An experimental study of standing waves

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[Plates 1 to 4]

The experiments here described were designed to test experimentally some conclusions about free standing waves recently reached analytically by Penney & Price. A close approximation to free oscillations was produced in a tank by wave makers operating with small amplitude and at frequencies where great amplification occurred, owing to resonance. The amplitude-frequency curve proved to consist of two non-intersecting branches, a result which can be explained theoretically.

A striking prediction made by Penney & Price was that when the height of the crests of standing waves reaches about 0·15 wave-length they will become pointed, in the form of a 90° ridge. Higher waves were expected to be unstable because the downward acceleration of the free surface near the crest would exceed that of gravity. The experimental conditions necessary for producing a crest in the form of an angled ridge were found and the wave photographed in this condition. Good agreement was found with the calculated form of the profile of the highest wave, which had an angle very near to 90°.

The predicted instability for two-dimensional waves was found to begin at the moment the crest became a sharp ridge. It rapidly assumed a three-dimensional character which was revealed by two photographic techniques. Even when the amplitude of oscillation of the wave makers was only 0·85°, violent types of instability developed which produced effects that are here recorded.

1. Introduction

In the mathematical analysis of water waves of small amplitude there is little difference between progressive waves, which travel without change of form in one direction, and standing waves which can be regarded as the result of superposing two sets of progressive waves of equal small amplitude travelling in opposite directions. It can be shown that free-standing waves of this type can exist in a rectangular tank with vertical ends. When the amplitude of waves is not small mathematical discussion becomes difficult, but it has been shown that finite progressive waves can be generated irrotationally and that the highest possible waves of this type would have sharp crests contained within an angle of 120°.

No analysis of standing waves of finite amplitude was available till Penney & Price (1952) developed a method for analyzing the free oscillations which can exist in water contained between two parallel vertical plane walls. They found an expansion in Fourier series for the velocity potential of the motion such that each term satisfied the boundary condition for standing waves and was multiplied by a factor which varied with time. This factor was also expressed as a Fourier series containing a fundamental frequency n and all integral multiples of it. These frequencies were common to all terms in the velocity potential expansion, so that the motion was periodic with frequency n.

The analysis of Penney & Price contained one arbitrary number, A, which determined both the frequency, n, and the ratio amplitude/wave-length. The frequency of standing waves of small amplitude and wave-length λ is $(g/2\pi\lambda)^{\frac{1}{2}}$.

The lowest frequency of oscillation between walls separated by a distance L is the same as that of waves of length $2L$, but these oscillations are anti-symmetrical. The lowest frequency for symmetrical oscillations of small amplitude is

$$n_0 = \left(\frac{g}{2\pi L}\right)^{\frac{1}{2}}.$$

If H is the height of the crest above the undisturbed surface when it reaches its maximum height, and D is the depth below this surface of the trough at its maximum, Penney & Price's calculations give H/L, D/L and n/n_0 in terms of the arbitrary parameter A. Taking their expansions to the fifth term in the Fourier expansions they obtained the expressions

$$\frac{H}{L} = \frac{1}{2\pi}\{A + \tfrac{1}{2}A^2 + \tfrac{13}{32}A^3 + \tfrac{145}{672}A^4 + 0{\cdot}116A^5\}, \tag{1}$$

$$(n/n_0)^2 = 1 - \tfrac{1}{4}A^2 - \tfrac{13}{128}A^4. \tag{2}$$

From (1) and (2) the values given in table 1 were calculated.

TABLE 1

A	n/n_0	H/L
0	1·000	0
0·1	0·9987	0·0168
0·2	0·9949	0·0356
0·3	0·9883	0·0570
0·4	0·9785	0·0816
0·5	0·9650	0·1103
0·592	0·9487	0·1411

The figures in the last line of table 1 for $A = 0{\cdot}592$ correspond with the highest possible periodic wave which, according to Penney & Price, has the following properties:

(*a*) The crest rises to a sharp edge of angle 90°.

(*b*) The downward acceleration of the fluid at the crest when it is at its highest point is equal to g, the acceleration of gravity. When this occurs the surface may become unstable near the pointed crest.

(*c*) At the instant when the wave crest reaches its maximum height the fluid is everywhere at rest. This is also true for waves of all amplitudes.

(*d*) The wave profile is that which is reproduced in figure 1.

In discussing the possibility that the crest of the highest wave will reach a sharp angle Penney & Price (1952, pp. 272–3) give a general argument, which does not seem to depend on the manner in which the angle is formed, to show that if such an angle is formed it must be a right angle. They point out that if the distribution of pressure near the point can be expressed in ascending integral powers of the co-ordinates, free surfaces meeting at the point must meet at right angles, each being at 45° to the vertical. While this is undoubtedly true I have been unable to follow Penney & Price's arguments tending to show that the pressure distribution will necessarily have the mathematical form they have assumed.

It is clearly possible to imagine that a wedge of water with any given vertical angle can be released from rest. If at some later instant the velocity at every point in the field is reversed, a field of flow will then exist which will subsequently produce the same angle as that which existed originally. For this reason it is difficult to understand how an argument which does not appear to depend on the mode of formation of the sharp-angled crest can be used to show that the angle must be a right angle when it is formed in the course of a free oscillation.

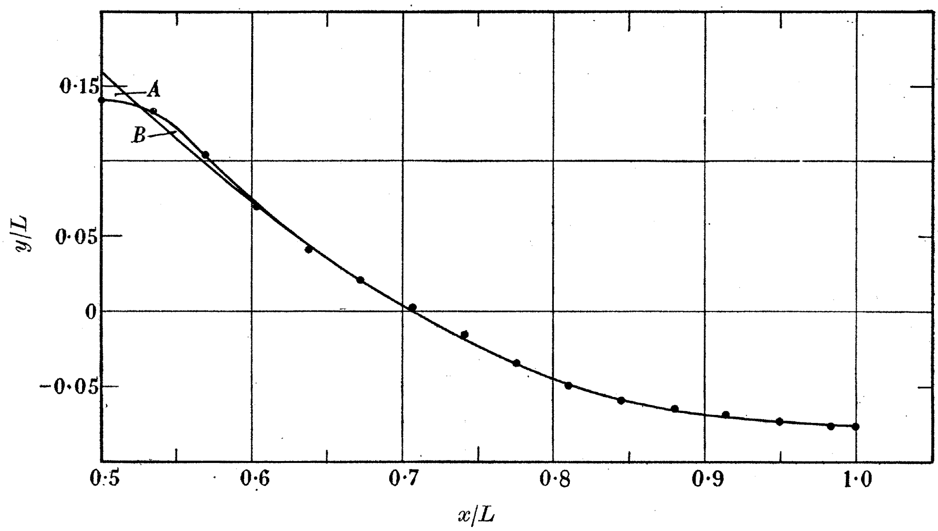


FIGURE 1. Half of wave profile calculated by Penney & Price. The points represent the five-term approximation.

Penney & Price's profile of the highest standing wave

The five-term approximation used by Penney & Price gives the profile shown in figure 1, p. 272, of their paper. Points taken from that figure are here reproduced in figure 1. If a curve is drawn through these points it is found to have a slightly wavy appearance, but a smooth curve with only slowly changing curvature, such as that shown in figure 1, can be drawn so that it passes very close to the five-term profile except near the crest, which cannot be pointed in this approximation. Penney & Price continued the smooth profile in the neighbourhood of the crest in the full line of figure 1. They found that this continuation must slope at an angle which is very close to 45° when it reaches the wave crest. Though at first sight it might be thought that this process leaves open the possibility of considerable artistic licence, it should be remembered that the smooth curve must be drawn so that the total area between the wave profile and the undisturbed water surface is zero. Since the five-term approximation necessarily has this property, the smooth curve must be so drawn that the area shown in figure 1 as A must be equal to the area shown as B. This consideration limits the possible maximum value of H/L to a small range of values close to $H/L = 0{\cdot}160$. Since the acceleration at the crest, calculated by the five-term approximation, with $A = 0{\cdot}592$, was equal to g at $H/L = 0{\cdot}141$, it seems that a particle which rises during a periodic oscillation above $H/L = 0{\cdot}141$ is likely to experience a downward acceleration greater than g. It seems therefore that to determine the pointed profile accurately the process described above for finding the height of the crest should be applied to a five-term profile corresponding to a value of A slightly less than $0{\cdot}592$; but it would not be possible to improve on the accuracy of Penney & Price's method without using more than five terms.

The slope of the smooth profile shown in figure 1 was measured using a protractor and found to be 44° at the crest, corresponding to a crest angle of 92°. This is so near to a right angle that it seems highly probable that if an analysis of the motion near the point could be carried out Penney & Price's prediction would be found to be correct.

It has been pointed out that if the crest becomes pointed the acceleration there is equal to g, since the pressure gradient must be zero. If higher periodic waves can exist the downward acceleration at the crest must be greater than g during a short time when it is near its maximum value. There is nothing in Penney & Price's analysis to show that such waves cannot be calculated using their method and assuming a value for A greater than $0{\cdot}592$. Such waves would be unstable in the sense that disturbances would grow during that part of each period in which the downward acceleration near the crest was greater than g, but it is not obvious that any particular disturbance would be greater at the end of a complete period than it was at the beginning. When a liquid with a free surface is subjected to vertical vibrations corrugations appear. The theory of this kind of instability has recently been analyzed by Dr F. Ursell, who pointed out to me that instability can occur under conditions when the greatest vertical acceleration is much less than g. If instability of the type associated with vertical oscillation of a plane horizontal surface can occur in a fluid which is oscillating as a standing wave, the waves calculated by Penney & Price might be unstable when the maximum value of H/L was less than the value for the pointed crest.

In view of these uncertainties, it seemed worth while to try to produce experimentally the wave contemplated analytically by Penney & Price. In real fluids the free oscillations imagined by mathematicians cannot be produced, owing to the damping effects of viscosity, but forced waves of constant amplitude can be produced; and if the frequency of the forcing agent is very close to that of a free oscillation, the mode will be very nearly the same as that of a free wave.

It is curious that though many experiments have been carried out using wave makers to produce progressive waves, no one seems to have succeeded in producing the highest wave for which a sharp 120° (Stokes 1880) crest is predicted. This is probably because a progressive wave of large amplitude could only be produced by a wave maker which would not only oscillate through a large amplitude, but its shape would have to be controlled to conform with the variable motions of particles at different depths and it would not oscillate in a simple harmonic motion. On the other hand, if a wave maker in a tank is made to oscillate with frequency close to that of a free mode, this mode should be excited even if the motion of the wave maker approximates only very roughly to that of the fluid particles during a free oscillation.

2. Experimental apparatus

A tank 4 m long, 14·8 cm wide and 23 cm deep was available. Since most of Penney & Price's results were concerned with water of infinite depth it was necessary to shorten the tank till the ratio of depth to length was so great that the effect of the bottom could be neglected. This was effected by making movable vertical partitions which could be laid on the bottom of the tank so as to form, with the glass sides, a wave tank of variable length.

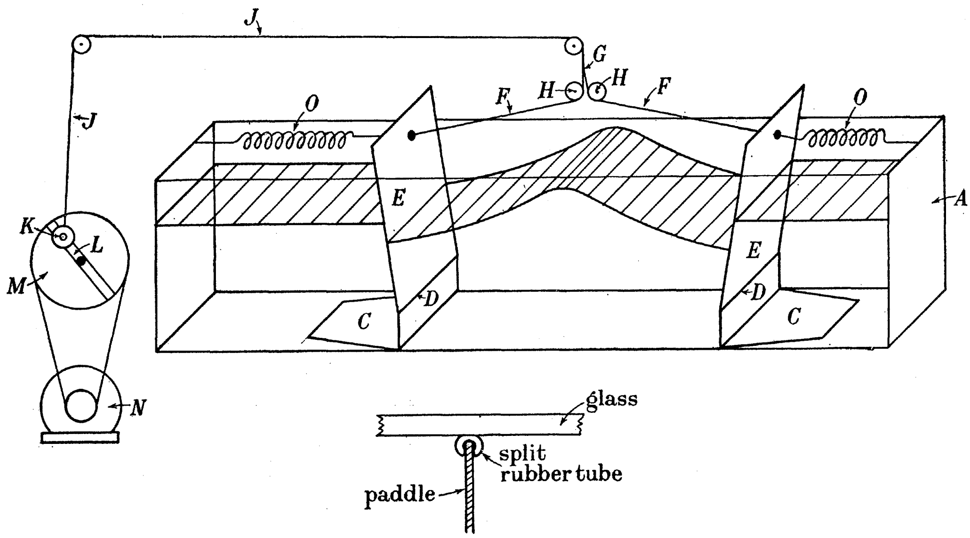


Figure 2. Wave tank.

The apparatus is shown in figure 2. The tank A is shown shorter than its actual length. The two partitions consist of two L-shaped pieces C, hinged at D to wave-making paddles E. The height of the vertical limb of the L-shaped pieces C was 5 cm. The width of the paddles was 14·3 cm, so that there was a gap of 2·5 mm each side between the glass walls of the tank and the edge of the partitions. It was soon found that waves produced in the section between the paddles were affected by the oscillating flow of water through these gaps. This difficulty was overcome by slitting a rubber tube down one side and slipping it down between the partition and the glass side. In this way a water-lubricated seal was formed which completely stopped the fluctuating flow round the partition. These water seals are not shown in figure 2, but a section of the edge of a partition and the seal is shown in a small inset below the main sketch.

The wave-making paddles E were caused to oscillate by means of wires F which joined at G after passing over pulleys H. These wires were pulled back and forth by a wire J, the other end of which was connected through a ball-bearing collar to a pin K. This pin could be fixed at any point of a slide L which formed the diameter of a wheel M. This wheel was driven by a motor N, the speed of which could be varied. The wires F, F and J were kept taut by springs O. A revolution counter, not shown in figure 2, was attached to the axle of the wheel M so that the frequency, n, could be measured by taking the time for 100 oscillations.

In the course of the work it was found that the damping was very small so that sharp resonance occurred near the speed of free oscillations. It was necessary to maintain the speed constant to about one part in 250 during a run of 100 oscillations. When the length of the oscillating basin was 32·9 cm the frequency of the natural oscillations was 2·17 c/s, so that the time for 100 oscillations was 46·0 s. It was necessary therefore to ensure that the variation in timing of successive runs of 100 should not be greater than $\frac{1}{5}$ s. Slow variations in the voltage of the electric supply to the motor N made it necessary to control the field current by hand during a run so that an instrument giving instantaneous readings of the angular velocity of M was needed. This was devised for me by Dr T. M. Ellison. A brass disk was fixed to the axis of M, and a powerful permanent magnet was fixed so that rotation of the disk gave rise to an e.m.f. between the centre and the circumference. The resulting current was tapped off by a sliding contact of brass which was identical with that of the disk and passed through a galvanometer. The readings of this instrument made it possible to control the frequency of the oscillations by means of a hand-operated rheostat to the desired degree of constancy.

When the wave maker was started in still water and its oscillations were maintained at constant frequency and constant small amplitude, the resulting forced waves at first grew with each succeeding stroke. Later they decreased till finally the beats died away and an oscillation of constant amplitude was produced. To measure the profiles of these waves three methods were used.

Method (*a*). Probes which just touched the wave at the highest point attained and under-water probes which just emerged from the surface at the lowest point were made and attached with appropriate vertical scales to a carriage which could slide on the glass walls of the tank. This method was not found to be very accurate.

Method (*b*). Photographs were taken by means of a camera 4 ft. from the side of the tank. The background was illuminated by means of a strong light and $\frac{1}{500}$ s exposures were taken at random. In some cases the instant of exposure was close to that at which the fluid was at rest and the crest at its maximum height.

Method (*c*). Glass plates ground on one side were inserted vertically before the instant when the maximum height was attained, and removed after that instant. The water wetted the ground glass and revealed the profile of the maximum height attained. Sometimes these plates were left in position during several oscillations. In each experiment two plates were inserted. The first was in the vertical plane parallel to, and midway between, the glass sides of the trough. The second was in the vertical plane at right angles to the length of the tank. These two plates were then hung on a frame, illuminated so as to show the wetted area, and photographed. The maximum height of the wetted area above the mean level of the water was also measured directly.

3. Resonance experiments

The usual theory of simple harmonic oscillations leads to the expectation that in order to produce a forced mode which shall be as nearly as possible identical with a free mode the frequency of the wave maker should be as nearly as possible equal to that of the free oscillation. It is not possible to apply this principle directly in the case of standing waves, because the frequency of free oscillations depends on the wave amplitude and the amplitude which will be attained is not known beforehand. For this reason the eccentric pin (K, figure 2) was set so that the amplitude of movement of the wave maker was constant and the maximum height of the wave at its crest was measured for a range of frequencies using the wetted plate method (c). The time taken for 100 oscillations was measured with a stop-watch.

In all the experiments here described the length L of the tank was 32·9 cm. The depth of the hinge of the wave makers was 10 cm and the depth, d, of the water 15·5 cm. The amplitude was found by measuring the maximum extent of the movement of the top of the wave makers, which were 22·6 cm long. The angular amplitude of their oscillations in degrees was therefore

$$\theta_0 = \frac{1}{2}\frac{\text{total travel of top of wave maker (cm)}}{22{\cdot}6}\left(\frac{180}{\pi}\right). \tag{3}$$

Reduction of results to non-dimensional form

The maximum height H of the crest above the level of the water before starting the wave maker can be measured. For comparison between results obtained in tanks of varying sizes as well as for comparison with theory, all length measurements were divided by the length L of the tank. Thus the maximum height H is expressed in the non-dimensional form H/L and horizontal distances, x, measured from one end of the tank are given as x/L. The crests occurred at $x/L = 0$, 0·5 and 1·0, and in some of the photographs (figures 14, 16, 19, plate 3) a scale for x/L is given. Vertical heights are expressed as y/H. These co-ordinates are used in figure 1.

To express the measured frequency n in non-dimensional form it can be divided by n_0, the frequency for oscillations of small amplitude. The frequency n_0 can be calculated, using the formula for symmetrical oscillations of small amplitude

$$n_0^2 = \frac{g}{2\pi L}\tanh\frac{2\pi d}{L}. \tag{4}$$

In most of the experiments $d = 15{\cdot}5$ cm, $L = 32{\cdot}9$ cm, $g = 981$, so that

$$n_0 = 2{\cdot}173 \text{ c/s}. \tag{5}$$

Oscillations of small amplitude could be excited by making the paddle perform two or three oscillations. After stopping the wave maker in its central position about 150 free oscillations could be counted. Several counts of 100 oscillations were timed, all of them at 46·1 or 46·0 s. These correspond to $n_0 = 2{\cdot}169$ or 2·174. Thus, the observed frequency was within the limit of accuracy of the observation equal to that calculated in the classical manner, assuming that viscosity has no effect and that the water slips without resistance over solid boundaries. For comparison with Penney & Price's theoretical results the observed values of n were divided by $n_0 = 2{\cdot}173$.

Experimental results

Several series of measurements were made with a fixed setting of the amplitude pin (K, figure 2). Crest heights H were measured over ranges of values of frequencies given by $0{\cdot}85 < n/n_0 < 1{\cdot}15$. The results for one such set in which the amplitude of the wave maker was $\theta_0 = 0{\cdot}716°$ are shown in figure 3. Starting with $n/n_0 = 0{\cdot}885$ for which $H/L = 0{\cdot}006$, the frequency was very gradually increased. As would be expected H/L rose with increasing rapidity as the resonance frequency ($n/n_0 = 1$) was approached. It rose steadily till at $n/n_0 = 0{\cdot}968$, $H/L = 0{\cdot}025$.

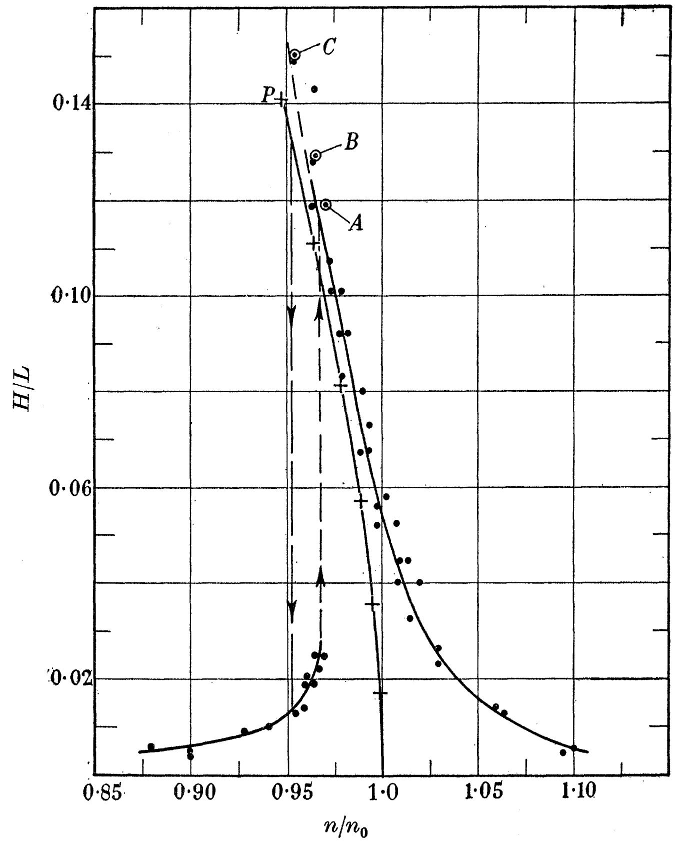


FIGURE 3. Resonance curve when wave makers oscillate with amplitude $\theta_0 = 0{\cdot}716°$. Crosses are calculated points for free oscillations (table 1); points A, B and C are for motions shown in figures 15, 16, 17 (plate 3).

At this stage the slightest increase in frequency led to a very great increase in amplitude. This is represented on the resonance diagram of figure 3 by the upward-pointing arrow at $n/n_0 = 0{\cdot}968$. The increase at $n/n_0 = 0{\cdot}968$ from $H/L = 0{\cdot}025$ to $0{\cdot}12$ was observed to be accompanied by a reversal of phase of the forced wave in relation to that of the wave maker. If when the system was oscillating with $n/n_0 = 0{\cdot}968$ and $H/L = 0{\cdot}12$ the frequency was increased, it was found that H/L decreased till at the frequency of resonance for small oscillations, $n/n_0 = 1{\cdot}0$, H/L was only $0{\cdot}058$. Further increase in n/n_0 gave rise to further reduction in H/L

till at $n/n_0 = 1{\cdot}10$, H/L was again reduced to $0{\cdot}006$. If, on the other hand, n/n_0 was very slowly reduced below the value at $0{\cdot}968$, it was found that H/L increased till at $n/n_0 = 0{\cdot}953$ it reached a value which was usually about $0{\cdot}15$. The smallest decrease in n/n_0 below $0{\cdot}953$ caused H/L to decrease from $0{\cdot}15$ to $0{\cdot}012$. This is indicated by the downward-pointing arrow at $n/n_0 = 0{\cdot}953$. It will be noticed in figure 3 that the part of the resonance curve which lies above $H/L = 0{\cdot}125$ is represented by a broken line. This is because the waves for which $H/L > 0{\cdot}125$ were observed to be three-dimensional in character, the crest at the centre of the tank being slightly higher than near the walls. This kind of instability is discussed in § 4.

It will be seen in figure 3 that the resonance curve consists of two branches which do not intersect. With the present experimental tank, and with wave-maker amplitude $0{\cdot}716°$, there is a range $0{\cdot}953 < n/n_0 < 0{\cdot}968$ in which either large waves corresponding with points on the upper branch or small waves corresponding with points on the lower branch can be generated. Which of these two alternative régimes is actually set up at any frequency in this range depends on whether the frequency has been attained by continuous decrease through $n = 0{\cdot}968n_0$ or increase through $0{\cdot}953n_0$.

Theoretical interpretation

These experimental results can be interpreted in the light of Penney & Price's calculations. Theoretical values of H/L for free oscillations and the corresponding values of n/n_0 found by using equations (1) and (2) are shown in figure 3 by means of crosses. A curve has been drawn through these crosses which runs from the point O ($n/n_0 = 1{\cdot}0$, $H/L = 0$) to the point P ($n/n_0 = 0{\cdot}9487$, $H/L = 0{\cdot}141$). The point P represents the highest crest for waves in which the downward acceleration is calculated to be everywhere less than g when the first five terms in the Fourier expansion are used.

Comparing the observed resonance curve with the theoretical frequency-amplitude curve for free undamped oscillations, it will be seen that they are related in much the same way that the well-known resonance curve for linear oscillating systems is related to the straight line at $n/n_0 = 1{\cdot}0$, which represents the fact that in such systems all amplitudes are possible for free undamped oscillations, and all have the same frequency.

In figure 4 two resonance curves are shown. Curve I has been calculated from the resonance equation for a linear oscillator with damping coefficient α which may be written

$$\frac{H}{L} = \gamma\left[\left(1-\frac{n^2}{n_0^2}\right)^2 + \frac{\alpha^2}{4\pi^2}\frac{n^2}{n_0^2}\right]^{-\frac{1}{2}}. \tag{6}$$

The constants γ and α have been so chosen that the band width for any particular value of H/L is near that observed when the wave-maker amplitude was $0{\cdot}716°$. In fact α was taken as $0{\cdot}063$ and γ as $0{\cdot}0012$.

Curve II, figure 4, is derived from curve I by displacing the band width at each ordinate through a distance equal to the displacement of the corresponding point on the line OP, which is identical with the line OP in figure 3. Thus, curve II on which lie the points $AKBCDJEFG$ is obtained from curve I by a strain, which involves only shear parallel to the abscissae. Though it is not suggested that the resonance curve for any particular non-linear oscillator can be obtained exactly in this way, it seems reasonable to suppose that the small departure from linearity in the restoring force which gives rise to the curvature of the line OP in figures 3 and 4 would, when damping is small, distort the resonance curve in the way indicated in figure 4.

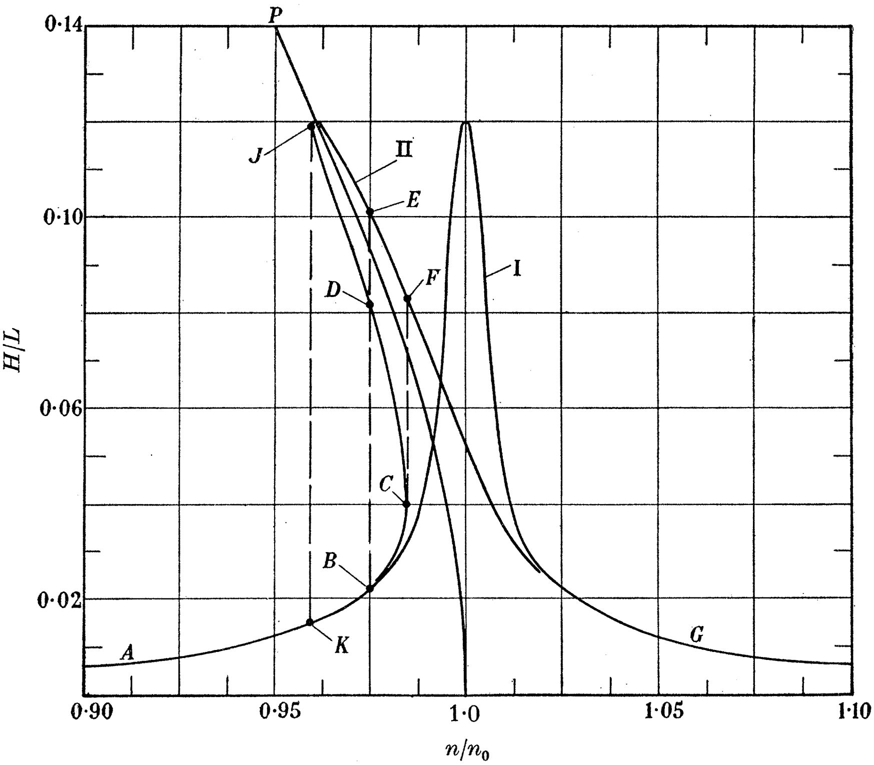


FIGURE 4. Resonance curves calculated for idealized systems. Curve I for linear restoring force and damping proportional to velocity. Curve II for non-linear restoring force but damping proportional to velocity.

Such a distortion will, if the damping is sufficiently small, ensure the existence of a small range of frequencies in which three wave amplitudes are possible for any given amplitude and frequency of the wave maker corresponding with the three points B, D, E (figure 4) in which a line of constant frequency cuts the resonance curve. Of these it seems that B and E may represent stable states and D an unstable state. It is clear that if such a qualitative picture is correct, all the states represented on the line $AKBC$ (figure 4) could be produced by slowly increasing the frequency of the wave maker, but as soon as the point C is reached where $\mathrm{d}H/\mathrm{d}n$ becomes infinite, the mode will change to that represented by F. If the frequency is further increased the successive states will be represented by points on the curve FG. If the frequencies are diminished the states will be represented by points on the portion FEJ of the curve. J is the second point where dH/dn is infinite. If the frequency is reduced below that corresponding with J the amplitude will fall till it is represented by K and then the representative point will move back along the curve KA as the forcing frequency decreases. All the possible states of forced oscillations are represented by points on the two non-intersecting curves $AKBC$ and $JEFG$. The portion of the resonance curve between C and J does not represent possible steady states of resonance.

Referring to figure 3, it will be seen that this qualitative picture of resonance for standing waves of finite amplitude agrees very closely with what is observed. It explains why all points corresponding with large amplification ($H/L > 0{\cdot}026$) are on the high-frequency side of the free oscillation line OP.

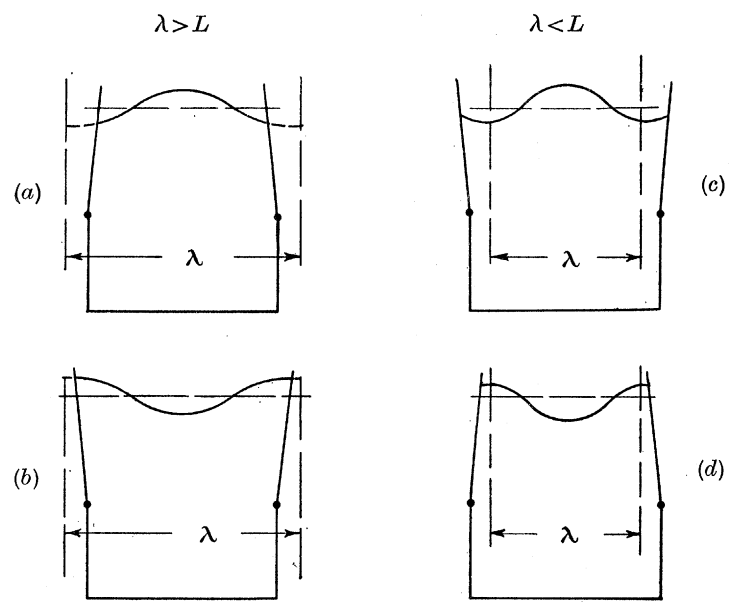


FIGURE 5. Sketch showing relationship between phases of wave and wave maker.

The consistency of this picture is emphasized by noting the relationship between the phase of the wave makers and that of the forced waves. It was observed that at the moment when the crest reaches its greatest height in the middle of the tank the wave makers were sloping towards one another when the representative front on the resonance curve was on the low-frequency branch and away from one another on the high-frequency branch. These states are represented diagrammatically in figure 5.

Consider the motion of initially vertical lines of particles during an undamped free oscillation. If they lie inside the node they will slope inwards when the crest is in the middle of the tank. If they lie near, but outside, this nodal plane they will slope outwards. If the wave makers coincide with these lines of particles, the motion will be as in figures 5(a) and (b) when the wave-length λ, corresponding with the free mode of frequency n, is greater than L, and as 5(c) and (d) if λ is less than L.

Thus the mode will be as in figure 5 (*a*) and (*b*) if the representative point is on the low-frequency branch of the resonance curve and as in (*c*) and (*d*) if on the high-frequency branch. For this reason high amplification occurs only when the mode is as in figure 5 (*c*) and (*d*).

Resonance experiments with reduced amplitude of wave makers

It will be noticed in figure 3 that when $\theta_0 = 0{\cdot}716°$ the crest rose, in some experiments, as high as $H = 0{\cdot}15L$. This is higher than the value $H = 0{\cdot}141L$ given by the five-term approximation for the height at which the greatest downward acceleration is equal to g. In order to find out whether there was any sign of instability when the acceleration is everywhere less than g the wave-maker amplitude was reduced till it was only $\theta_0 = 0{\cdot}355°$.

Even with this small amplitude the wave height rose at its maximum to $H = 0{\cdot}125L$. It was found that no instability developed and that the waves were, as far as the eye could tell, two-dimensional. The resonance curve was like that for $\theta_0 = 0{\cdot}716°$, but the range of frequencies for which two values of H/L are possible was slightly different. These ranges are given in table 2 together with the range calculated from the theoretical model illustrated in figure 4. It will be noticed that the theoretical model agrees very closely with the resonance phenomena observed when $\theta_0 = 0{\cdot}355°$.

TABLE 2

θ_0	maximum n/n_0 on lower branch of resonance	minimum n/n_0 on upper branch	maximum H/L
0·716°	0·968	0·953	0·15
0·335°	0·980	0·956	0·125
theoretical model (figure 4)	0·978	0·96	0·12

4. Development of instability in forced waves of large amplitude

It has been pointed out that the highest waves observed when the wave-maker amplitude was 0·716° were about equal in height to the highest wave for which the calculated downward acceleration is everywhere less than g. If the downward acceleration over any part of the surface is greater than g it will be unstable during the time that g is exceeded. This does not necessarily mean that instability will be very marked immediately g is exceeded, for this excess may only be in existence for a very small proportion of each wave period. It would be difficult to investigate mathematically what happens when surface instability appears; accordingly, an experimental study, particularly of the early stages of instability, was undertaken.

The wave maker was set with amplitude 0·75° and frequency $n = 2{\cdot}10$ c/s corresponding with $n/n_0 = 0{\cdot}965$. The oscillation was in the mode represented by the upper branch of the resonance curve. In this condition a number of photographs (method (*b*), §2) were taken at random and those reproduced as figures 7, 8, 9, plate 1, have been selected as representing interesting phases of the motion. The motion appears to be very nearly two-dimensional. Figure 7, which showed the highest crest, was measured, and it was found that $H/L = 0{\cdot}12$. The value of H/L corresponding with $n/n_0 = 0{\cdot}965$ in figure 3 is also $H/L = 0{\cdot}12$, so that this photograph must have been taken at or very close to the position of greatest displacement. Figure 8 shows the form of the surface when it is nearly flat, and in figure 9 the central trough has nearly reached its greatest depth.

The frequency was then reduced, keeping $\theta_0 = 0{\cdot}75°$, till n/n_0 was between 0·95 and 0·96. The amplitude increased till the crest appeared to become nearly pointed, and if the wave-maker oscillation continued it ceased to be two-dimensional.

Comparison of profiles with calculation

To compare the highest two-dimensional waves with Penney & Price's calculations a number of runs were made. The waves were watched and photographs taken as soon as they appeared to reach their greatest amplitude as two-dimensional waves, i.e. just before they began to develop three-dimensional characteristics. Figures 10 to 13, plate 2, were taken under these conditions. It will be seen in figures 11 and 12 that the crest is nearly sharp and is showing signs of instability in the sense that small protuberances are appearing at the crest. For this reason these profiles have been compared with Penney & Price's calculations. The centres of the circles in figures 11 and 12 are points on the calculated profile which have been superposed on the photographs. These were obtained by altering the scale of figure 1 so that it could be superposed on the photographs. Since the photographs were taken from a distance 121 cm from the nearer glass wall and 135 cm from the further wall, the length L would appear on different scales in the photograph according as it is measured on the nearer or farther wall. For comparison with Penney & Price's profile, the scale of figure 1 was reduced so that the length $x = L$ on the photograph corresponded with a length 32·9 cm placed in the mid-plane at a distance of $121 + 7{\cdot}4 = 128{\cdot}4$ cm from the camera. In each photograph only the far edge of the wave maker is seen, the nearer edge being just outside the picture. The nearly vertical black lines seen on the edges of each photograph are the water-lubricated rubber seals. The breaks in them are, of course, due to the refraction of the water. The horizontal lines seen are black threads placed on the outside of the nearer glass wall to show approximately the position of the undisturbed water surface.

The scale length L on the photographs was found by multiplying the length between the farther edges of the wave makers by $135{\cdot}8/128{\cdot}4 = 1{\cdot}057$. When Penney & Price's profile was reduced to this scale it was found that it was so close to the wave profile of photographs in figures 11 and 12 that it was not possible to superpose it without confusion. For this reason the theoretical profiles were drawn on tracing paper and points pricked through on to the photographs. Circles were then drawn on figures 11 and 12 round the pricked points.

It will be seen that the photographic profile in the case of figure 12 is very close indeed to that calculated by Penney & Price, and that at the moment when instability just begins to appear while the wave is still two-dimensional the crest seems to be pointed with an angle which is very close to 90°.

5. Development of lateral instability

It has already been stated that two-dimensional unstable motion can only be attained as a transitory phenomenon. If the wave makers are kept going at constant speed after the sharpest crest has been attained lateral motion sets in and the crest becomes wavy in the transverse direction perpendicular to the sides of the tank. The photographic technique (*b*) in which the profile was photographed from one side of the tank cannot be used in describing transverse instability. For this purpose recourse was had to the technique of method (*c*), § 2, in which dry ground-glass plates were lowered into the water and the wetted area photographed. This method reveals the maximum height attained by the water surface during the time the plate was partially immersed. It could only give an instantaneous profile if the plate were immersed very rapidly and then immediately withdrawn.

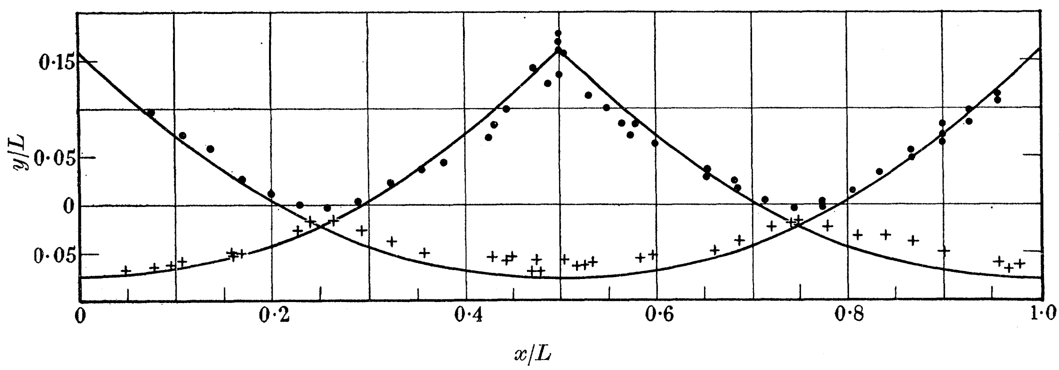


Figure 6. Measurements with probes (method (*a*)). x, measured from one end of tank; y, measured from water surface before starting wave maker. Black circles, ●, downward pointing probe; crosses, +, upward pointing probe.

If the plate is lowered and kept in position for at least one period the true profiles of the crests which form at intervals of half a period at the middle and ends of the tank are revealed. Near the points $x = \frac{1}{4}L$ and $x = \frac{3}{4}L$ the water attains its maximum height at times which are intermediate between those at which the crests are at their highest.

This can be appreciated by referring to measurements made by method (*a*), in which the heights of probes which just touched the surface at its highest and lowest points were recorded. In figure 6 measurements made with the downward-pointing probe are shown as circles and those made with the upward-pointing probe as crosses. The calculated profiles at the two instants when the water is at rest are also shown. Though the method is not very accurate on account of the difficulty of seeing when a fine probe just touches a surface which is only instantaneously at rest, it shows that the wetted plate, which traces the same curve as the downward-pointing probe, gives a correct instantaneous profile for the crests and for distances of about $0{\cdot}2L$ on each side of them. The troughs will not be revealed by the wetted plates, and the form of the wave in the two regions $0{\cdot}2L < x < 0{\cdot}3L$, $+0{\cdot}7L < x < +0{\cdot}8L$ is not revealed either by method (*a*) or by (*c*).

Results using method (c)

A frame was constructed which fitted over the tank and contained slots by means of which the ground-glass slides could be placed in a predetermined position. The ground-glass slides were usually kept in position for two or three oscillations and then withdrawn and placed in a frame set up at a fixed distance from a fixed camera. After withdrawing the longitudinal glass slide the transverse one was inserted, held in position for one or two oscillations, and then withdrawn and placed in the fixed frame above the wet longitudinal slide. The two were then photographed. Several examples of the results using this technique are shown in figures 14 to 20, plate 3. In each case the transverse profile at the centre of the tank, i.e. with the transverse plate along the crest, is shown to the right of the longitudinal wetted area. The two photographs are placed at the same level in each case. In some of them scales showing x (cm) and x/L are given.

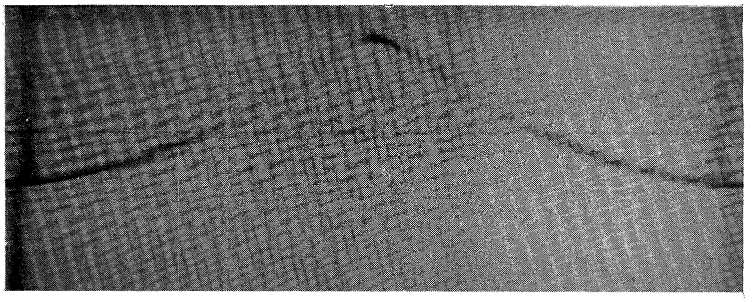
The first measurements were made under conditions which were expected to be stable. The amplitude of the wave maker was cut down to 0·455° and the frequency was maintained at $n/n_0 = 0{\cdot}960$. A number of photographs were taken in which the measured values of H/L were 0·131, 0·130, 0·130, 0·129. In all of them the transverse section, taken at the mid-point of the tank, was practically flat so that the motion was two-dimensional. Figure 14 shows one of these photographs.

The amplitude of the wave maker was then increased to 0·716°. Photographs were taken at $n/n_0 = 0{\cdot}970$, 0·964 and 0·955. These are shown in figures 15, 16, 17. The values of H/L were 0·119, 0·129, 0·150. There is a slight sign of lateral motion in figure 15 ($H/L = 0{\cdot}119$) and this becomes more pronounced as H/L increases. Figure 17 shows that instability in the form of a ridge rises from the crest. The transverse section in figure 17 is very definitely highest in the middle, but a horizontal section of the ridge would be longer in the transverse direction than in the longitudinal direction. Since $\theta_0 = 0{\cdot}716°$ was the amplitude at which the resonance experiments recorded in figure 3 were made, the points corresponding to figures 15, 16 and 17 are marked as A, B and C on that diagram. Figure 18 shows the result of putting a ground-glass slide in and then withdrawing it as rapidly as possible. The longitudinal photograph shows a figure which is probably intermediate in character between the photographs of method (*b*) and the wetted area figures 14 to 17 produced by method (*c*).

The wave-maker amplitude was then increased till $\theta_0 = 0{\cdot}885°$. A number of photographs were taken at frequency $n/n_0 = 0{\cdot}962$. Instability similar to that noticed with $n = 0{\cdot}716°$ was observed. Figure 19 shows the instability at $H/L = 0{\cdot}155$. It will be seen that it is of the same general nature as that shown in figure 17, but the crest is higher and does not extend so far in a transverse direction. In figure 20 where $H/K = 0{\cdot}173$ the same type of instability is increasing.

With the amplitudes so far described the waves were essentially periodic with the frequency of the wave maker, but the slightest decrease in n/n_0 below 0·962 (retaining $\theta_0 = 0{\cdot}885°$) brought a new phenomenon. The lateral motion began to increase very rapidly. The unstable crest which had consisted of a mass of fluid

7



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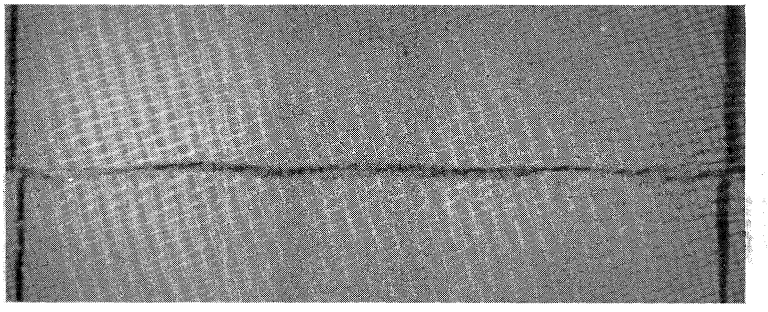

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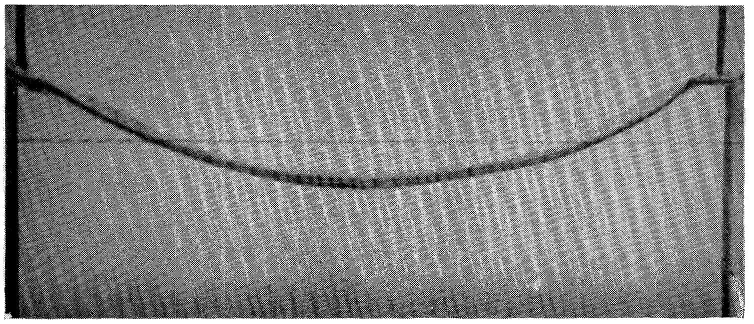
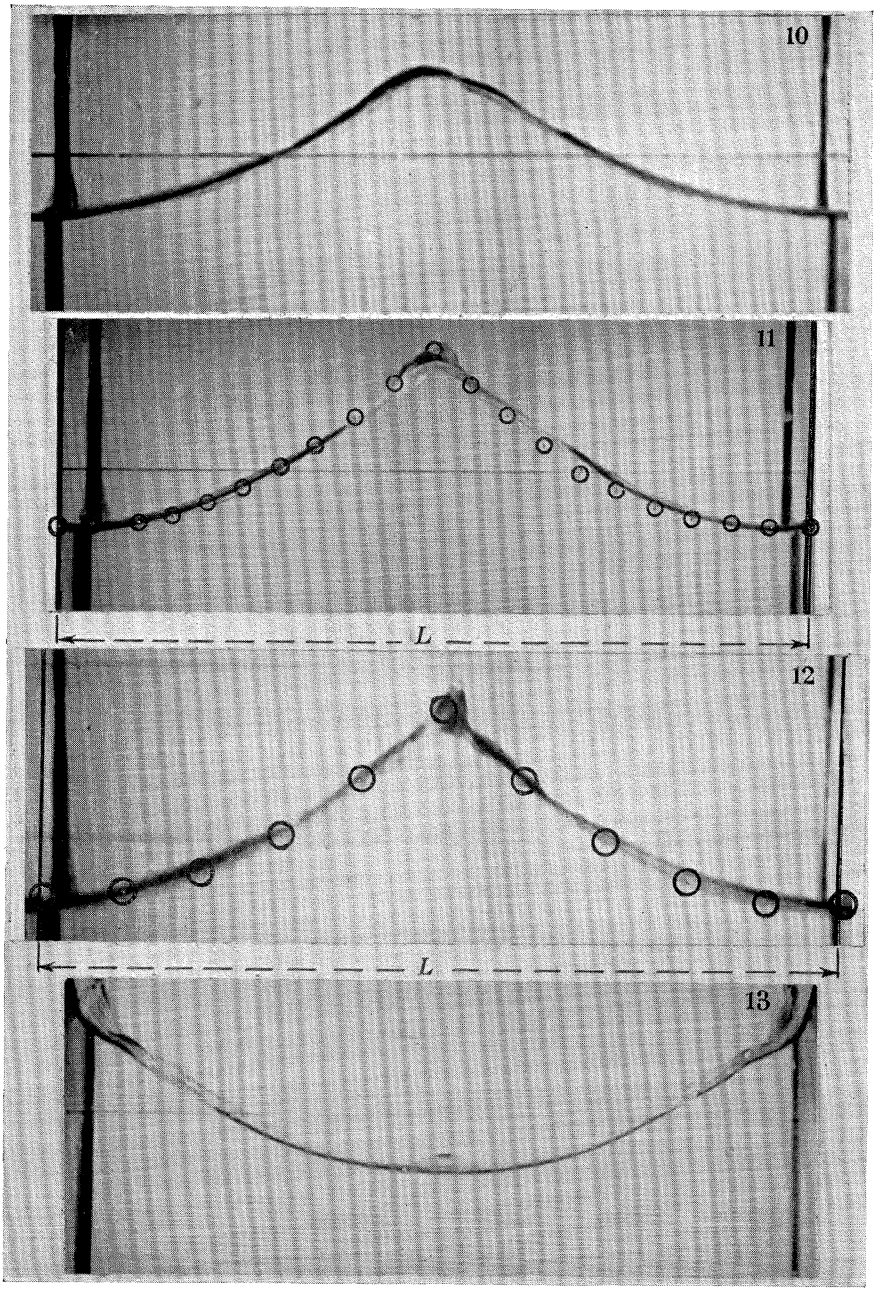


FIGURES 7 to 9. Various phases of oscillation when $\theta_0 = 0{\cdot}75^\circ$, $n/n_0 = 0{\cdot}965$.



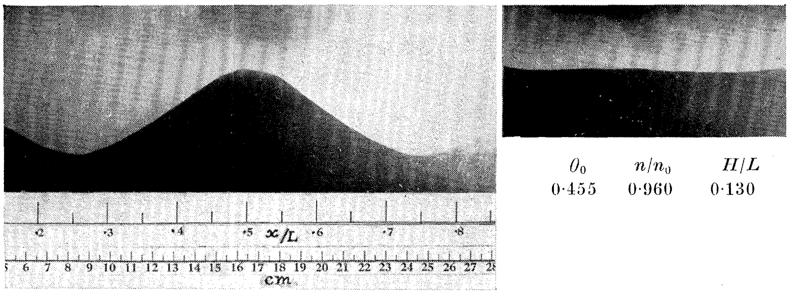
FIGURES 10 to 13. Transient condition before three-dimensional motion starts, when $\theta_0 = 0{\cdot}75°$, $n/n_0 = 0{\cdot}955$.

FIGURE 10. Crest approaching maximum height.

FIGURES 11, 12. Circles show Penney & Price's curve transferred on the correct scale from figure 1.

FIGURE 13. Trough near greatest depth.

14

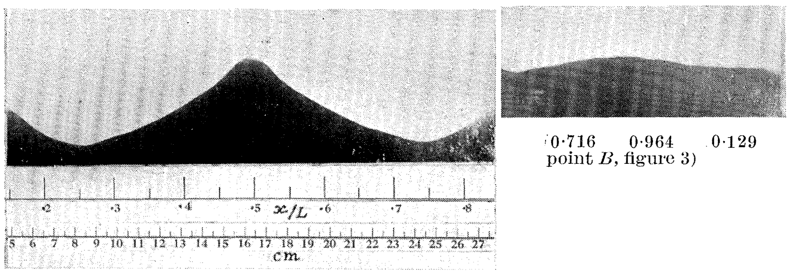


θ_0	n/n_0	H/L
0·455	0·960	0·130

15

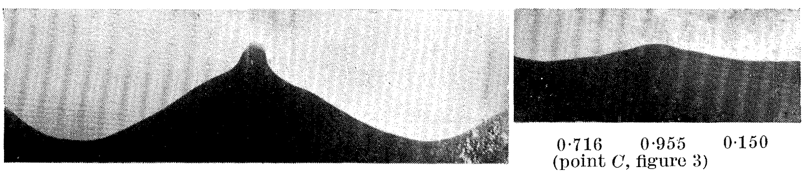
0·716 0·970 0·119
(point *A*, figure 3)

16



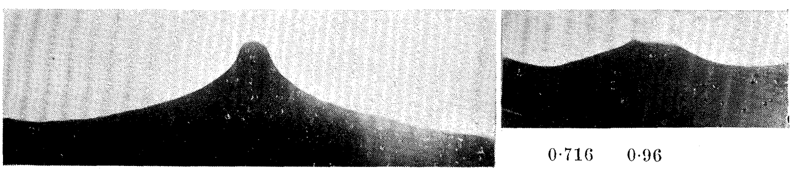
0·716 0·964 0·129
point *B*, figure 3)

17



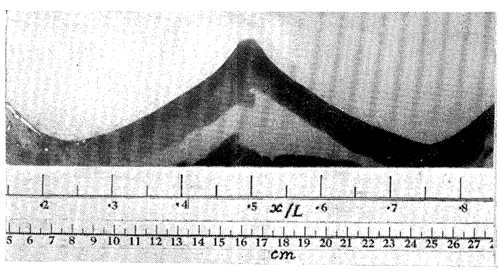
0·716 0·955 0·150
(point *C*, figure 3)

18



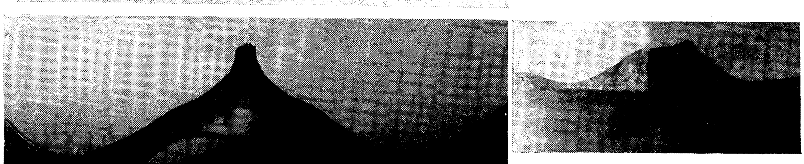
0·716 0·96

19



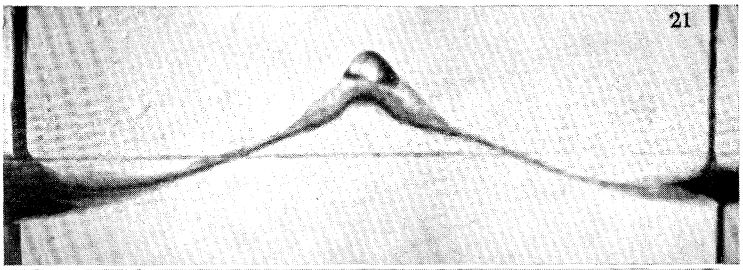
0·885 0·97 0·155

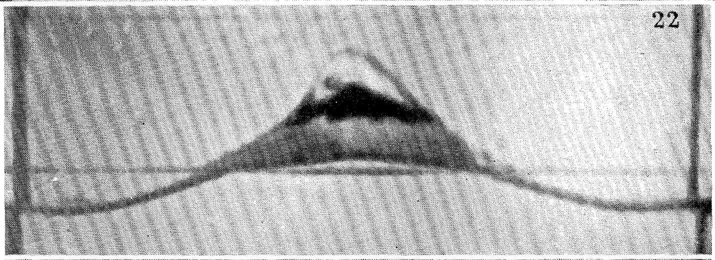
20

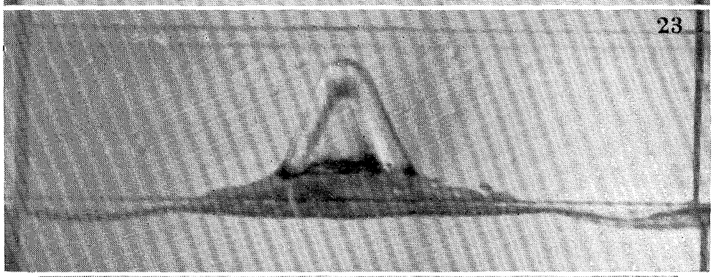


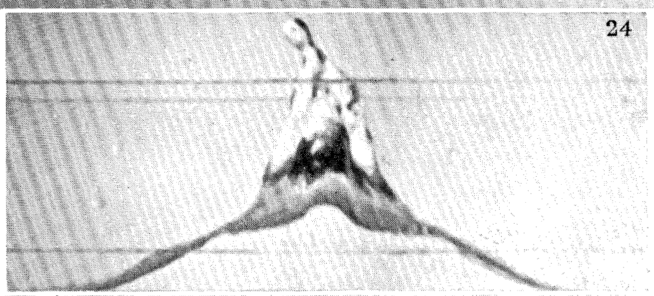
0·885 0·962 0·173

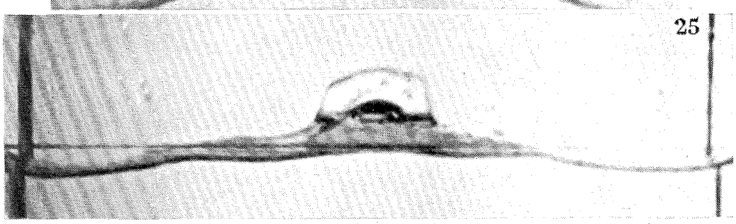
FIGURES 14 to 20. Wetted areas on vertical ground-glass plates (method (*c*)). Left, plate parallel to sides of tank. Right, plate parallel to end of tank. Photographs set at correct relative levels. Figure 18 shows the result of dipping the slide and quickly removing it.











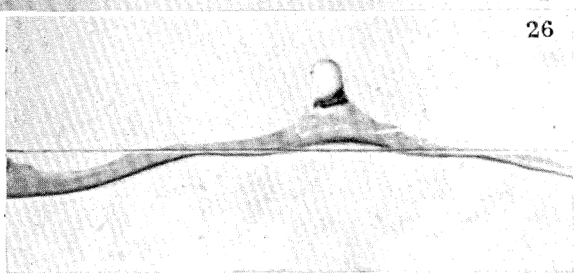


FIGURE 21. $\theta_0 = 0{\cdot}885$, $n/n_0 < 0{\cdot}962$, violent lateral motion beginning.
FIGURES 22 to 26. Later phases of the violent motion with $\theta_0 = 0{\cdot}885$, $n/n_0 < 0{\cdot}962$.

narrow in the direction of the length of the tank and broad in the transverse direction, became more nearly conical. The crests alternated, rising in the centre into a cone or on the two walls in the form of half-cones in successive periods of the wave maker. The frequency of the disturbance was therefore halved. The motion became very violent. Figure 21, plate 4, shows the beginning of this phase of the motion. This photograph (method (*b*)) shows the central conical crest rising in the middle of the tank. At the same time there are two transverse crests at the two walls at the two ends of the tank. These appear as a broadening of the black line representing the profile near the wave maker. When the same phase of the oscillation of the wave maker next occurs the highest crest will be at the two walls in the middle of the tank and transverse crests will appear at the middle of the ends of the tank. This motion rapidly developed a violent character, but it retained its property of being periodic with twice the period of the wave maker. Figures 22 to 26, plate 4, show various stages of this motion. The impression gained by observing this phenomenon was that when the height to which the unstable region at the crest was thrown became sufficiently great it moved as though it were a freely rising and falling body, and when a mass of fluid was thrown up in one oscillation it would meet, during its descent, the crest rising in the next oscillation, thus damping its motion. This explanation, however, is not complete, for the half-frequency motion with its lateral crests was so persistent that it seemed likely to be associated with some possible type of free oscillation. The amplitude of the wave makers was only 0·88 of a degree, and for so small a forcing movement to excite so violent a motion it is difficult to imagine any cause but synchronism between free and forced oscillations.

Though I have not been able to imagine any type of free oscillation with a mode similar to that excited at half the frequency of the wave maker, it was found necessary to avoid conditions suitable for setting up resonance in transverse modes. When, for instance, the length of the tank was made twice the width, the lowest transverse mode had a frequency equal to that of the lowest symmetrical longitudinal oscillation, and it was found impossible to excite two-dimensional waves of large amplitude.

These experiments were carried out in the Cavendish Laboratory, Cambridge.

References

Penney, W. G. & Price, A. T. 1952 *Phil. Trans.* A, **244**, 254.
Stokes, G. G. 1880 *Collected Papers*, **5**, 62. Cambridge University Press.