Analysis of Inhomogeneous Wave Number Spectra

M. A. TAYFUN, C. Y. YANG, V. KLEMAS, AND H. WANG

University of Delaware, Newark, Delaware 19711

As waves propagate from deep to shallow water, various effects of the shallowing depth of water spatially modify spectral characteristics of the sea surface. In such spatially inhomogeneous regimes, spectral computations based on a finite field size are subject to errors due to an inherent spatial smoothing in addition to resolution errors of a more conventional nature. The conflicting requirements of resolution and smoothing errors for an accurate analysis prescribe an optimal field size for which the combined magnitude of such errors is minimal. The optimal field size, the associated errors in resolution and smoothing, and the conditions for limiting accuracy are derived for computations in a shallow water wave field in which refraction and shoaling constitute the predominant inhomogeneity effects. Bottom friction, percolation, and reflection are neglected, and calculations are based on a first-order approximate shallow water wave theroy.

INTRODUCTION

Aerial photographic techniques have recently proved to be entirely appropriate and effective in the study of the ocean surface. These techniques offer certain advantages over various other techniques for the computation of two-dimensional wave number spectra: the advantages of economy and of covering large areas in a short time. *Stilwell* [1969] and *Stilwell and Pilon* [1974] developed and demonstrated an operational photographic system capable of providing quantitative estimates of surface wave spectra for nonstationary and spatially homogeneous wave fields and suggested further applications of these techniques in the study of wave propagation, interaction, and generation.

Although several limitations resulting from wave slopes, solar angles, sky luminance, haze, etc., influence the accuracy of spectral estimates obtained with photographic techniques, the applicability of these methods is not severely restricted unless a fundamental assumption embedded in such techniques is violated. This is the assumption of spatial homogeneity, i.e., the requirement that the statistical properties of the sea surface remain essentially the same over the area to be spectrally analyzed. The violation of this assumption means that if the size of the area to be photographed is too large, the spectral characteristics of a given location may be smoothed or smeared out by contamination from the neighboring areas in the estimation process. On the other hand, if the size of the area photographed is too small, the accuracy of the resulting estimates will be limited in terms of spectral resolution.

Recently, in applying photooptical techniques in shallow water, *Klemas et al.* [1974] pointed out that the spatial inhomogeneity of the shallow water wave field requires choosing an optimal field size. Consequently, questions arise as to the rational basis for such a selection, the criteria governing the accuracy of the associated shallow water spectral estimates, and the limiting conditions under which the applicability of photographic techniques becomes doubtful. *Polis* [1974], using Heisenberg's uncertainty principle, showed the possibility of choosing an optimal field size on the basis of a trade off that must be made between the intrinsic spatial variation of a wave number in shallow water and the increase in the resolution of the wave number from a larger field size. Although this concept is interesting as an optical analogy and obviously useful in examining a regular train of waves in shallow water, its applicability seems less certain to photooptical spectral computations in a random wave field where the primary objective is to determine the spectral magnitude associated with a prescribed wave number locally rather than the optimal resolution or separation of the wave number itself. In the photooptical analysis of directional wave spectra it is plausible in principle to use a two-dimensional generalization of the recent techniques developed for the univariate spectral analysis of nonstationary and inhomogeneous processes [*Priestley*, 1966; *Tayfun et al.*, 1975]. However, the complicated nature of the procedures involved in these techniques likewise suggests that the necessity of using them would eliminate some of the present advantages of the photooptical techniques.

This study examines the limitations on the accuracy of spectral computations based on a finite field size to determine a rationale for the selection of an optimal field size in shallow water. Under the assumptions of a first-order shallow water wave theory and for spectra that can be characterized with a bandwidth measure, approximate expressions are derived for the resolution and smoothing errors associated with the general form of spectral estimates in a shallow water wave field in which refraction and shoaling constitute the predominant inhomogeneity effects. Other incompletely understood effects of the varying water depth such as bottom friction and percolation are neglected. Attention is specifically directed to the accuracy limitations on the sample or raw spectral estimates without any special regard to their sampling fluctuations (or stability) simply because such a consideration becomes irrelevant if the accuracy of the sample estimates themselves is not warranted.

SPECTRAL DEFINITIONS AND GENERAL FORM OF AN ESTIMATE

Consider a spatially homogeneous area of the ocean in which the random surface displacement η from the mean water level is observed as a function of space $\mathbf{r} = (x, y)$. The mean product

$$Z(\mathbf{r}) = Z(\mathbf{r}, 0) = \langle \eta(\mathbf{r}^*, t_0)\langle\eta\rangle(\mathbf{r}^* + \mathbf{r}, t_0)\rangle \quad (1)$$

in which the angle brackets denote a complex conjugate, is defined as the two-dimensional spatial autocorrelation for the instantaneous free surface displacements [*Phillips*, 1969].

The Fourier transform of $Z(\mathbf{r})$ defines the wave number spectrum $S(\mathbf{k})$:

$$S(\mathbf{k}) = (2\pi)^{-2} \int_{\mathbf{r}} Z(\mathbf{r}) \exp(i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{r} \quad (2)$$

where $\mathbf{k} = (l, n)$ denotes the wave number vector $d\mathbf{r} = dx\, dy$ as a shorthand notation, and an integration over the whole surface is implied. For the inverse of (2) we can write

$$Z(\mathbf{r}) = \int_{\mathbf{k}} S(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{k} \qquad (3)$$

in which $d\mathbf{k} = dl\, dn$, and the integration is over all of the wave number space. It is seen from (2) that $S(\mathbf{k})$ is a real nonnegative function, and from (3) with $\mathbf{r} = 0$,

$$Z(0) = \langle |\eta|^2 \rangle = \int_{\mathbf{k}} S(\mathbf{k})\, d\mathbf{k} \qquad (4)$$

Therefore the spectrum represents the density of contributions to the mean square displacement $\langle |\eta|^2 \rangle$ per unit area of the $\mathbf{k}$ space.

The spectrum can also be defined in terms of the Fourier components of the surface displacement itself. As a homogeneous process, $\eta(\mathbf{r}, t_0)$ admits a Fourier-Stieltjes representation in the form [*Phillips*, 1969]

$$\eta(\mathbf{r}) = \int_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r})\, dA(\mathbf{k}) \qquad (5)$$

where it was assumed that $t_0 = 0$ for simplicity and $dA(\mathbf{k})$ is a zero-mean random process in the $\mathbf{k}$ space with orthogonal increments, such that

$$\begin{aligned} \langle dA(\mathbf{k}) \langle dA \rangle(\mathbf{k}^*) \rangle &= 0 \qquad & \mathbf{k} \neq \mathbf{k}^* \\ \langle dA(\mathbf{k}) \langle dA \rangle(\mathbf{k}^*) \rangle &= S(\mathbf{k})\, d\mathbf{k} \qquad & \mathbf{k} = \mathbf{k}^* \end{aligned} \qquad (6)$$

Hence the equation

$$S(\mathbf{k}) = \langle |dA(\mathbf{k})|^2 \rangle / d\mathbf{k} \qquad (7)$$

defines the spectrum as the mean square Fourier amplitude of $\eta(\mathbf{r})$ per unit area of $\mathbf{k}$ space, and this definition is entirely consistent with (2), (3), and (4).

The estimation of the spectrum $S(\mathbf{k})$ through aerial photographic or stereophotogrammetric techniques [*Stilwell and Pilon*, 1974] is based on the analysis of a single photograph or a pair of overlapping stereophotographs of an area of the ocean surface. The analysis is therefore restricted to a finite domain or boundaries corresponding in the simplest case to a square wave field of dimensions $(L \times L)$. With respect to an x, y coordinate system placed at the center of the wave field, the sample $\{\eta(\mathbf{r}) = \eta(x, y); -L/2 \leq x, y \leq L/2\}$ is regarded as one of the many possible realizations of the process $\eta(\mathbf{r})$. The general form of an estimate $\hat{S}(\mathbf{k}^*)$ of $S(\mathbf{k})$ at $\mathbf{k}^*$ constructed from this sample can be expressed in terms of the squared modulus of a modified Fourier transform in the form

$$\hat{S}(\mathbf{k}^*) = \left| \int_{\mathbf{r}} g(\mathbf{r})\eta(\mathbf{r}) \exp(-i\mathbf{k}^*\cdot\mathbf{r})\, d\mathbf{r} \right|^2 \qquad (8)$$

in which $g(\mathbf{r})$ represents a real positive weighting function that incorporates the effects of the finite field size properly normalized so that

$$(2\pi)^2 \int_{\mathbf{r}} [g(\mathbf{r})]^2\, d\mathbf{r} = 1 \qquad (9)$$

The simplest form of $g(\mathbf{r})$, corresponding to a square field, is the two-dimensional boxcar function

$$\begin{aligned} g(\mathbf{r}) &= g(x, y) = (2\pi L)^{-1} \qquad & |x|, |y| < L/2 \\ g(\mathbf{r}) &= g(x, y) = 0 \qquad & |x|, |y| > L/2 \end{aligned} \qquad (10)$$

If we let

$$G(\mathbf{k}) = \int_{\mathbf{r}} g(\mathbf{r}) \exp(-i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{r} \qquad (11)$$

represent the Fourier transform of $g(\mathbf{r})$, then it follows from (5), (6), (8), and (11) that the expected value of $\hat{S}(\mathbf{k}^*)$ can be written in the equivalent forms

$$\begin{aligned} \langle \hat{S}(\mathbf{k}^*) \rangle &= \int_{\mathbf{k}} |G(\mathbf{k})|^2 S(\mathbf{k}^* - k)\, d\mathbf{k} \\ &= \int_{\mathbf{k}} |G(\mathbf{k}^* - \mathbf{k})|^2 S(\mathbf{k})\, d\mathbf{k} \end{aligned} \qquad (12)$$

Therefore the estimator $\hat{S}(\mathbf{k}^*)$ is equivalent on the average to a smoothed or weighted integral of $S(\mathbf{k})$ in the vicinity of $\mathbf{k}^*$ and with weights proportional to $|G|^2$. For the boxcar function (10) the explicit form of these weights, given by

$$|G(\mathbf{k})|^2 = (L/2\pi)^2 \left\{ \frac{\sin lL/2}{lL/2} \right\}^2 \left\{ \frac{\sin nL/2}{nL/2} \right\}^2 \qquad (13)$$

corresponds to a bivariate generalization of the well-known Bartlett spectral window [*Jenkins and Watts*, 1969] and is characterized by a major central lobe over the range $-2\pi/L \leq l, n \leq 2\pi/L$ and by minor side lobes that decay as $O(l^{-2})$ and $O(n^{-2})$.

FINITE FIELD SIZE AND RESOLUTION ERRORS

The general objective in the spectral analysis of the sample $\{\eta(x, y); |x|, |y| \leq L/2\}$ is to estimate the function $S(\mathbf{k})$ as accurately as possible. In a manner of speaking, an estimate of $S(\mathbf{k})$ at a prescribed wave number $\mathbf{k}^*$ corresponds to looking at the spectrum in the vicinity of $\mathbf{k}^*$ through a slit or window with a variable transmission $|G(\mathbf{k})|^2$. The accuracy of the transmitted image as an estimate of $S(\mathbf{k}^*)$ therefore depends very much on the effective width of the window, in that the narrower the width, the less the difference of the observed image from $S(\mathbf{k}^*)$. On the other hand, the width of the window $|G|^2$ is inversely related to the field size L^2; i.e., the larger the field size, the narrower the window width. In this manner, for very large values of L^2, the spectral window $|G|^2$ in effect behaves as a pseudo-delta function with respect to S, and it follows from (7) that as $L^2 \rightarrow \infty$, $\langle \hat{S}(\mathbf{k}^*) \rangle \rightarrow S(\mathbf{k}^*)$ asymptotically. However, the field size L^2 is always limited by various considerations such as data processing costs and the absence of homogeneity in addition to several other physical constraints involved in aerial photographic techniques [*Stilwell*, 1969; *Stilwell and Pilon*, 1974; *Klemas et al.*, 1974]. Therefore the estimator $\hat{S}(\mathbf{k}^*)$ is in general an impression of $S(\mathbf{k}^*)$ contaminated or distorted by the neighboring values over the finite width of the window $|G|^2$. The error due to this distortion is the bias b of the estimator $\hat{S}$ defined at a given $\mathbf{k}$ by

$$b[\hat{S}] = \langle \hat{S} \rangle - S \qquad (14)$$

Another term that is used for the same effect is resolution in analogy with a similar yet different concept in optics associated with the problem of resolving lines in a spectrum. In that instance the spectrum consists of delta functions, and one is mainly interested in the separation or the resolution of the corresponding frequencies. In spectral analysis, however, the main concern is whether the quantity $\hat{S}(\mathbf{k}^*)$ is an accurate estimate of $S(\mathbf{k})$ at the prescribed wave number $\mathbf{k}^*$. The separation of the wave number $\mathbf{k}^*$ itself is irrelevant. Depending on how much or how little distortion is contributed to $\hat{S}(\mathbf{k}^*)$ from the neighboring values of $S(\mathbf{k}^*)$ the estimator can be regarded as a poorly resolved or well resolved image of $S(\mathbf{k}^*)$. Hence in this sense the term resolution error is used as a synonym for the bias (14) [*Blackman and Tukey*, 1959; *Priestley*, 1962].

It is evident that in order to examine specifically the resolution error (14) of an estimator corresponding to a particular window an explicit knowledge of the spectrum $S(\mathbf{k})$ is required. Obviously, this is possible only in an artificially constructed hypothetical case. In empirical spectral analysis, the most we can hope to gather a priori is some rough information on the general characteristics of the spectrum. It is therefore natural to seek a usable expression for the resolution (14) based on an approximate knowledge of the spectrum. Hence if it is assumed that $S(\mathbf{k})$ is smooth relative to the spectral window $|G|^2$ over its effective width, we may expand $S(\mathbf{k}^* - k)$ in (12) as a joint Taylor series in l and n, neglecting the powers higher than the second order. Then recognizing that $|G(\mathbf{k})|^2$ is always an even function, (12) is approximately reduced to

$$\langle \hat{S}(\mathbf{k}^*) \rangle \simeq S(\mathbf{k}^*) + \tfrac{1}{2} B_g^2 \nabla^2 S(\mathbf{k}^*) \quad (15)$$

where

$$\nabla^2 = \partial^2/\partial l^2 + \partial^2/\partial n^2 \quad (16)$$

and the parameter

$$B_g = \left(\int_{\mathbf{k}} l^2 |G|^2 \, d\mathbf{k} \right)^{1/2} = \left(\int_{\mathbf{k}} n^2 |G|^2 \, d\mathbf{k} \right)^{1/2} \quad (17)$$

is a measure of the effective width of $|G|^2$. The Bartlett spectral window (13) taken as a whole has an infinite width. However, since its contribution to (12) is essentially from the major central lobe, we will use the conventional approximation [*Priestley*, 1966; *Polis*, 1974]

$$B_g \simeq \frac{L}{2\pi} \left\{ \int_{-2\pi/L}^{2\pi/L} u^2 \left(\frac{\sin uL/2}{uL/2} \right)^2 du \int_{-\infty}^{\infty} \left(\frac{\sin vL/2}{vL/2} \right)^2 dv \right\}^{1/2} = 2/L \quad (18)$$

The function

$$E_R(\mathbf{k}^*) = |\langle \hat{S}(\mathbf{k}^*) \rangle - S(\mathbf{k}^*)| / S(\mathbf{k}^*) \simeq \tfrac{1}{2} [B_g / B(\mathbf{k}^*)]^2 \quad (19)$$

defines relative resolution error in which

$$B(\mathbf{k}) = |S(\mathbf{k}) / \nabla^2 S(\mathbf{k})|^{1/2} \quad (20)$$

Finally, the maximum relative resolution error over all wave numbers $\mathbf{k}$ is given from (18) and (19) by

$$E_R = \max_{\mathbf{k}} E_R(\mathbf{k}) \simeq \tfrac{1}{2} (B_g / B)^2 \quad (21)$$

where

$$B = \min_{\mathbf{k}} B(\mathbf{k}) \quad (22)$$

An examination of (19) and (21) indicates that the most serious resolution error will occur where the curvature of $S(\mathbf{k})$ is pronounced, as it is at local maxima and minima. In general the exact value of the quantity B is unknown. However, a generalization of the bandwidth concept in the univariate spectral analysis suggests that for a spectrum characterized by a dominant peak, B is related to the physical bandwidth defined in terms of the half-power points of the dominant spectral peak, as is schematically illustrated in Figure 1 [*Priestley*, 1962; *Priestley*, 1966; *Tayfun et al.*, 1975]. Therefore for a sharply peaked spectrum, B is small (narrow band), and for a relatively smooth spectrum, B is correspondingly large (wide band). For spectra that can be characterized as such, (21) implies that the resolution accuracy of the associated sample spectral estimates is $O(B_g^2/B^2)$. In other words, by noting from (18) that $B_g \simeq 2/L$ we can say that a spectral computation in which the field size satisfies the condition $(LB)^2 \gg 2$ will provide useful estimates with high resolution, whereas hardly any meaning can be attached to those for which $(LB)^2 \leq 2$, or equivalently $E_R \geq 100\%$. Thus the condition

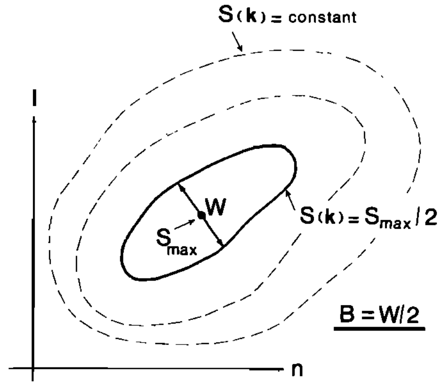

Fig. 1. Contours of $S(\mathbf{k})$ = const and schematic illustration of the spectral bandwidth B.

$$(LB)^2 \geq 2 \quad (23)$$

constitutes a limiting accuracy criterion for spectral computations in homogeneous wave fields and in inhomogeneous regimes as well if the spatial variation of wave characteristics is reasonably slow over the field size.

INHOMOGENEITY AND SMOOTHING ERRORS IN SHALLOW WATER

In shallow water, where wave characteristics change continuously with the varying depth of water, the wave number spectrum has a spatially inhomogeneous nature. For a shallow water wave field in which refraction and shoaling constitute the predominant bottom effects, the nature of this inhomogeneity is well-defined [*Longuet-Higgins*, 1956; *Collins*, 1972]. If $S'(\mathbf{k}') = S'(l', n')$ denotes the homogeneous deepwater spectrum before refraction and $S(\mathbf{k}) = S(l, n)$ denotes the spectrum after refraction in shallow water, then S is related to S' by

$$S(\mathbf{k}) = S'(\mathbf{k}') \quad (24)$$

where the shallow water wave number $k = (l^2 + n^2)^{1/2}$ is a function of the deepwater $k' = (l'^2 + n'^2)^{1/2}$ and the local water depth $D = D(\mathbf{r})$ in the form

$$k \tanh kD = k' \quad (25)$$

The direction θ with respect to the local isobaths obeys Snell's law,

$$k \cos \theta = \text{const} \quad (26)$$

The preceding definitions physically imply that a contour of energy density $S'(\mathbf{k}')$ = const in deep water is transformed into a contour of $S(\mathbf{k})$ = const in shallow water, though the shape and the magnitude of the area enclosed by the contour may change. Therefore the shallow water wave energy $S(\mathbf{k})\,d\mathbf{k}$ associated with the wave numbers in a small region $d\mathbf{k}$ of the local $\mathbf{k}$ space is no longer the same as $S'(\mathbf{k}')\,d\mathbf{k}'$ but is related to it by the transformation

$$S(\mathbf{k})\,d\mathbf{k} = S'(\mathbf{k}')\,J(\mathbf{k}, \mathbf{k}')\,d\mathbf{k}' \quad (27)$$

in which the Jacobian

$$J(\mathbf{k}, \mathbf{k}') = \partial(l, n)/\partial(l', n') \quad (28)$$

represents the local stretching of **k** coordinates. It follows then that the mean square displacement or the total wave energy per unit of horizontal area

$$\langle|\eta(\mathbf{r})|^2\rangle = \int_{\mathbf{k}} S(\mathbf{k})\, d\mathbf{k} = \int_{\mathbf{k}'} S'(\mathbf{k}') J(\mathbf{k}, \mathbf{k}')\, d\mathbf{k}' \quad (29)$$

is space dependent.

Consider now the problem of estimating the spectrum $S(\mathbf{k})$ in shallow water from a sample of finite field size ($L \times L$). As an estimator of $S(\mathbf{k})$ at a prescribed $\mathbf{k}$ we can write, similar to (8),

$$\hat{S}(\mathbf{k}) = \left|\int_{\mathbf{r}} g(\mathbf{r})\eta(\mathbf{r}) \exp(-i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{r}\right|^2 \quad (30)$$

in which it is assumed that the sample $\eta(\mathbf{r})$ is symmetrically located with respect to the point of interest where $D(\mathbf{r}) = \tilde{D}$. An application of Parseval's theorem to (30) yields

$$||\eta||^2 = (2\pi)^2 \int_{\mathbf{r}} |g(\mathbf{r})\eta(\mathbf{r})|^2\, d\mathbf{r} = \int_{\mathbf{k}} \hat{S}(\mathbf{k})\, d\mathbf{k} \quad (31)$$

It is evident that the expected value of this quantity physically corresponds to a weighted integral or a smoothing of the wave energy over the sample space ($L \times L$). This, however, is given from (29) by

$$\langle||\eta||^2\rangle = (2\pi)^2 \int_{\mathbf{r}} \int_{\mathbf{k}} [g(\mathbf{r})]^2 S(\mathbf{k})\, d\mathbf{k}\, d\mathbf{r}$$

$$= \int_{\mathbf{k}} S'(\mathbf{k}')\left\{(2\pi)^2 \int_{\mathbf{r}} [g(\mathbf{r})]^2 J(\mathbf{k}, \mathbf{k}')\, d\mathbf{r}\right\} d\mathbf{k}' \quad (32)$$

A comparison of (29), (31), and (32) indicates that $\hat{S}(\mathbf{k})\, d\mathbf{k}$ is an estimator of

$$[S(\mathbf{k})\, d\mathbf{k}]_{\text{avg}} = S'(\mathbf{k}')\left\{(2\pi)^2 \int_{\mathbf{r}} [g(\mathbf{r})]^2 J(\mathbf{k}, \mathbf{k}')\, d\mathbf{r}\right\} d\mathbf{k}' \quad (33)$$

rather than $S(\mathbf{k})\, d\mathbf{k} = S'(\mathbf{k}')J(\mathbf{k}, \mathbf{k}')\, d\mathbf{k}'$. Since $S'(\mathbf{k}')\, d\mathbf{k}'$ is space independent, $[S(\mathbf{k})\, d\mathbf{k}]_{\text{avg}}$ corresponds to a spatially smoothed image or a weighted average of the true energy $S(\mathbf{k})\, d\mathbf{k}$ over the sample space ($L \times L$). Therefore in addition to the aforementioned errors in resolution a shallow water spectral estimator is subject to inherent errors as a result of this spatial smoothing. In an obvious manner, the function

$$|[S(\mathbf{k})\, d\mathbf{k}]_{\text{avg}} - S(\mathbf{k})\, d\mathbf{k}|/S(\mathbf{k})\, d\mathbf{k} \quad (34)$$

represents a relative measure for such errors.

To obtain a usable approximation to (34) in the following, we consider a shallow water topography with isobaths locally parallel to the shore and with a mean slope $m = -dD/dx$ as shown in Figures 2 and 3 and use the boxcar form (10) in the smoothing function $[g(\mathbf{r})]^2$. Under these conditions, $J(\mathbf{k}, \mathbf{k}') = \partial l/\partial l'$, and (34) is simplified to

$$\left|\left(L\frac{\partial l}{\partial l'}\right)^{-1} \int_{-L/2}^{L/2} \left(\frac{\partial l}{\partial l'}\right) dx - 1\right| \quad (35)$$

Now by using the identity

$$\partial l/\partial l' = \partial l/\partial k\ \partial k/\partial k'\ \partial k'/\partial l' \quad (36)$$

with

$$\partial l/\partial k = k/l = [1 - (k'/k)^2 \cos^2\theta']^{-1/2} \quad (37)$$

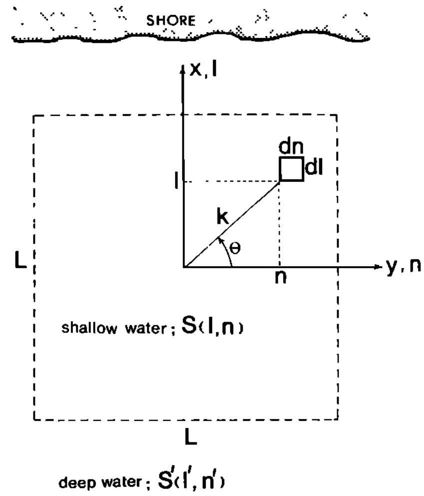

Fig. 2. Definition sketch (plan).

$$\partial k/\partial k' = (\tanh kD + kD \operatorname{sech}^2 kD)^{-1} \quad (38)$$

$$\partial k'/\partial l' = l'/k' = \sin\theta' \quad (39)$$

it is easily verified that the maximum (≥ 0) and the minimum ($=0$) of (35) occur, respectively, with the spectral components propagating normal ($\theta' = \pi/2$) and parallel ($\theta' = 0$) to the isobaths, as is expected. Obviously, the maximum smoothing error that is of interest here is associated with the shallow water waves characterized by small $k'D$ values. Therefore by using (36)–(39) with $\theta' = \pi/2$ and the shallow water wave approximation [*Longuet-Higgins*, 1956]

$$\partial k'/\partial k = \tanh kD + kD \operatorname{sech}^2 kD \simeq 2(k'D)^{1/2} \quad (40)$$

in (35), the maximum spatial smoothing error denoted by E_S is approximately given by

$$E_S \simeq \left|\frac{\tilde{D}^{1/2}}{L} \int_{-L/2}^{L/2} D^{-1/2}\, dx - 1\right| \quad (41)$$

Noting from Figure 3 that $D(x) = -mx + \tilde{D}$, we finally obtain

$$E_S \simeq \left|\frac{2\tilde{D}}{Lm}\left[\left(1 + \frac{Lm}{2\tilde{D}}\right)^{1/2} - \left(1 - \frac{Lm}{2\tilde{D}}\right)^{1/2}\right] - 1\right| \quad (42)$$

Equation (42) indicates that as L increases from zero up to a physically feasible maximum of $2\tilde{D}/m$, the smoothing error E_S increases nonuniformly from zero to 40%, approximately. For $m \to 0$ or $\tilde{D} \to \infty$, corresponding to a spatially homogeneous shallow water field with uniform depth or a homogeneous deepwater wave field, respectively, the error $E_S \to 0$, as is expected.

Optimal Shallow Water Field Size

The preceding discussion suggests that errors in resolution and spatial smoothing should both be considered as criteria for the accuracy of a sample spectral estimate in shallow water.

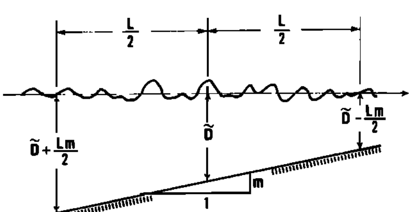

Fig. 3. Definition sketch (profile).

Moreover, it is evident that these criteria have conflicting requirements concerning the sample size for an accurate analysis; i.e., in order to decrease errors in resolution E_R we must take as large a field size as feasible, whereas to reduce the smoothing error E_S we must consider as small a field size as possible. Therefore choosing the sample size L so as to minimize both errors simultaneously suggests itself as an overall optimality criterion. Using (18), (21), and (42), we can write this criterion in terms of a dimensionless field size $L^* = Lm/2\tilde{D}$ and a dimensionless bandwidth $B^* = B\tilde{D}/m$ as

$$E_{\min}(B^*) = \min_{0<L^*<1} [E_S(L^*) + E_R(L^*, B^*)]$$

$$= \min_{0<L^*<1} \{L^{*-1}[(1 + L^*)^{1/2} - (1 - L^*)^{1/2}] - 1 + \tfrac{1}{2}(L^*B^*)^{-2}\} \quad (43)$$

The numerical solution of the preceding equation for the optimal value of L^* and the associated errors $E_{\min} = (E_R + E_S)$, E_R, and E_S are presented in Figure 4 as functions of B^*. The results indicate that for a shallow water wave situation in which $B^* > 10$ the errors $E_R[\simeq(4B^*)^{-1}]$ and $E_S[\simeq(4B^*)^{-1}]$ have equal but negligible significance, implying that an accurate computation of $\tilde{S}(\mathbf{k})$ is very much assured. The corresponding optimal value of the dimensionless field size is approximately given by

$$L_{\text{opt}}^* \simeq (2/B^*)^{1/2} \quad (44)$$

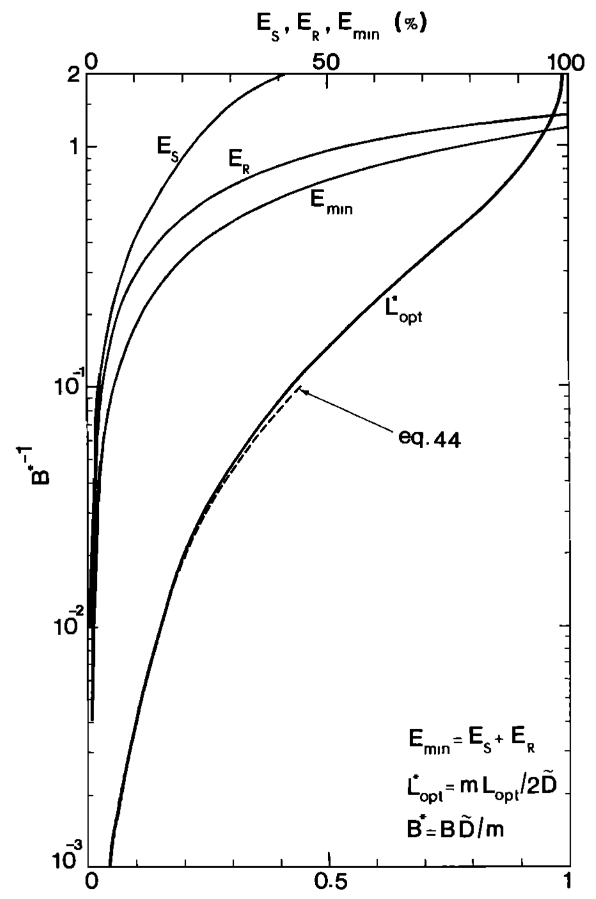

Fig. 4. Dimensionless optimal shallow water field size (L_{opt}^*) and the associated errors in resolution (E_R) and spatial smoothing (E_S) as functions of the dimensionless spectral bandwidth (B^*).

On the other hand, for decreasing values of B^*, both errors become increasingly significant. For $B^* < 1$ and approximately corresponding to the values $L_{\text{opt}}^* > 0.95$, the combined effect ($E_{\min}$) of the errors E_R and E_S becomes so serious that a sample spectral estimator $\tilde{S}(\mathbf{k})$ is no longer meaningful. For a given depth to slope ratio ($\tilde{D}/m$) the preceding results based on the dimensionless bandwidth $B^* = B\tilde{D}/m$ apply to wide band and narrow band spectra characterized with the large and small B values, respectively. In the limiting case, for the bandwidth $B \to 0$, corresponding to a wave field that consists of a regular wave train or a dominant swell, the estimation of the wave number spectrum via spectral analysis is clearly unwarranted.

SUMMARY AND CONCLUSIONS

In homogeneous wave fields, resolution constitutes the basic accuracy criterion for spectral computations based on a finite field size. As such this criterion embodies a condition of limiting accuracy which in principle suggests that the accuracy of spectral estimates increases as the field size is chosen progressively larger than a lower bound that is dependent on the bandwidth measure of the estimated spectrum.

In shallow water wave regimes in which the wave number spectrum has a spatially inhomogeneous nature the accuracy of spectral computations is limited further by an inherent spatial smoothing error. Under conditions in which refraction and shoaling are the primary inhomogeneity effects the smoothing error increases with the increasing field size. The fact that spectral resolution and spatial smoothing place conflicting requirements on the choice of the field size for an accurate analysis prescribes the optimal field size as that which minimizes the combined effects of these errors.

The numerical solutions for the optimal shallow water field size and the associated errors in resolution and spatial smoothing have been presented in a readily usable graph form. The general character of these results indicates that the feasibility and limiting accuracy of shallow water spectral computations depend very much on the bandwidth B of the spectrum in question, the local depth $\tilde{D}$, and the mean bottom slope m. For the values of $(B\tilde{D}/m)$ approximately smaller than 1 the overall magnitude of the error ($\geq 100\%$) associated with the estimates implies that the spectral analysis of the shallow water wave field is hardly meaningful. It should, however, be pointed out that these conclusions are conservatively based on the shallow water peak spectral components. A spectral computation that is feasible and accurate for such major spectral components implicitly warrants better quality estimates for the other components of the spectrum.

Acknowledgment. Financial support for this work was provided by the Geography Programs of the Office of Naval Research under contract N00014-69-A0407 with the University of Delaware.

REFERENCES

Blackman, R. B., and J. W. Tukey, *The Measurement of Power Spectra*, p. 175, Dover, New York, 1959.

Collins, J. I., Prediction of shallow water spectra, *J. Geophys. Res.*, *77*(15), 2693–2707, 1972.

Jenkins, G. M., and G. G. Watts, *Spectral Analysis and Its Applications*, p. 242, Holden-Day, San Francisco, Calif., 1969.

Klemas, V., J. Borchardt, L. Hsu, M. A. Tayfun, and N. Jensen, Photo-optical determination of shallow water wave spectra, paper presented at International Symposium on Ocean Wave Measurement and Analysis, Amer. Soc. of Civil Eng., New Orleans, La., Sept. 9–11, 1974.

Longuet-Higgins, M. S., The refraction of sea waves in shallow water, *J. Fluid Mech.*, *1*, 163–176, 1956.

Phillips, O. M., *The Dynamics of the Upper Ocean*, pp. 72–79, Cambridge University Press, London, 1969.

Polis, D. F., Optimal field size for wave spectra determination, *J. Geophys. Res.*, *79*(18), 2733–2734, 1974.

Priestley, M. B., Basic considerations in the estimation of spectra, *Technometrics*, *4*(4), 551–564, 1962.

Priestley, M. B., Design relations for non-stationary processes, *J. Roy. Statist. Soc., Ser. B*, *28*(1), 228–240, 1966.

Stilwell, D., Jr., Directional energy spectra of the sea from photographs, *J. Geophys. Res.*, *74*(8), 1974–1986, 1969.

Stilwell, D., Jr., and R. O. Pilon, Directional spectra of surface waves from photographs, *J. Geophys. Res.*, *79*(18), 1277–1284, 1974.

Tayfun, M. A., C. Y. Yang, and G. C. Hsiao, Optimal design for wave spectrum estimates, *J. Geophys. Res.*, *80*(15), 1937–1947, 1975.

(Received February 10, 1975;
accepted March 17, 1975.)