\mathbf{u}_2$ represents the second-order difference between the velocity vector $\mathbf{u}_\omega$ and the depth-averaged horizontal velocity $\hat{\mathbf{u}}$. Hence $\mathbf{V}_3$ represents a forcing on the vorticity ω due to dispersive effects.

Terms $\mathbf{V}_{12}$ can be re-written in a more physically clear way through the use of (0.1.18), (0.1.22) in (0.1.24):

$$\mathbf{V}_{12} = \nabla\left\{\mathbf{u}_\omega\cdot\mathbf{u}_{12} - \frac{1}{2}(\mathsf{w})^2 + \underbrace{\frac{\partial(\eta\mathsf{w})}{\partial t} + \nabla\cdot(\eta\mathsf{w}\mathbf{u}_\omega)}_{T_p} + \frac{\eta^2}{2}\underbrace{\left[\frac{\partial(\nabla\cdot\mathbf{u}_\omega)}{\partial t} + \mathbf{u}_\omega\cdot\nabla(\nabla\cdot\mathbf{u}_\omega)\right]}_{T_d}\right\}, \tag{0.1.27}$$

where:

$$\mathsf{w} = \frac{\partial\eta}{\partial t} + \mathbf{u}_\omega\cdot\nabla\eta. \tag{0.1.28}$$

The first term on the r.h.s. of equation (0.1.27) represents the dispersive effects due to the interaction between the velocity $\mathbf{u}_\omega$ and the deviation respect $\mathbf{u}_\alpha$ and indirectly $\hat{\mathbf{u}}$. It should be pointed out that in the shallow water framework, NSWE, because of it is usually assumed that the horizontal velocity is depth-uniform, $\mathbf{u}_\omega \equiv \mathbf{u}_\alpha \equiv \hat{\mathbf{u}}$ and both $\mathbf{u}_2$ and $\mathbf{u}_{12}$ tends to zero, therefore this term disappear as well as $\mathbf{V}_3$. However, while assuming $\mathbf{u}_\omega \equiv \mathbf{u}_\alpha \equiv \hat{\mathbf{u}}$ equation (0.1.26) falls into the usual non-dissipative vorticity shallow water equation:

$$\frac{\partial\omega}{\partial t} + \hat{\mathbf{u}}\cdot\nabla\omega = -\omega\nabla\cdot\hat{\mathbf{u}} + O\left(\mu^2\frac{gh_0}{l_0^2}\right), \tag{0.1.29}$$

the same does not happen for the momentum equations (0.1.23) because $\mathbf{V}_{12}$ does not disappear. This means that some dispersive terms, straightly dependent on η, remain (i.e. $2nd$, $3rd$ and $4th$ terms on the r.h.s of (0.1.27)).

The second term of (0.1.27) represents the "wave energy" associated with the velocity w.

T_p and T_d are two total derivative respectively for term $(\eta\mathsf{w})$, which is a sort of *pseudomomentum* associated with the wave oscillation, and $(\nabla\cdot\mathbf{u}_\omega)$.

Finally it can be derived that both T_p and T_d fall to zero at the shoreline where $h = h_s + \eta \to 0$ (i.e. $h_s \to -\eta$ and $z_\alpha \to \eta$).

A further analyses has been made looking at the velocity $\mathbf{u}_{12}$ and $\mathbf{u}_2$, and trying to get a better understanding onto their physical meaning. In particular it has been found:

$$\mathbf{u}_{12} = \nabla \left[z_\alpha \left(E + \frac{z_\alpha}{2} G \right) \right] - (E + z_\alpha G) \nabla z_\alpha, \qquad (0.1.30)$$

with:

$$E = \nabla \cdot (h_s \mathbf{u}_\omega) \qquad \text{and} \qquad G = \nabla \cdot \mathbf{u}_\omega, \qquad (0.1.31)$$

The formal structure is similar to the previous one (here not shown) found with η instead of z_α and leading to the velocity w and the quantity $\eta\mathsf{w}$, therefore a qualitative draft description of $\mathbf{u}_{12}$ let us guess that correctly it represents a perturbation velocity of $\mathbf{u}_\alpha$ with respect to $\mathbf{u}_\omega$, see (0.1.18).

More difficult is looking inside the physics described by $\mathbf{u}_2$. After some manipulation and by using the continuity equation, it turns out that:

$$\mathbf{u}_2 = \frac{1}{2} \left\{ \underbrace{(\eta - h_s)\nabla \mathsf{w}}_{S.I} - \underbrace{(\eta - h_s)\nabla \cdot \mathbf{u}_\omega \nabla \eta}_{S.II} + \underbrace{\frac{1}{3}\left(h_s^2 - 2\eta^2 + 2\eta h_s\right) \nabla(\nabla \cdot \mathbf{u}_\omega)}_{S.III} \right\} \qquad (0.1.32)$$

Here $S.I$ is the effect due to the gradient of the velocity w while $S.II$ is mainly due to the gradient of the wave η. (Both are multiplied by $(\eta - h_s)$ which seems taking into account the dynamical aspect). $S.III$ is more complicated to analyse but it can be shown keeping inside contribution relating to the vorticity ω with both the topography gradient ∇h_s and $\nabla \eta$.

In the following we study the wave-mean interaction in the the nearshore zone making use of the fully nonlinear Boussinesq model, previously presented. In the first part of this study we performed an asymptotic expansion in small wave amplitude. Even if this assumption is quite restrictive, it give us an idea on the physical processing involved and on the influence of the dispersion terms on the nearshore hydrodynamics. The analysis is performed for idealized non-dissipative waves. In particular we are interested on looking at the forcing of the potential vorticity, due to the dispersion effects. A more complete analyses on their effects is afterwards performed through the use of the Generalized Lagrangian Mean theory (i.e. Andrews and McIntyre, 1978*a*), which gives results no more dependent on the small wave assumption and valid for finite amplitude waves.

The small wave amplitude approximation looks at the mean-field response to slowly varying small-amplitude gravity waves. Therefore, considering a background state of rest at $O(1)$, we assume that the flow fields can be uniquely decomposed into their mean part $\bar{\phi}$ (Eulerian mean is denoted with the overbar) and a disturbance part ϕ' at the relevant orders in small wave amplitude a, such that $\phi = \phi' + \bar{\phi}$ and $\bar{\phi}' = 0$. Making these assumptions we set the generic velocity field $\mathbf{u}$ and the free stream surface η such that:

$$\mathbf{u} = \mathbf{u}' + \bar{\mathbf{u}} + O(a^3) \qquad \text{and} \qquad \eta = \eta' + \overline{\Delta h} + O(a^3), \qquad (0.1.33)$$

where the disturbance quantities $\mathbf{u}'$ and η' are $O(a)$ while the mean-flow response quantities $\bar{\mathbf{u}}$ and $\overline{\Delta h}$, representing the depth set-up, are $O(a^2)$. Note that the still water depth h_s represents a background field and, therefore, is $O(1)$.

Gravity waves

Substituting equations (0.1.33) in the continuity equation (0.1.1)-(0.1.22) and momentum equation (0.1.23) we achieve the following continuity and momentum equations at the first order $O(a)$

valid for gravity waves:

$$\frac{\partial \eta'}{\partial t} + \nabla \cdot \mathbf{M}' = O(\mu^4 \sqrt{g h_0}) \tag{0.1.34}$$

$$\frac{\partial \mathbf{u}'_\omega}{\partial t} + g \nabla \eta' = O(\mu^4 \frac{g h_0}{l_0}) \tag{0.1.35}$$

$$\text{with} \qquad \mathbf{M}' = h_s \left[\mathbf{u}'_\omega - \mu^2 \left\{ \underbrace{-\frac{h_s}{2} \nabla(\nabla \cdot (h_s \mathbf{u}'_\omega)) + \frac{h_s^2}{6} \nabla(\nabla \cdot \mathbf{u}'_\omega)}_{\mathbf{u}'_2} \right\} \right] + O(\mu^4 h_0 \sqrt{g h_0}). \tag{0.1.36}$$

Following Bühler and Jacobson (2001), it becomes useful now introduce a linear particle displacement $\boldsymbol{\xi}' = (\xi'_a, \xi'_b)$ such that:

$$\frac{\partial \boldsymbol{\xi}'}{\partial t} = \hat{\mathbf{u}}' \qquad \text{hence} \qquad \eta' + \nabla \cdot (h_s \boldsymbol{\xi}') = 0, \tag{0.1.37a, b}$$

This set of equations, (0.1.34)-(0.1.35), is useful for determining the wave properties at $O(a^2)$ which will be important studying the equation for the mean-flow response. Those wave properties depend only on the previous linearized equations being averaged squares of the $O(a)$ solutions. Of particular interest is the wave energy per unit, defined as:

$$E = \frac{1}{2} \left(\overline{u'^2_w} + \overline{v'^2_w} + g \frac{\overline{\eta'^2}}{h_s} \right) h_s \tag{0.1.38}$$

Now performing the scalar product between multiplying (0.1.35) and $(h_s \mathbf{u}'_\omega)$, averaging over the wave period and invoking (0.1.34) we can obtain the following transport equation for the energy E:

$$\frac{\partial E}{\partial t} + \nabla \cdot (g h_s \overline{\eta' \mathbf{u}'_\omega}) - \mu^2 \, g \, \overline{\eta' \nabla \cdot (h_s \mathbf{u}'_2)} = 0 \tag{0.1.39}$$

In (0.1.39) the energy E is affected by the dispersion terms which are represented by the last term on the l.h.s.

From (0.1.20) we can see that inside $\mathbf{u}_\omega$ there are some $O(\mu^2)$ contributions taking into account dispersion effects. Hence we split the wave energy E into part, such that:

$$E = \hat{E} + 2\mu^2 \hat{E}_2 - 2\mu^2 \, g \overline{\eta'^2} \tag{0.1.40}$$

$$\text{with} \qquad \hat{E} = \frac{1}{2} \left(\overline{\hat{u}'^2} + \overline{\hat{v}'^2} + g \frac{\overline{\eta'^2}}{h_s} \right) h_s; \qquad \text{and} \qquad \hat{E}_2 = \frac{1}{2} \left(\overline{\hat{u}' u'_2} + \overline{\hat{v}' v'_2} + g \frac{\overline{\eta'^2}}{h_s} \right) h_s. \tag{0.1.41}$$

Now substituting equations (0.1.40) and (0.1.20) in 0.1.38, we get the following transport equation, valid for the wave energy $\hat{E}$ at order $O(\mu^2)$:

$$\frac{\partial \hat{E}}{\partial t} + \nabla \cdot (g h_s \overline{\eta' \hat{\mathbf{u}}'}) + 2 \, \mu^2 \left\{ \frac{\partial \hat{E}_2}{\partial t} - g \, \frac{\partial \overline{\eta'^2}}{\partial t} \right\} = 0, \tag{0.1.42}$$

where the last term is $O(\mu^2)$ and is completely due to the dispersion effects. $\hat{E}_2$ represents the wave energy per unit of area associated to the interaction between the depth-averaged part of the wave velocity and the corresponding depth-varying part. If this interaction is weak as expected in shallow water, equation (0.1.42) conserve the area integral of $\hat{E}$ if non-dissipative waves on a background state of rest are considered.

Another useful wave property we can get at this step of our analysis, using the velocity field $\mathbf{u}'_\omega$ and η', is the Stokes drift $\bar{\mathbf{u}}^S_\omega$. This wave property is usually defined as the difference between the Lagrangian mean velocity $\bar{\mathbf{u}}^L_\omega$ and the Eulerian mean velocity $\bar{\mathbf{u}}_\omega$, note that the overbar denote averaging over the fast time scales. Although it is general formulation is quite complex, at $O(a^2)$, following Andrews & McIntyre (1978) and Bühler & Jacobson (2001) , it reduces to:

$$\bar{\mathbf{u}}^S_\omega = \overline{(\boldsymbol{\xi}' \cdot \nabla)\mathbf{u}'_\omega} = \frac{1}{h_s}\overline{(h_s\boldsymbol{\xi}' \cdot \nabla)\mathbf{u}'_\omega} = -\frac{1}{h_s}\overline{\nabla \cdot (h_s\boldsymbol{\xi}')\mathbf{u}'_\omega} + O(\varepsilon\ a^2) \approx \frac{1}{h_s}\ \overline{\eta'\mathbf{u}'_\omega}. \qquad (0.1.43)$$

Equation (0.1.43) has been obtained using (0.1.37a, b) and neglecting terms of order $O(\varepsilon a^2)$, where it has been assumed that if the gradients of a disturbance field are $O(1)$ the mean-flow response has gradients of $O(\varepsilon)$, with $\varepsilon \ll 1$ representing a small suitable parameter used to describe the scale separation. Expressions similar to (0.1.43) valid for Stokes correction, i.e. $\bar{\phi}^S = \bar{\phi}^L - \bar{\phi}$, can be found.

Mean-flow response

Analyzing now, the continuity and momentum equations at order $O(a^2)$, we get a description of the mean-flow response to the gravity waves. This set of equations is obtained time-averaging over the fast time scale Boussinesq model equations (0.1.1)-(0.1.23) and retaining all terms up to order $O(a^2)$. Using also (0.1.33) we get:

$$\frac{\partial\overline{\Delta h}}{\partial t} + \nabla \cdot \overline{\mathbf{M}} = O(\mu^4\sqrt{gh_0}) \qquad (0.1.44)$$

$$\frac{\partial\bar{\mathbf{u}}_\omega}{\partial t} + \overline{(\mathbf{u}'_\omega \cdot \nabla)\mathbf{u}'_\omega} + g\nabla\overline{\Delta h} + \mu^2\overline{\mathbf{V}_{12}} + \mu^2\overline{\mathbf{u}'_2 \times (\nabla \times \mathbf{u}'_\omega)} = O(\mu^4\frac{gh_0}{l_0}) \qquad (0.1.45)$$

$$\text{with} \quad \overline{\mathbf{M}} = h_s\left(\bar{\mathbf{u}}^L_\omega - \mu^2\ \bar{\mathbf{u}}^L_2\right) + O(\mu^4 h_0\sqrt{gh_0}). \qquad (0.1.46)$$

We note that $\mu^2\ \overline{\mathbf{V}_{12}}$ in (0.1.45), invoking (0.1.24), is a gradients of mean-flow quantities, therefore it is at order $O(\varepsilon\ a^2)$ and can be neglected. Furthermore this term is irrotational and does not influence the vorticity dynamics.

The advective term is handle in the same way as did by Bühler & Jacobson (2001) and it holds to:

$$\overline{(\mathbf{u}'_\omega \cdot \nabla)\mathbf{u}'_\omega} = \frac{1}{h_s}\nabla \cdot \left(h_s\overline{\mathbf{u}'_\omega\mathbf{u}'_\omega} + \delta\frac{g}{2}\ \overline{\eta'^2}\right) + \frac{\partial\bar{\mathbf{u}}^S_\omega}{\partial t} - \mu^2\frac{1}{h_s}\overline{\nabla \cdot (h_s\mathbf{u}'_2)\mathbf{u}'_\omega}, \qquad (0.1.47)$$

where both continuity (0.1.34) and momentum (0.1.35) equations have been used and δ is the unit tensor associated to the Kronecker's delta. Introducing now the radiation-stress tensor S as described by Longuet-Higgins (1970), the advective term becomes:

$$\overline{(\mathbf{u}'_\omega \cdot \nabla)\mathbf{u}'_\omega} = \frac{1}{h_s}\nabla \cdot \mathsf{S}\ + \frac{\partial\bar{\mathbf{u}}^S_\omega}{\partial t} - \mu^2\frac{1}{h_s}\overline{\nabla \cdot (h_s\mathbf{u}'_2)\mathbf{u}'_\omega}. \qquad (0.1.48)$$

Looking now at the last term in the momentum equation (0.1.45) it turns out that is equal zero. In fact considering the vorticity equation for the gravity waves, which can be easily derived taking the curl of the momentum equation (0.1.35) it comes out that:

$$\frac{\partial}{\partial t}\left(\nabla \times \mathbf{u}'_\omega\right) = 0. \qquad (0.1.49)$$

By integrating this vorticity equation in time and assuming that there was no disturbance at the initial time, it results that $\nabla \times \mathbf{u}'_\omega = 0$. Hence, even the dispersion term $\overline{\mathbf{u}'_2 \times (\nabla \times \mathbf{u}'_\omega)}$ does not give any contributions.

The momentum equation, we get after these considerations and using (0.1.48), reads:

$$\frac{\partial \bar{\mathbf{u}}^L_\omega}{\partial t} + g\nabla\overline{\Delta h} = -\frac{1}{h_s}\nabla \cdot \mathbf{S} + \mu^2 \frac{1}{h_s}\overline{\nabla \cdot (h_s \mathbf{u}'_2)\mathbf{u}'_\omega} + O(\mu^4 \frac{g h_0}{l_0}) \quad . \tag{0.1.50}$$

This equation together with (0.1.44) completely describes the mean-flow response to the gravity waves because depends only on waves quantities, which are derived directly from the linear solutions.

We note that in (0.1.50) the radiation-stress tensor $\mathbf{S}$ is not the only forcing of the mean flow, as it happens in shallow water framework (i.e. Bühler & Jacobson, 2001), but there are also dispersion contributions represented by the last term on the r.h.s.. The mean-flow is expected to respond to a different number of physical effect such as wave dissipation, transience and mean pressure changes due to waves and radiation-stress describes only part of these effects.

Another important wave property is the so-called pseudomomentum per unit of mass $\mathbf{p}$, associated at the velocity field $\mathbf{u}_\omega$, which at order $O(a^2)$ and considering equation (0.1.49) is:

$$\mathsf{p}_i = -\overline{\xi'_{j,i} u'_{wj}} \approx \overline{\xi'_j u'_{wi,j}} = \bar{u}^S_i, \tag{0.1.51}$$

whose evolution equation can be easily derived multiplying equation (0.1.45) with η'/h_s, averaging and using the corresponding continuity equation (0.1.44). After some manipulation, invoking (0.1.51) and following Bühler & Jacobson (2001) , we get:

$$\frac{\partial\, \mathbf{p}}{\partial t} + \frac{1}{h_s}\nabla \cdot \mathbf{S} - \frac{1}{2}\nabla\overline{|\mathbf{u}'_\omega|^2} = \mu^2 \frac{1}{h_s}\overline{\nabla \cdot (h_s \mathbf{u}'_2)\mathbf{u}'_\omega}, \tag{0.1.52}$$

where the dispersive term on the r.h.s., although in absence of dissipative force, captures the decay of $\mathbf{p}$. Furthermore substituting (0.1.53) back in (0.1.50) and taking the curl results in:

$$\frac{\partial}{\partial t}\nabla \times (\bar{\mathbf{u}}^L_\omega - \mathbf{p}\ ) = 0, \tag{0.1.53}$$

being the gradient term irrotational. This equation is exactly the same found by Bühler & Jacobson (2001) in a shallow water framework making a similar small wave approximation and make clear the central role of the pseudomomentum on the mean-flow response. In particular it shows that the pseudomomentum take into account dispersion terms which are not kept by the radiation-stress tensor $\mathbf{S}$.

0.1.2 Generalize Lagrangian-Mean theory

One of the difficulties in the Eulerian description is to identify the mean motion in an otherwise oscillating field (in the splitting between currents and waves.). One way to obtain this splitting is used in the time-averaged, depth-integrated models, in which essentially the mean motion is depth-uniform. When depth-varying currents are considered (the mean motion is still a function of the depth) one way to split up mean and oscillating motion is through the Generalized Lagrangian Mean theory (GLM). The GLM gives the advantage that it is formed over the displaced locations, while the Eulerian mean is formed (as normally done) at the undisplaced locations. This theory was developed by Andrews and McIntyre to separate in a more rational way waves from mean flow and to describe wave-mean interaction.

Andrews and McIntyre's Generalized Lagrangian-Mean is an exact and very general Lagrangian-mean description of the effect of oscillatory disturbances upon the mean flow. GLM theory is based upon an exact Lagrangian-mean operator $\overline{(\quad)}^L$, corresponding to any given Eulerian-mean operator $(\quad)$ through an exact disturbance-associated particle displacement field and provided the mapping $\mathbf{x} \Rightarrow \mathbf{x} + \xi$ is invertible. GLM theory derives the general equations for its evolution from the equation of motion (continuity and momentum equations). Two important vector fields ξ and $\mathbf{p}$ are used in GLM. The first generalizes the disturbance particle displacement field. The second is a wave property, and it is called *pseudomomentum*, per unit of mass $\mathbf{p}$ can be considered as the nonlinear forcing of the mean flow by the waves. In the GLM description the conservation of pseudomomentum is straightforward connected with the invariance to a translation of the disturbance pattern while *mean* particle positions are kept fixed.

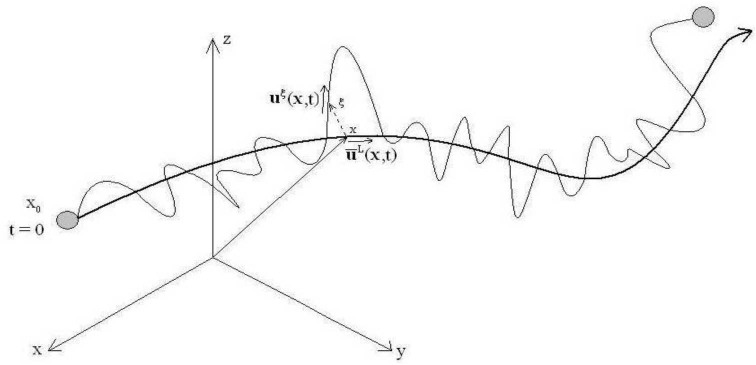

Figure 1: GLM theory scheme: the bold line represents the *mean* particle trajectory, while the thin line represents the *actual* particle trajectory. Both trajectories are supposed to start at the same point x_0 at time $t = 0$ and the position $\mathbf{\Xi} = \mathbf{x} + \xi(\mathbf{x}, t)$ is the actual position of the particle whose mean position is $\mathbf{x}$.

Now if we consider a material particle trajectory, solution of $\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t)$, from figure 1 we can see that its motion can be decomposed into a slow and a fast part by averaging over the fast timescale. Through this averaging process we associate two different trajectory with each particle: the thin line, representing the *actual*, rapidly varying trajectory, and the thick line, the mean, slowly varying trajectory. The GLM theory provides the tools to link these two different trajectories, introducing an unique disturbance particle displacement field $\xi(\mathbf{x}, t)$. Assuming mean particle trajectories must not cross each other, and therefore that each point $\mathbf{x}$ is touched only by a trajectory at a given time t (otherwise the field $\xi(\mathbf{x}, t)$ is not uniquely defined), the *actual* particle position can be defined such that:

$$\mathbf{\Xi}(\mathbf{x}, t) = \mathbf{x} + \xi(\mathbf{x}, t) \tag{0.1.54}$$

while $\mathbf{x}$ is the *mean* position, being the time average over the fast timescale of $\xi(\mathbf{x}, t)$ defined as follows:

$$\overline{\xi(\mathbf{x}, t)} = 0. \tag{0.1.55}$$

We can observe that the particle whose *mean* position is $\mathbf{x}$ at time t is not necessary the same particle that it is *at* $\mathbf{x}$ at time t. Now it can be defined an exact Lagrangian-mean velocity as

$$\overline{\mathbf{u}}^L(\mathbf{x}, t) = \overline{\mathbf{u}(\mathbf{x} + \xi(\mathbf{x}, t), t)} \tag{0.1.56}$$

$\overline{\mathbf{u}}^L$ being the mean velocity of the particle whose mean position is $\mathbf{x}$ at time t. It should be noted that the Lagrangian-mean operator is applied on the *actual* particle positions, as it can be seen in figure 1. The Lagrangian-mean operator applied to a generic function φ can be defined in analogy with (0.1.56). Now it useful to introduce the following notation

$$\varphi^{\xi}(\mathbf{x},t) = \varphi\{\mathbf{x} + \xi(\mathbf{x},t), t\}, \tag{0.1.57}$$

in which the upper symbol $^{\xi}$ means the generic quantity φ is evaluated at time t in $\mathbf{x} + \xi(\mathbf{x},t)$, to pass from mean to actual particle positions, and viceversa. For this intention we have two important properties of φ^{ξ} arising from the chain rule of differentiation:

$$(\varphi^{\xi})_{,t} = (\varphi_{,t})^{\xi} + (\varphi_{,j})^{\xi}\Xi_{j,t} \quad \text{and} \quad (\varphi^{\xi})_{,i} = (\varphi_{,j})^{\xi}\Xi_{j,i} \tag{0.1.58a,b}$$

where $\;_{,j}$ denotes the covariant differentiation with respect to x_j, the time-differentiation is denoted with $\;_{,t}$ and Ξ_i are the component of the actual position vector $\mathbf{\Xi}$. Repeated indices are summed. The relation between the Lagrangian-mean and the usual Eulerian-mean is represented by the 'Stokes correction' for every mean field and it is written as

$$\overline{\varphi}^{S} \equiv \overline{\varphi}^{L} + \overline{\varphi}. \tag{0.1.59}$$

In the special case of the velocity the Stokes correction is defined as the Stokes drift $\overline{\mathbf{u}}^{S}$.
Lagrangian-mean trajectories can be defined as integral curves of the lagrangian-mean velocity $\overline{\mathbf{u}}^{L}$ (i.e. solution of $\frac{d\mathbf{x}}{dt} = \overline{\mathbf{u}}^{L}(\mathbf{x},t)$) and this leads at the following definition of the Lagrangian-mean time derivative

$$\overline{D}^{L} = \frac{\partial}{\partial t} + \overline{\mathbf{u}}^{L} \cdot \nabla, \tag{0.1.60}$$

The Lagrangian-mean material derivative $\overline{D}^{L}$ of the actual particle position (0.1.54) along a mean trajectory is equal to the actual fluid velocity $\mathbf{u}^{\xi}$:

$$\overline{D}^{L}\Xi = \left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}^{L} \cdot \nabla\right)\Xi = \mathbf{u}^{\xi}. \tag{0.1.61}$$

This equation has as consequence an important rule connecting the material derivative along mean and actual trajectories, valid for any field φ:

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}^{L} \cdot \nabla\right)(\varphi^{\xi}) = \left(\frac{D\varphi}{Dt}\right)^{\xi}. \tag{0.1.62}$$

By simply averaging (0.1.62) we obtain the following useful relation:

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}^{L} \cdot \nabla\right)\overline{(\varphi)}^{L} = \overline{\left(\frac{D\varphi}{Dt}\right)}^{L} \tag{0.1.63}$$

We note that the previous relation (0.1.61) may be written as

$$\overline{D}^{L}\xi = \mathbf{u}^{\ell} \tag{0.1.64}$$

where the quantity

$$\mathbf{u}^{\ell} = \mathbf{u}^{\xi} - \overline{\mathbf{u}}^{L} \tag{0.1.65}$$

is the mean material rate of change of ξ and it is called Lagrangian disturbance velocity, obviously from (0.1.65) it results

$$\overline{\mathbf{u}^{\ell}} = 0. \tag{0.1.66}$$

Postulate (**viii**) in Andrews and McIntyre states that each Lagrangian-mean trajectory passes through at least one point $(\mathbf{x}_0, t)$ in the neighbourhood of which there is no disturbance. A consequence of this postulate, given $\mathbf{u}(\mathbf{x}, t)$ and assuming that the initial flow has no disturbances (i.e. $\xi(\mathbf{x}, 0) = 0$) from equations (0.1.56), (0.1.60) and (0.1.61) it is possible to determine ξ and, therefore, $\overline{\mathbf{u}}^L$ at all later times.

Another important characteristic, which distinguishes the GLM theory from the standard Eulerian-mean theories, is the so called 'divergence effect', or rather the fact that in the GLM theory $\nabla \cdot \mathbf{u} = 0$ does not imply that $\nabla \cdot \overline{\mathbf{u}}^L = 0$. This divergence effect was discussed by Andrews and McIntyre, who emphasized that a mean density field $\tilde{\rho}$ satisfying the continuity equation is not in general equal to $\overline{\rho}^L$. They gave a definition of this 'new' mean density field

$$\tilde{\rho} = \rho^\xi J \tag{0.1.67}$$

where J is the Jacobian of the jacobian matrix $\Xi_{i,j}$ mapping $\mathbf{x} \Rightarrow \mathbf{x} + \xi$:

$$J = det\{\Xi_{i,j}\} = det\{\delta_{ij} + \xi_{i,j}\}. \tag{0.1.68}$$

For later manipulations it also is useful to introduce the cofactors K_{ij} of J, as Andrew and McIntyre (1978) did, which satisfy

$$\Xi_{i,k} K_{ij} = J\delta_{kj} \tag{0.1.69a}$$

$$\Xi_{k,i} K_{ji} = J\delta_{kj} \tag{0.1.69b}$$

from this relation, since $\Xi_{i,k} = \delta_{ik} + \xi_{i,k}$, the cofactor K_{ij} can be written as

$$K_{kj} = J\delta_{ij} - \xi_{i,j} K_{ij} \tag{0.1.70}$$

or in the more useful form

$$K_{ij} = \partial J / \partial \Xi_{i,j} = \frac{1}{2} \epsilon_{ilm} \epsilon_{jpq} \Xi_{l,p} \Xi_{m,q} \tag{0.1.71}$$

which leads to

$$K_{ij,j} = 0, \qquad \text{and} \qquad J_{,\mu} = K_{ij} (\Xi_{i,j})_{,\mu}. \tag{0.1.72a,b}$$

Furthermore, now we can write the inverse of (0.1.58b) by using (0.1.69b), as:

$$(\varphi_{,j})^\xi = (\varphi^\xi)_{,i} \frac{K_{ji}}{J}; \tag{0.1.73}$$

The continuity equation written in terms of the density ρ is

$$D\rho/Dt + \rho \nabla \cdot \mathbf{u} = 0, \tag{0.1.74}$$

and in the GLM theory, using the 'new' mean density field $\tilde{\rho}$, can be re-written in same way of the previous (0.1.74) as follows

$$D\tilde{\rho}/Dt + \tilde{\rho} \nabla \cdot \mathbf{u} = 0. \tag{0.1.75}$$

The physical meaning of $\tilde{\rho}$ is the same as the ordinary density, but while ρ measures the dilatation or contraction of the *actual* material volumes, $\tilde{\rho}$ measures the dilatation or contraction of the *mean* material volumes as it is straightforward shown by equation (0.1.67) as defined in Andrews and McIntyre. Note that $\tilde{\rho}$ is a mean quantity, as can be shown by averaging (0.1.75) and therefore $\overline{\tilde{\rho}} = \tilde{\rho}$.

Before we apply the GLM theory to the two-dimensional Boussinesq equation it is useful to introduce a new vector field as Andrews & McIntyre (1978) did: the *pseudomomentum* or 'wave-momentum' $\mathbf{p}$ per unit of mass. It is a wave property, therefore may be evaluated to a consistent first approximation from linearized theory and depends only on the way in which 'disturbance' and 'mean flow' are defined.

In a more generic form, Andrews & McIntyre (1978) first introduced a fundamental measure of the 'wave activity' for finite-amplitude disturbances on arbitrary mean flows, the 'wave-action' A:

$$A \equiv \overline{\xi_{,\alpha} \cdot (\mathbf{u}^{\ell} + \mathbf{\Omega} \times \xi)} \tag{0.1.76}$$

where $(\ )_{,\alpha}$ stands for $\partial/\partial\alpha$ and $\mathbf{\Omega}$ is the angular velocity of a rotating frame of reference. They derived from the momentum equation, after some manipulation, an appropriate wave-action conservation relation:

$$\overline{D}^{L}\mathbf{A} + \tilde{\rho}^{-1}\nabla \cdot \mathbf{B} = \mathscr{F} \tag{0.1.77}$$

where $\mathbf{B}$ is the non-advective flux of wave-action, and $\mathscr{F}$ represents the rate of generation or dissipation of wave-action.

Later, for slowly-varying wave fields, they link the wave-action to the *pseudomomentum* per unit of mass, replacing in (0.1.76) the $\partial/\partial\alpha$ with $-\partial/\partial x_i$, the ith component results

$$\mathsf{p}_i(\mathbf{x}, t) \equiv -\overline{\xi_{j,i}\{u_j^{\ell} + (\mathbf{\Omega} \times \xi)_j\}} \tag{0.1.78}$$

where, because of (0.1.55) and (0.1.64), the disturbance velocity u_j^{ℓ} can be replaced by the actual velocity u_j^{ξ}. The *pseudomomuntum* represents the nonlinear forcing of the mean motion by the waves. In our case of study of nearshore circulation we can neglect the angular velocity of the frame of reference so that $\mathbf{\Omega} = 0$, therefore the *pseudomomentum* vector is reduced to the following form:

$$\mathsf{p}_i(\mathbf{x}, t) = -\overline{\xi_{j,i}u_j^{\xi}}. \tag{0.1.79}$$

GLM-Boussinesq theory

The GLM equations describe the back effect of oscillatory disturbances upon the mean state. Of course the GLM still describes a mean motion, but it describes the Lagrangian aspects of the motion from an Eulerian framework and is consequently able to capture structural aspects of the flow. Therefore, the GLM approach appears to be quite useful in some classes of problems.
In the nearshore zone studies an accurate description of the wave evolution is provided by Boussinesq-type equations. Therefore, to get a good and "more complete" description of the wave-current interactions in the nearshore flow it is useful to write a GLM-Boussinesq model.

In the classical Eulerian approach the radiation stress as introduced by Longuet-Higgins (1962), described the "excess flux of momentum due to the presence of waves", and changes in the radiation stress (momentum flux) of the waves are compensated for by changes in the mean field, so the overall momentum is conserved. In the GLM, as already seen previously, the *pseudomomentum*, introduced as the nonlinear forcing of the mean motion by the waves, differs from the classical "radiation stress" and provides a more complete description of the wave-current interactions (see Andrew McIntyre, 1978b).

The GLM theory is now applied to the continuity equation (0.1.1) reported in the following with the (0.1.22), for completeness:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \mathbf{M}^{\omega} = O(\mu^4 \sqrt{g h_0}) \tag{0.1.80}$$

with

$$\mathbf{M}^{\omega} = (h_s + \eta) \left[\mathbf{u}_{\omega} - \mu^2 \mathbf{u}_2\right] + O(\mu^4 h_0 \sqrt{g h_0}), \tag{0.1.81}$$

where $\mathbf{u}_2$ is given by (0.1.32). Defining an opportune mean layer depth $\tilde{h} = \overline{\tilde{h}}$ in analogy with the mean density $\tilde{\rho}$, the GLM continuity equation can be achieved easily multiplying $(0.1.80)^{\xi}$, which means equation (0.1.80) evaluated at the position $\mathbf{x} + \xi(\mathbf{x}, t)$, by J:

$$\overline{D}^L_{\wedge} \tilde{h} + \tilde{h} \nabla \cdot \bar{\hat{\mathbf{u}}}^L = 0 \tag{0.1.82}$$

with

$$\tilde{h} = h^{\xi} J = h^{\xi} \left(1 + \nabla \cdot \boldsymbol{\xi} + \frac{\partial(\xi_a, \xi_b)}{\partial(x, y)}\right), \tag{0.1.83}$$

where (0.1.20) has been used and $\overline{D}^L_{\wedge} = \partial/\partial t + \bar{\hat{\mathbf{u}}}^L \cdot \nabla$. Now using again (0.1.20) in (0.1.82) we can get the continuity equation valid for the velocity field $\mathbf{u}_{\omega}$ as follows:

$$\overline{D}^L \tilde{h} + \tilde{h} \nabla \cdot \bar{\mathbf{u}}^L_{\omega} - \mu^2 \bar{\mathbf{u}}^L_2 \cdot \nabla \tilde{h} - \mu^2 \tilde{h} \nabla \cdot \bar{\mathbf{u}}^L_2 = 0, \tag{0.1.84}$$

where $\overline{D}^L = \partial/\partial t + \bar{\mathbf{u}}^L_{\omega} \cdot \nabla$.

The momentum equation is obtained following Andrews & McIntyre (1978) in a such way that the gradient character of the pressure term is maintained. The j-th component of $(0.1.23)^{\xi}$ (i.e. evaluated at $\mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$) is multiply by $\Xi_{j,i}$ and then a time average over the fast time scale is taking. The result is:

$$\overline{D}^L(\overline{u}^L_i - \mathsf{p}_i) + (\overline{u}^L_k)_{,i}(\overline{u}^L_k - \mathsf{p}_k) + g\overline{\eta}^L_{,i} - \frac{1}{2}\overline{(u^{\xi}_j u^{\xi}_j)}_{,i} + \mu^2\, \overline{\Xi_{j,i} \left\{(V_{12})^{\xi}_j + (V_3)^{\xi}_j\right\}} = 0 \tag{0.1.85}$$

where the *pseudomomentum* is defined, following (0.1.79), as:

$$\mathsf{p}_i = -\overline{\xi_{j,i} u^{\xi}_{\omega\, j}}. \tag{0.1.86}$$

and the 1*st*, 2*nd* and 3*th* are derived following exactly the same manipulation of Andrews and McIntyre (1978).

Because of we are interested on looking at the effects that dispersive terms have on the macrovortices dynamics (i.e. Kennedy et al., 2006) we need manage more explicitly the dispersive terms. In particular we have seen in (0.1.27) that $\mathbf{V}_{12}$ is an irrotational term which does not give any contributions to the vorticity equation, therefore much more attention will be give to $\mathbf{V}_3$ from which arise a clear "Vortex Force effect", involved into the vorticity equation.

In the following we analyse term $\mathbf{V}_3$, with $\boldsymbol{\omega} = \nabla \times \mathbf{u}_{\omega} = (0, 0, \omega)$:

$$\begin{aligned}
\overline{\Xi_{j,i}(V_f)^{\xi}_j} &= \overline{\Xi_{j,i}(\mathbf{u}_2 \times \boldsymbol{\omega})^{\xi}_j} = \overline{\Xi_{j,i}(\mathbf{u}^{\xi}_2 \times \boldsymbol{\omega}^{\xi})_j} = \overline{\Xi_{j,i}\, \varepsilon_{jqk}\, (u^{\xi}_{2q}\, \omega^{\xi}_k)} = \\
&= \overline{(\delta_{ij} + \xi_{j,i})\, \varepsilon_{jqk}\, (\overline{u}^L_{2q} + u^{\ell}_{2q})\, \omega^{\xi}_k} = \\
&= \varepsilon_{iqk} \overline{u}^L_{2q}\, \overline{\omega}^L_k \;+\; \varepsilon_{iqk}\, \overline{u^{\ell}_{2q}\, \omega^{\ell}_k} \;+\; \varepsilon_{jqk}\, \overline{u}^L_{2q}\, \overline{\omega^{\ell}_k \xi_{j,i}} \;+\; \varepsilon_{jqk}\, \overline{\omega}^L_k \overline{u^{\ell}_{2q} \xi_{j,i}} \;+\; \varepsilon_{jqk}\, \overline{u^{\ell}_{2q}\, \omega^{\ell}_k \xi_{j,i}}.
\end{aligned} \tag{0.1.87}$$

Detailed expressions for $\boldsymbol{\omega}^\xi$ and the corresponding Lagrangian mean $\overline{\boldsymbol{\varpi}}^L$ are given in Appendix A.

The other pure dispersive term $\mathbf{V}_{12}$ is analyse in the following even if we aspect it should maintain its gradient character without consequently influencing the vorticity dynamics.

$$\overline{\Xi_{j,i}(V_g)_j^\xi} = \overline{\Xi_{j,i}\left(\left[\mathbf{u}_\omega \cdot \mathbf{u}_{12} - \frac{1}{2}\mathsf{w}^2 + T_p + \frac{\eta^2}{2}T_d\right]_{,j}\right)^\xi}$$

$$= \left[\bar{\mathbf{u}}_\omega^L \cdot \bar{\mathbf{u}}_{12}^L + \overline{(\mathbf{u}_\omega^\ell \cdot \mathbf{u}_{12}^\ell)} - \overline{\left(\frac{1}{2}\mathsf{w}^2\right)^\xi} + \overline{T_p^\xi} + \overline{\left(\frac{\eta^2}{2}T_d\right)^\xi}\right]_{,i} \qquad (0.1.88)$$

with

$$T_p = \frac{\partial(\eta \mathsf{w})}{\partial t} + \nabla \cdot (\eta \mathsf{w} \mathbf{u}_\omega), \qquad (0.1.89)$$

$$T_d = \left[\frac{\partial(\nabla \cdot \mathbf{u}_\omega)}{\partial t} + \mathbf{u}_\omega \cdot \nabla(\nabla \cdot \mathbf{u}_\omega)\right]. \qquad (0.1.90)$$

Here w/μ^2 is the vertical velocity evaluated in $z = \eta$ and consequently term $-\frac{1}{2}\mathsf{w}^2$ is related with the dynamic boundary condition on the free surface, imposed to derive the Boussinesq model. Furthermore T_p describes the transport of a quantity, $(\eta \mathsf{w})$, which is a pseudomomentum related to the free surface disturbance η with respect to the rest conditions represented by still waver depth h_s.

Now terms $\mathbf{u}_{12}$ and $\mathbf{u}_2$, respectively in equation (0.1.88) and (0.1.87), can be further expanded, their detailed derivations are given in Appendix B, and read:

$$\mathbf{u}_{12}^\xi = z_\alpha^\xi \nabla\left(\nabla \cdot (h_s^\xi \mathbf{u}_\omega^\xi)\right) + \frac{(z_\alpha^\xi)^2}{2}\nabla\left(\nabla \cdot \mathbf{u}_\omega^\xi\right) + z_\alpha^\xi\, \mathscr{K}\left(h_s^\xi \mathbf{u}_\omega^\xi\right) + \frac{(z_\alpha^\xi)^2}{2}\mathscr{K}\left(\mathbf{u}_\omega^\xi\right), \qquad (0.1.91)$$

$$\mathbf{u}_2^\xi = \frac{1}{2}\left(\eta^\xi - h_s^\xi\right)\nabla\left(\nabla \cdot (h_s^\xi \mathbf{u}_\omega^\xi)\right) + \frac{1}{6}\left((\eta^\xi)^2 - \eta^\xi h_s^\xi + (h_s^\xi)^2\right)\nabla\left(\nabla \cdot \mathbf{u}_\omega^\xi\right)$$
$$+\frac{1}{2}\left(\eta^\xi - h_s^\xi\right)\, \mathscr{K}\left(h_s^\xi \mathbf{u}_\omega^\xi\right) + \frac{1}{6}\left((\eta^\xi)^2 - \eta^\xi h_s^\xi + (h_s^\xi)^2\right)\mathscr{K}\left(\mathbf{u}_\omega^\xi\right). \qquad (0.1.92)$$

Because of we are interested on describing how dispersive effects can influence the macrovortices dynamics we analyse the vorticity equation obtained now by the GLM theory. In particular the potential vorticity equation is derived in the following. A "local" description of the vorticity dynamics is obtain making directly use of the velocity field given by the Boussinesq model, $\mathbf{u}_\omega$ as transport velocity. A more "global" description is achieve and discuss using as transport velocity the depth-averaged field $\hat{\mathbf{u}}$.

The potential vorticity, $\mathsf{q} = \omega/h$, equation in the Eulerian framework can be obtained taking the curl of the momentum equation (0.1.23) and invoking the continuity equation (0.1.1):

$$\frac{D}{Dt}\,\mathsf{q} = -\mu^2\left\{\frac{1}{h}\nabla \times (\mathbf{u}_2 \times \omega) + \mathsf{q}\ \nabla \cdot \mathbf{u}_2 + \mathsf{q}\ \mathbf{u}_2 \cdot \frac{\nabla h}{h}\right\} \qquad (0.1.93)$$

where $D/Dt = (\partial/\partial t + \mathbf{u}_\omega \cdot \nabla)$. In (0.1.93) we can still identified on the r.h.s the dispersive contribution given by the vortex force written for the potential vorticity, a 2D PV stretching term due only to the $\mathbf{u}_2$ velocity contribution and a correction term taking into account the interaction with the topography. All these dispersive effects contribute to change the potential vorticity with respect to the velocity field $\mathbf{u}_\omega$. Now the GLM potential vorticity equation can be obtained from (0.1.93) by using LEMMA 1 in Bühler (2000). This lemma can be applied not only to the Lagrangian-mean of the definition of q but to all other quantities in (0.1.93) and leads to:

$$\boxed{\overline{D}^L\tilde{\mathsf{q}} = -\mu^2\frac{\nabla\times\widetilde{\mathbf{V}}_3}{\tilde{h}} - \mu^2\overline{(\mathsf{q}\ \nabla\cdot\mathbf{u}_2)}^L - \mu^2\overline{\left(\mathsf{q}\ \mathbf{u}_2\cdot\frac{\nabla h}{h}\right)}^L}, \tag{0.1.94}$$

with

$$\tilde{\mathsf{q}} = \frac{\nabla\times(\bar{\mathbf{u}}^L_\omega - \mathbf{p}\ )}{\tilde{h}}, \tag{0.1.95}$$

$$\widetilde{V}_{3\ i} = \overline{V_{3i}}^L + \overline{\xi_{j,i}V^\ell_{3\ i}}, \tag{0.1.96}$$

note that assuming no initial disturbance we can assume that $\tilde{\mathsf{q}} = \overline{\mathsf{q}}^L$. In (0.1.94) the forcing term for the potential vorticity is identified in the Vortex force term (i.e. the first on the r.h.s.).This mean force $\widetilde{\mathbf{V}}_3$ depends only on the local flow and, in particular, on both the local vorticity $\boldsymbol{\omega}$ and the velocity $\mathbf{u}_2$. The second and third terms in (0.1.94) represent once more the 2D stretching terms for the potential vorticity given by the velocity $\mathbf{u}_2$ and written in their lagrangian form. A more accurate, but complex, expression for the Vortex force term can be obtained if (0.1.113) and (0.1.117) are used in (0.1.96), showing that $\mathbf{V}_3 = \overline{(\mathbf{u}_2\times\boldsymbol{\omega})}^L$, which represent a dispersive lagrangian mean quantity can be further split in terms depending on the lagrangian mean field $\bar{\mathbf{u}}^L_\omega$ and the corresponding disturbance part $\mathbf{u}^\ell_\omega$. More interesting is the second term on the l.h.s. in (0.1.96) which can be seen in analogy to the pseudomomentum. In particular in (0.1.94) it represents the rate of generation or dissipation of the potential vorticity associated with the dispersion described by the departure of $\mathbf{u}_\omega$ from the depth averaged velocity. Hence $\overline{\xi_{j,i}V^\ell_{f\ i}}$ is the nonlinear forcing of the mean flow by dispersive characteristics of the wave field.

Now it is useful to derive the evolution equation for the pseudomomentum $\mathbf{p}$. Following Andrews and McIntyre (1978*b*) we multiply the j-th component of the Lagrangian disturbance part of the momentum equation (0.1.23) by $-\xi_{j,i}$ and afterward averaged. It turns out that:

$$\overline{D}^L\mathsf{p}_i + \overline{u}^L_{k,i}\mathsf{p}_k + \tfrac{1}{2}\overline{(u^\xi_j u^\xi_j)}_{,i} - g\ \overline{\xi_{j,i}(\eta_{,j})^\ell} - \mu^2\ \overline{\xi_{j,i}(V^\omega_{12})^\ell_j} + \mathscr{V}_i = 0 \tag{0.1.97}$$

where

$$\mathscr{V}_i \equiv -\mu^2\ \overline{\xi_{j,i}V_{3j}^{\ \ell}}\ . \tag{0.1.98}$$

Note that $\mathscr{V}$ is the same term we found developing $\mathbf{V}_3$ in the Lagrangian momentum equation, (0.1.87). Now the pseudomomentum-forcing terms $-\mu^2\ \overline{\xi_{j,i}(V_g)^\ell_j}$ and $\mathscr{V}_i$ are wave properties and describe the generation or destruction of pseudomomentum due to dispersive effects. In particular $\mathscr{V}_i$ is equal to the second term found in (0.1.96), and leads to:

$$\widetilde{V}_{3\ i} = \overline{V_{3i}}^L - \mathscr{V}_i. \tag{0.1.99}$$

This results, such as the one found by Bühler (2000) for dissipative waves in shallow water, becomes important because connect directly the *"forcing"* $\widetilde{\mathbf{V}}_3$ of the potential vorticity in (0.1.94) with the rate of dissipation of pseudomomentum due to dispersive wave effect.

As already discussed the potential vorticity equation can be written "globally" also in terms of the depth-averaged velocity transport. In the Eulerian framework it is given by (0.1.16) and leads to the conservation of the potential vorticity q, reported in the following:

$$\frac{D_\wedge}{Dt}\ \mathsf{q}\ = 0. \tag{0.1.100}$$

Invoking once more LEMMA 1 in Bühler (2000), the potential vorticity eqaution (0.1.102) can be written in GLM theory as:

$$\overline{D}^L_\wedge\overline{\mathsf{q}}^L = \overline{D}^L_\wedge\frac{\nabla\times(\bar{\mathbf{u}}^L_\omega - \mathbf{p}\ )}{\tilde{h}} = 0. \tag{0.1.101}$$

where $\overline{D}^L = \partial/\partial t + \bar{\hat{\mathbf{u}}}^L \cdot \nabla$. Apparently no dispersive terms appear in (0.1.102), but both $\mathbf{u}_\omega$ and p bring inside, if written in term of $\hat{\mathbf{u}}$, dispersive effects. Making use of (0.1.32), the potential vorticity equation (0.1.102) becomes:

$$\boxed{\overline{D}^L_\wedge \frac{\nabla \times (\bar{\hat{\mathbf{u}}}^L - \mu^2 \overline{\mathbf{u}_2}^L - \mathsf{p}_\wedge + \mu^2 \mathsf{p}_2)}{\tilde{h}} = 0}, \tag{0.1.102}$$

where

$$\mathsf{p}_{\wedge i} = -\overline{\xi_{j,i} \hat{u}^\xi_j} \qquad \text{and} \qquad \mathsf{p}_{2i} = -\overline{\xi_{j,i} u^\xi_{2\,j}}. \tag{0.1.103}$$

From (0.1.102) it becomes evident that the dispersive effects, acting on the depth averaged velocity fields, are described in term of both lagrangian mean velocity $\overline{\mathbf{u}_2}^L$ and disturbance particle field through the associated pseudomomentum p_2. Once more we note that in shallow water approximation, where it is usually assumed depth-averaged velocity, equation (0.1.102) turns in the same potential vorticity equation found by Bühler & McIntyre (1998) for non-dissipative waves.

The GLM Boussinesq equations are now evaluated in the small amplitude limit, making the same assumption did for the Eulerian equations. In particular we assume a small-amplitude parameter $a \ll 1$ and expand the flow fields in a such way that the background state is $O(1)$, the disturbance fields are $O(a)$ and the mean-flow response fields are $O(a^2)$. The derivative of the disturbance fields are assumed to be $O(1)$ while the spatial derivatives of mean fields are $O(\varepsilon)$, assuming as suitable scale parameter $\varepsilon \ll 1$. The generic Lagrangian disturbance field and the Stokes corrections can be respectively approximated, following Andrews and McIntyre (1978*a*), as:

$$\phi^\ell \approx \phi', \tag{0.1.104}$$

$$\overline{\phi}^S \approx -\overline{\xi'_{j,j}\phi'}. \tag{0.1.105}$$

where ϕ' and the disturbance ξ' conventionally indicate respectively the $O(a)$ component of ϕ and ξ. The Lagrangian mean field $\overline{\phi}^L$ can be now easily evaluated as sum of Eulerian mean $\overline{\phi}$ and Stokes correction $\overline{\phi}^S$.

Making the same assumption did in the previous section, i.e. (0.1.37a), we get analogous results holding to:

$$\overline{\phi}^S \approx \frac{1}{h_s}\overline{\eta'\phi'}, \tag{0.1.106}$$

$$\overline{\phi}^L \approx \overline{\phi} + \frac{1}{h_s}\overline{\eta'\phi'}, \tag{0.1.107}$$

$$\mathsf{p} \approx \bar{\mathbf{u}}^S_\omega \qquad \text{to } O(a^2). \tag{0.1.108}$$

These simple relation, valid in small ave amplitude approximation reduce the generality of the previous founding in GLM theory to the ones gotten in the Eulerian framework for under the same approximation. Under these approximation become easy to extract from the Eulerian Boussinesq numerical model the Lagrangian mean quantities, avoiding the choose of an explicit disturbance field ξ, which require a "closure".

0.1.3 Appendix A

Here ω_k^ξ is manipulated in the following two way:

$$\omega_k^\xi = \varepsilon_{k\ell m}(u_{\omega\ m,\ell})^\xi = \varepsilon_{k\ell m}(u_{\omega\ m})^\xi_{,p}\,\frac{K_{\ell p}}{J} = \varepsilon_{k\ell m}(u_{\omega\ m})^\xi_{,\ell} - \mathscr{D}_k \tag{0.1.109}$$

$$\boldsymbol{\omega}^\xi = \frac{h^\xi}{h^\xi}(\nabla\times\mathbf{u}_\omega)^\xi = \frac{(\nabla\times h\mathbf{u}_\omega)^\xi}{h^\xi} - \frac{(\nabla h)^\xi\times\mathbf{u}^\xi_\omega}{h^\xi}, \tag{0.1.110}$$

with

$$\mathscr{D}_k = \varepsilon_{k\ell m}(u_{\omega\ m})^\xi_{,p}\,\mathsf{B}_{ps}, \tag{0.1.111}$$

$$\mathsf{B}_{ps} = \xi_{s,\ell}\,\frac{K_{sp}}{J}. \tag{0.1.112}$$

The first expression has been obtained invoking (0.1.73) and (0.1.70) and can be easily written in the compact form as $\boldsymbol{\omega}^\xi = \nabla\times\mathbf{u}^\xi_\omega - \mathscr{D}$ and it becomes useful for dealing with averaged terms in (0.1.87) such as the $2nd$, $3rd$ and $5th$. The second expression (0.1.110) becomes very useful to describe $\overline{\boldsymbol{\varpi}}^L = \overline{(\nabla\times\mathbf{u}_\omega)^\xi}$. Applying a time average and using Lemma 1 in Bühler (2000) we can achieve the following result:

$$\overline{\boldsymbol{\varpi}}^L = \frac{\nabla\times(\widetilde{h\mathbf{u}_\omega}) - \overline{[(\nabla h^\xi\cdot\mathbf{K})\times\mathbf{u}^\xi_\omega]}}{\tilde{h}} = \nabla\times\bar{\mathbf{u}}^L_\omega - \overline{\mathscr{D}}, \tag{0.1.113}$$

with

$$\begin{aligned}
(\widetilde{h\mathbf{u}_\omega})_i &= \overline{(h\mathbf{u}_\omega)}^L_i + \overline{\xi_{j,i}\,(h\mathbf{u}_\omega)^\ell_j} = \overline{(h_s\mathbf{u}_\omega)}^L_i + \overline{\xi_{j,i}\,(h_s\mathbf{u}_\omega)^\ell_j} + \overline{(\eta\mathbf{u}_\omega)}^L_i + \overline{\xi_{j,i}\,(\eta\mathbf{u}_\omega)^\ell_j}\\
&= \overline{h}^L(\bar{\mathbf{u}}^L_\omega)_i + \overline{(h^\ell\mathbf{u}^\ell_\omega)}_i + \overline{\xi_{j,i}\,h^\xi(\mathbf{u}_\omega)^\ell_j} - \overline{h}^L\overline{\xi_{j,i}\,(\mathbf{u}_\omega)^\ell_j}\\
&= \overline{h}^L[(\bar{\mathbf{u}}^L_\omega)_i - \mathsf{p}_i] + \overline{(h^\xi\mathbf{u}^\ell_\omega)}_i + \overline{\xi_{j,i}\,h^\xi(\mathbf{u}_\omega)^\ell_j}\\
&= \overline{h}^L[(\bar{\mathbf{u}}^L_\omega)_i - \mathsf{p}_i] + \overline{h^\xi[(\mathbf{u}^\ell_\omega)_i + \xi_{j,i}\,(\mathbf{u}_\omega)^\ell_j]}\\
&= \overline{h}^L[(\bar{\mathbf{u}}^L_\omega)_i - \mathsf{p}_i] + \overline{h^\xi[(\mathbf{u}^\ell_\omega)_j\ \Xi_{j,i}]}
\end{aligned} \tag{0.1.114}$$

0.1.4 Appendix B

Here $\mathbf{u}_{12}^{\xi}$ and $\mathbf{u}_{2}^{\xi}$ are developed as follows:

$$\mathbf{u}_{12}^{\xi} = [z_\alpha \nabla(\nabla\cdot(h_s\mathbf{u}_\omega))]^{\xi} + \left[\frac{z_\alpha^2}{2}\nabla(\nabla\cdot\mathbf{u}_\omega)\right]^{\xi}$$

$$\Downarrow$$

$$\begin{aligned}
u_{12\,i}^{\xi} &= \left[z_\alpha \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}(h_s u_{\omega\, m})\right]^{\xi} + \left[\frac{z_\alpha^2}{2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}\; u_{\omega\, m}\right]^{\xi} \\
&= z_\alpha^{\xi}\left\{[(h_s u_{\omega\, m})_{,m}]^{\xi}_{,i} - [(h_s u_{\omega\, m})_{,m}]^{\xi}_{,\ell}\underbrace{\frac{K_{j\ell}}{J}\xi_{j,i}}_{\mathsf{B}_{\ell i}}\right\} + \frac{(z_\alpha^{\xi})^2}{2}\left\{[\,u_{\omega\, m,m}]^{\xi}_{,i} - [u_{\omega\, m,m}]^{\xi}_{,\ell}\underbrace{\frac{K_{j\ell}}{J}\xi_{j,i}}_{\mathsf{B}_{\ell i}}\right\} \\
&= z_\alpha^{\xi}\left\{(h_s^{\xi}u_{\omega\, m}^{\xi})_{,mi} - \mathsf{B}_{\ell i}\,(h_s^{\xi}u_{\omega\, m}^{\xi})_{,m\ell} - \left[\mathsf{B}_{pr}\,(h_s^{\xi}u_r^{\xi})_{,p}\right]_{,i} + \mathsf{B}_{\ell i}\left[\mathsf{B}_{pr}\,(h_s^{\xi}u_{\omega\, r}^{\xi})_{,p}\right]_{,\ell}\right\} \\
&\quad + \frac{(z_\alpha^{\xi})^2}{2}\left\{\,u_{\omega\, m,mi}^{\xi} - \mathsf{B}_{\ell i}\,u_{\omega\, m,m\ell}^{\xi} - \left(\mathsf{B}_{pr}\,u_{r,p}^{\xi}\right)_{,i} + \mathsf{B}_{\ell i}\left(\mathsf{B}_{pr}\,u_{\omega\, r,p}^{\xi}\right)_{,\ell}\right\}
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\mathbf{u}_{12}^{\xi} &= z_\alpha^{\xi}\left\{\nabla\left[\nabla\cdot(h_s^{\xi}\mathbf{u}_\omega^{\xi})\right] - \mathbf{B}\cdot\nabla\left[\nabla\cdot(h_s^{\xi}\mathbf{u}_\omega^{\xi})\right] - \nabla\left[(\mathbf{B}\cdot\nabla)(h_s^{\xi}\mathbf{u}_\omega^{\xi})\right] + \mathbf{B}\cdot\nabla\left[(\mathbf{B}\cdot\nabla)(h_s^{\xi}\mathbf{u}_\omega^{\xi})\right]\right\} \\
&\quad + \frac{(z_\alpha^{\xi})^2}{2}\left\{\nabla\left[\nabla\cdot\mathbf{u}_\omega^{\xi}\right] - \mathbf{B}\cdot\nabla\left[\nabla\cdot\mathbf{u}_\omega^{\xi}\right] - \nabla\left[(\mathbf{B}\cdot\nabla)\mathbf{u}_\omega^{\xi}\right] + \mathbf{B}\cdot\nabla\left[(\mathbf{B}\cdot\nabla)\mathbf{u}_\omega^{\xi}\right]\right\} \\
&= z_\alpha^{\xi}\nabla\left(\nabla\cdot(h_s^{\xi}\mathbf{u}_\omega^{\xi})\right) + \frac{(z_\alpha^{\xi})^2}{2}\nabla\left(\nabla\cdot\mathbf{u}_\omega^{\xi}\right) + z_\alpha^{\xi}\,\mathscr{K}\left(h_s^{\xi}\mathbf{u}_\omega^{\xi}\right) + \frac{(z_\alpha^{\xi})^2}{2}\mathscr{K}\left(\mathbf{u}_\omega^{\xi}\right), \qquad (0.1.115)
\end{aligned}$$

where (0.1.58b) (0.1.73) and (0.1.112) are used and the vector operator $\mathscr{K}$ is defined such that for a generic 2D vector $\mathbf{a}$:

$$\mathscr{K}(\mathbf{a}) = -\mathbf{B}\cdot\nabla\left[\nabla\cdot\mathbf{a}\right] - \nabla\left[(\mathbf{B}\cdot\nabla)\mathbf{a}\right] + \mathbf{B}\cdot\nabla\left[(\mathbf{B}\cdot\nabla)\mathbf{a}\right]. \qquad (0.1.116)$$

Analogous results can be find for the velocity $\mathbf{u}_2$ as shown in the following:

$$\begin{aligned}
\mathbf{u}_2^{\xi} &= \frac{1}{2}\left(\eta^{\xi} - h_s^{\xi}\right)\nabla\left(\nabla\cdot(h_s^{\xi}\mathbf{u}_\omega^{\xi})\right) + \frac{1}{6}\left((\eta^{\xi})^2 - \eta^{\xi}h_s^{\xi} + (h_s^{\xi})^2\right)\nabla\left(\nabla\cdot\mathbf{u}_\omega^{\xi}\right) \\
&\quad + \frac{1}{2}\left(\eta^{\xi} - h_s^{\xi}\right)\;\mathscr{K}\left(h_s^{\xi}\mathbf{u}_\omega^{\xi}\right) + \frac{1}{6}\left((\eta^{\xi})^2 - \eta^{\xi}h_s^{\xi} + (h_s^{\xi})^2\right)\mathscr{K}\left(\mathbf{u}_\omega^{\xi}\right). \qquad (0.1.117)
\end{aligned}$$

0.1.5 References

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