Wave Envelope and Related Spectra

By M. Aziz Tayfun[1] and Jen-Men Lo,[2] Members, ASCE

Abstract: Theoretical forms of spectra associated with the odd and even powers of the wave envelope are examined. It is shown that spectral densities for the even powers of the envelope admit exact forms, whereas those for the odd powers can only be expressed in series, involving first-, second-, and higher-order terms. The implications of these results are discussed and contrasted with a number of similar results stated in previous studies. Certain discrepancies and points of concern encountered are clarified. The two leading terms to the envelope spectral density are then examined in detail and compared with simulated data typical of wide- and narrow-banded waves. It is found that the sum of these two leading terms would represent the envelope spectral density with sufficient accuracy for most purposes. As an alternative to the relatively complex analytical forms implied by envelope-related spectra, the possibility of constructing more practical approximations is considered.

Introduction

In the analysis and synthesis of irregular sea waves, the sea surface displacement is treated as a Gaussian random process. The Gaussian model and its physical basis are well established at present, and therefore need no further elaboration here. One of the fundamental concepts that the Gaussian model leads to is the wave-envelope function, which provides a formal basis from which the statistical structure of wavefield parameters, e.g., amplitudes, crest-to-trough heights, and group properties of a wave record, can be deduced. For linear waves, and in certain cases for nonlinear waves, the envelope function has been used extensively and quite profitably to explore the nature of wave amplitudes and heights (see, e.g., Ochi 1982). Its application to the description of group properties of a wave record is somewhat more recent, and perhaps requires further research and development to bear fruit.

The formation and properties of wave groups are of interest and known to play a principal part in a multitude of coastal and ocean engineering problems. These include low-frequency motions and forces associated with moored vessels, the stability of fixed or floating structures, the occurrence of wave breaking, the formation of surf beats, the growth of low frequency energy and motions in the nearshore zone, and possible others (see, e.g., Pinkster 1984; Langley 1984, 1987; Donelan et al. 1972; Goda 1985; Bury 1987). When the sea surface has narrow-banded spectral characteristics, the statistical description of most wave group properties can be derived with relative ease from the probability structure of the associated envelope. This structure is well known in terms of one- and two-time probability density functions and related moments (Rice 1944, 1945; Middleton 1960). Longuet-

[1]Prof., Civ. Engrg., Coll. of Engrg. Petroleum, Kuwait Univ., P.O. Box 5969, 13060 Safat, Kuwait.

[2]Res. Sci., Kuwait Inst. for Scientific Res., P.O. Box 24885, 13109 Safat, Kuwait.

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Higgins (1984) gives a recent survey of the subject, and presents further improvements on the available results. By and large, the quantitative application of wave group statistics to actual problems remains to be shown. Nonetheless, it is evident in theory that the nature of such statistics crucially depends on the oscillatory characteristics of the surface profile and its envelope. These are, of course, described concisely by the corresponding spectral densities or, equivalently, covariance functions.

In contrast with the probabilistic approach, other studies follow somewhat differing lines of thought to describe the wave group phenomenon. For instance, Funke and Mansard (1980) give persuasive arguments in favor of using the smoothed instantaneous wave energy history (SIWEH) and its spectral density, as an alternative to the envelope concept. Bitner-Gregersen and Gran (1983) consider the slowly varying part of squared surface displacements as a still more convenient alternative to the SIWEH approach. The quantity of interest in this case is proportional to the square of the wave envelope function, and so has relevancy to the low-frequency motions and forces associated with moored vessels. Both concepts have physical appeal, and serve to delineate the presence of wave groups in a wave record quite effectively. Further, they can be extended to the more general case of directional seas (see, e.g., Pinkster 1984). Read and Sobey (1987) draw attention to the possibility of employing "phase spectra" of wave records in a search for potentially ordered structure. The aforementioned studies and similar others do not directly attempt to describe or deal with the probabilistic structure of wave groups, but they do rely on and utilize the spectral characteristics of the surface or its envelope as a basis to identify and/or quantify wave groupiness.

The preceding brief review is not intended to be a comprehensive survey of all possible uses or benefits of the envelope concept, but it may serve to establish to some extent its relevance in a variety of engineering problems, particularly, in terms of the spectral characterization of various quantities that functionally depend on the wave envelope. So far, relatively little has been done in this direction. With the possible exception of the spectral density associated with the squared wave envelope, which arise in connection with the low-frequency mooring forces (see, e.g., Pinkster 1974), theoretical forms of envelope-related spectra have not hitherto been examined systematically. In fact, the only other reference to the theoretical form of a spectral density appropriate to the wave envelope itself is by Nolte and Hsu (1972), who suggested that the spectral densities of the envelope and its square differ only by a constant, and are therefore equivalent in a normalized form. The particular form proposed has since then appeared or been utilized in some studies (see, e.g., Goda 1976; Bury 1987), but has also been challenged on the grounds that it predicts a main spectral peak at zero frequency (Funke and Mansard 1980).

The principal goal here is to examine the theoretical forms of spectral densities associated with various powers of the wave envelope, based on the premise that such theoretical forms correspond to the Fourier transforms of respective covariance kernels. The latter are known, but assume a power series form for the odd powers of the envelope. Therefore, the transform relations need to be applied in a termwise fashion to develop a series of approximations to the underlying spectral densities. The nature and implications of these are discussed and contrasted with various other similar or

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related results stated in some of the previous studies aforementioned. Certain discrepancies and points of concern which arise are clarified. In view of the relative complexity of analytical forms implied by envelope-related spectral densities, the possibility of constructing more practical approximations is explored.

Covariance Kernels

Definitions

The surface displacement from the mean sea level is described as a function of time t by the real part of

$$W = \eta + i\hat{\eta} = \sum_{j=1}^{N} c_j \exp\left[i(\omega_j t + \mu_j)\right] \quad (1)$$

where $i = (-1)^{1/2}$; ω_j = frequency in rad/s; and, μ_j = random phases, uniformly distributed over the interval $(0, 2\pi)$. When N is very large, then ω_j's become dense, and

$$c_j \simeq [2S(\omega_j)\Delta\omega_j]^{1/2} \quad (2)$$

where S = spectral density of η; and, $\Delta\omega_j$ = discrete intervals of frequency. The imaginary part $\hat{\eta}$ of Eq. 1 corresponds to the Hilbert transform of η (see, e.g., Tayfun and Lo 1989).

The jth spectral moment is denoted by m_j ($j = 0, 1, 2, \ldots$) so that $m_o = \langle\eta^2\rangle$; $\bar{\omega} = m_1/m_o$ = the central or mean frequency of S; and, $\nu = [(m_o m_2/m_1^2) - 1]^{1/2}$ represents a measure of spectral band width.

The wave envelope or amplitude function is given by the modulus of W, i.e.,

$$A = (\eta^2 + \hat{\eta}^2)^{1/2} \quad (3)$$

The odd and even statistical moments of A are (Papoulis 1965)

$$\langle A^{2n+1}\rangle = \left(\frac{\pi}{2}\right)^{1/2} 1.3 \ldots (2n+1) m_o^{(2n+1)/2} \quad (4)$$

$$\langle A^{2n}\rangle = n!(2m_o)^n \quad (5)$$

where $n = 0, 2, 2, \ldots$.

The joint statistical moments of $A_1 = A(t)$ and $A_2 = A(t + \tau)$ have the form (see, e.g., Middleton 1960)

$$\langle A_1^m A_2^n\rangle = (2m_o)^{m+n/2}\,\Gamma\left(1 + \frac{m}{2}\right)\Gamma\left(1 + \frac{n}{2}\right)F\left(-\frac{m}{2}, -\frac{n}{2}; 1; \kappa^2\right) \quad (6)$$

where $m, n = 1, 2, 3, \ldots$; Γ = gamma function; and F = Gauss hypergeometric series defined by (Abramowitz and Stegun 1965)

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)} \frac{zj}{j!} \quad (7)$$

The argument κ^2 corresponds to the correction coefficient between A_1^2 and A_2^2, and has the form

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$$\kappa^2 = m_o^{-2}(\rho^2 + \lambda^2) \quad \ldots\ldots (8)$$

where

$$\rho(\tau) = \int_0^\infty S(\omega) \cos(\omega - \bar{\omega})\tau d\omega \quad \ldots\ldots (9)$$

$$\lambda(\tau) = \int_0^\infty S(\omega) \sin(\omega - \bar{\omega})\tau d\omega \quad \ldots\ldots (10)$$

Cross-Covariance and Covariance Kernels

The cross-covariance kernel or function of A_1^m and A_2^n is defined by

$$C_{m,n}(\tau) = \langle A_1^m A_2^n \rangle - \langle A^m \rangle \langle A^n \rangle \quad \ldots\ldots (11)$$

Clearly, when $m = n$, the same expression gives the covariance kernal of A_1^n and A_2^n. It can be verified from Eqs. 4 through 7 that

$$C_{m,n}(\tau) = C_{n,m}(\tau) = C_{m,n}(-\tau) \quad \ldots\ldots (12)$$

The covariance kernel for the odd powers of A_1 and A_2 follows from the substitution of Eqs. 4 through 7 into Eq. 11 as

$$C_{n,n}(\tau) = \frac{1}{\pi^2}(2m_o)^n \Gamma^4\left(1 + \frac{n}{2}\right) \sum_{j=1}^{\infty} \left[\frac{\Gamma\left(j - \frac{n}{2}\right)}{(j+1)!}\right]^2 \kappa^{2j} \quad \ldots\ldots (13)$$

where $n = 1, 3, 5, \ldots$. In particular, setting $n = 1$ in the preceding expression will yield the covariance kernel of the envelope itself in the form

$$C_{1,1}(\tau) = \frac{1}{8} m_o \pi \kappa^2 \left(1 + \frac{1}{16}\kappa^2 + \frac{1}{64}\kappa^4 + \ldots\right) \quad \ldots\ldots (14)$$

The Gauss hypergeometric series of Eq. 6 reduces to a polynomial of degree j in κ^2 when $m/2$ or $n/2$ is equal to $j = 0, 1, 2, \ldots$ (see, e.g., Abramowitz and Stegun 1965). Consequently, the cross-covariance kernel for the even powers of A_1 and A_2 can be expressed in the form

$$C_{2m,2n}(\tau) = m!n!(2m_o)^{m+n} \sum_{j=0}^{m-1} \binom{m}{j}\binom{n}{m-j}(\kappa^2)^{m-j} \quad \ldots\ldots (15)$$

where $m = 1, 2, 3, \ldots$, and $n > m - 1$. When $m = 1$, one obtains

$$C_{2,2n}(\tau) = (2m_o)^{n+1} n\Gamma(n+1)\kappa^2 \quad (n = 1, 2, 3, \ldots) \quad \ldots\ldots (16)$$

In particular, setting $n = 1$ in this expression gives the covariance kernal of A_1^2 and A_2^2 as

$$C_{2,2}(\tau) = 4m_o^2\kappa^2 \quad \ldots\ldots (17)$$

In all the preceding results, it is understood that $\kappa = \kappa(\tau)$. What is more significant, however, is the observation that the covariance kernels for the even powers of A_1 and A_2 reduce to polynomials in κ^2, whereas those for the odd powers are in the form of a power series in κ^2. As $\tau \to 0$, $\kappa(0) \to 1$, and

$$F\left(-\frac{m}{2}, -\frac{n}{2}; 1; 1\right) = \frac{\Gamma\left(1 + \frac{m+n}{2}\right)}{\Gamma\left(1 + \frac{m}{2}\right)\Gamma\left(1 + \frac{n}{2}\right)} \quad \ldots\ldots (18)$$

in Eq. 6. Now $C_{1,1}$ can be used as a case in point to show that, as $\tau \to 0$, Eq. 14 converges to the variance of A exactly, i.e.,

$$\text{Var}(A) = \langle A^2 \rangle - \langle A \rangle^2 = \frac{1}{2} m_o(4 - \pi) \quad \ldots\ldots (19)$$

However, if the leading term of Eq. 14 is considered only, the variance is given by $m_o\pi/8$ as compared to Eq. 19. Evidently, the first term accounts for nearly 91.50% of the total variance. The inclusion of the second term makes this figure 97.22%, and that of third 98.64%. Therefore, the total contribution of all other terms not shown in Eq. 14 to the variance of A is less than 1.4%.

Transform Relations and Spectra

By virtue of Eq. 12, the (cross) spectral density, say, $\Psi_{m,n}$ associated with A_1^m and A_2^n corresponds to the Fourier cosine transform of $C_{m,n}$, i.e.,

$$\Psi_{m,n}(\omega) = \frac{2}{\pi} \int_0^\infty C_{m,n}(\tau) \cos \omega\tau \, d\tau \quad \ldots\ldots (20)$$

such that

$$C_{m,n}(\tau) = \int_0^\infty \Psi_{m,n}(\omega) \cos \omega\tau \, d\omega \quad \ldots\ldots (21)$$

The even moments of $\psi_{m,n}$ are

$$M_{2j}^{m,n} = \int_0^\infty \omega^{2j} \Psi_{m,n}(\omega) \, d\omega = (-1)^j C_{m,n}^{(2j)}|_{\tau = 0} \quad \ldots\ldots (22)$$

where $j = 0, 1, 2, \ldots$, and the last equality follows from Eq. 21, with $C_{m,n}^{(2j)}$ representing the $2j$th derivative with respect to τ.

It is evident that the explicit evaluation of $\Psi_{m,n}$ requires the cosine transform of $C_{m,n}$. The latter is either a polynomial or a power series of κ^2. Since a typical term in either case is proportional to κ^{2n} ($n = 1, 2, 3, \ldots$), let the corresponding transform be defined by

$$G_{2n}(\omega) = \frac{2}{\pi} \int_0^\infty \kappa^{2n}(\tau) \cos \omega\tau \, d\tau \quad \ldots\ldots (23)$$

It is shown in Appendix I that the preceeding can be rewritten in the recursive form

$$G_{2n} = \frac{1}{2} \int_0^\infty G_2(u)[G_{2n-2}(|u - \omega|) + G_{2n-2}(u + \omega)] \, du \quad \ldots\ldots (24)$$

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where $n = 2, 3, \ldots$, and

$$G_2(\omega) = \frac{2}{m_o^2} \int_0^\infty S(u)S(u + \omega)du \quad \ldots\ldots (25)$$

It is now a matter of straightforward substitution to develop explicit expression for $\Psi_{m,n}$. For instance, the simplest case of interest here would follow from Eqs. 16 and 25 as

$$\Psi_{2,2n}(\omega) = 2^{n+2} m_o^{n-1} n\Gamma(n + 1) \int_0^\infty S(u)S(u + \omega)du \quad \ldots\ldots (26)$$

where $n = 1, 2, 3, \ldots$. When $n = 1$, this gives the spectral density associated with A^2 as

$$\Psi_{2,2}(\omega) = 8 \int_0^\infty S(u)S(u + \omega)du \quad \ldots\ldots (27)$$

This is entirely consistent with previous results (Pinkster 1984). In contrast, the spectral density $\Psi_{n,n}$ ($n = 1, 3, 5, \ldots$) associated with the odd powers of A can only be expressed in the form

$$\Psi_{n,n}(\omega) = \sum_{j=1}^{\infty} {}_n\Psi_j(\omega) \quad \ldots\ldots (28)$$

where ${}_n\Psi_j$ ($j = 1, 2, 3, \ldots$) denotes the cosine transform of the jth of Eq. 13, which is proportional to κ^{2j}. As a result, ${}_n\Psi_j$ can be expressed in terms of ${}_n\Psi_{j-1}$ and ${}_n\Psi_1$ through the recursive use of Eqs. 24 and 25. For example, the spectral density $\Psi_{1,1}$ of A itself becomes

$$\Psi_{1,1}(\omega) = {}_1\Psi_1(\omega) + {}_1\Psi_2(\omega) + {}_1\Psi_3(\omega) + \ldots \quad \ldots\ldots (29)$$

where ${}_1\Psi_1$, ${}_1\Psi_2$, etc. correspond to the transforms associated with the first, second, etc. terms of Eq. 14, so that

$${}_1\Psi_1 = \frac{\pi}{4m_o} \int_0^\infty S(u)S(u + \omega)du \quad \ldots\ldots (30)$$

$${}_1\Psi_2 = \frac{1}{4\pi m_o} \int_0^\infty {}_1\Psi_1(u)[{}_1\Psi_1(|u - \omega|) + {}_1\Psi_1(u + \omega)]du \quad \ldots\ldots (31)$$

$${}_1\Psi_3 = \frac{1}{\pi m_o} \int_0^\infty {}_1\Psi_1(u)[{}_1\Psi_2(|u - \omega|) + {}_1\Psi_2(u + \omega)]du \quad \ldots\ldots (32)$$

Clearly, ${}_1\Psi_1$ is a convolution type functional of the surface spectral density S. The remaining terms ${}_1\Psi_2$, ${}_1\Psi_3$, etc. correspond in turn to further convolutions of ${}_1\Psi_1$. In practice, all of these need to be computed numerically. Note, however, that the contribution of each term to the total spectral content diminishes fairly rapidly as its order increases. To be precise, e.g.,

$$\int_0^\infty {}_1\Psi_1 d\omega = \frac{\pi}{8} m_o \quad \ldots\ldots (33)$$

$$\int_0^\infty {}_1\Psi_2 d\omega = \frac{\pi}{128} m_o \quad \text{(34)}$$

$$\int_0^\infty {}_1\Psi_3 d\omega = \frac{\pi}{512} m_o \quad \text{(35)}$$

$$\int_0^\infty ({}_1\Psi_4 + {}_1\Psi_5 + \ldots) d\omega = 2m_o\left(1 - \frac{325\pi}{1{,}024}\right) \quad \text{(36)}$$

These represent respectively 91.50%, 5.72%, 1.42%, and 1.36% of the total spectral content, as was previously mentioned. This observation coupled with the nature of convolution operations, which tend to spread the spectral mass over all frequencies, suggests that it may in essence be sufficient to confine attention to the first few terms of Eqs. 28 and 29.

Comparisons with Previous Results

Nolte and Hsu (1972) suggested that the spectral densities of A and A^2 differ by a constant, and are therefore equivalent in the normalized form

$$\frac{\int_0^\infty S(u)S(u+\omega)du}{\int_0^\infty S^2(u)du} \quad \text{(37)}$$

In view of Eqs. 27 and 30, it is clear that the suggested form is identical with $\Psi_{2,2}$ and ${}_1\Psi_1$, normalized by the maxima at $\omega = 0$. However, ${}_1\Psi_1$ represents only the first approximation to $\Psi_{1,1}$. Therefore, Eq. 37 is strictly appropriate to A^2, not to A.

Funke and Mansard (1980) challenged the validity of Eq. 37 on the grounds that it predicts a maximum at $\omega = 0$. They prefer to consider the SIWEH spectral density, and observe that highly periodic wave group phenomena lead to a peak in that density at non-zero frequencies. They argue further that the SIWEH spectral density is also affected by the phase differences $\mu_{j+n} - \mu_j$ between adjacent frequency components ω_{j+n} and ω_j in Eq. 1, suggesting how the "phase spectrum" of wave trains may be related to the grouping phenomeon. The SIWEH is defined by

$$E(t) = \int_{-\infty}^{\infty} h(\tau)\eta^2(t+\tau)d\tau \quad \text{(38)}$$

where h represents a window or weighting function such that $h \geq 0$, $h(\tau) = h(-\tau)$, and

$$\int_{-\infty}^{\infty} h(\tau)d\tau = 1 \quad \text{(39)}$$

Clearly, $\langle E \rangle = m_o$. If η is zero-mean Gaussian and admits a representation of the same form as the real part of Eq. 1, then it can be shown after some algebra that the covariance kernel of E is given by

$$C_E(\tau) = \langle E(t)E(t+\tau)\rangle - m_o^2$$

$$= 2m_o^2 \int_{-\infty}^{\infty} h(u)h(v)R^2(\tau + v - u)\,du\,dv \quad \ldots\ldots (40)$$

where

$$R(\tau) = \langle \eta(t)\eta(t+\tau)\rangle = \int_0^{\infty} S(\omega)\cos\omega\tau\, d\omega \quad \ldots\ldots (41)$$

The SIWEH spectral density follows from the Fourier transform of Eq. 40, after some straightforward algebra, as

$$\Psi_E(\omega) = |H(\omega)|^2 \int_0^{\infty} S(u)[S(|u-\omega|) + S(u+\omega)]\,du \quad \ldots\ldots (42)$$

where $\omega \geq 0$, and

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau)\exp(-i\omega\tau)\,d\tau \quad \ldots\ldots (43)$$

It may be of interest to note that the integral part of Eq. 42 is simply twice the spectral density of η^2. What is more significant, however, is the fact that, since $H(\omega), \leq H(0)$, Ψ_E has a maximum at $\omega = 0$ exactly, and is not affected by any phase differences of any sort.

Bitner-Gregersen and Gran (1983) considered the slowly varying part of η^2 as an alternative to the SIWEH. In terms of A and the phase ϕ of W, $\eta = A\cos\phi$, and so

$$\eta^2 = (A\cos\phi)^2 = \frac{1}{2}A^2 - \frac{1}{2}A^2\cos 2\phi \quad \ldots\ldots (44)$$

Hence, the slowly varying part is represented by the first term on the right, i.e., $A^2/2$, whose covariance kernel follows from Eq. 17 as $m_o^2\kappa^2(\tau)$. The corresponding spectral density has the form

$$2\int_0^{\infty} S(u)S(u+\omega)\,du \quad \ldots\ldots (45)$$

which is identical with the result previously given by Pinkster (1984). Interestingly, the same concept can easily be extended to nonlinear waves under certain simplifying assumptions. Specifically, if the free surface is long-crested and has narrow-band spectral characteristics, then (Tayfun 1986; Tayfun and Lo 1989)

$$\eta \simeq A\cos\phi + \frac{1}{2}A^2\bar{k}\cos 2\phi \quad \ldots\ldots (46)$$

to second-order or approximation, with $\bar{k} = \bar{\omega}^2/g$ correct to $O(\nu)$. The slowly varying part of η^2 in this case is

$$\frac{1}{2}A^2\left(1 + \frac{1}{4}A^2\bar{k}^2\right) \quad \ldots\ldots (47)$$

with the covariance kernel

$$\frac{1}{4}\left[C_{2,2}(\tau) + \frac{1}{2}\bar{k}^2 C_{2,4}(\tau) + \frac{1}{16}\bar{k}^4 C_{4,4}(\tau)\right] \quad (48)$$

The latter can be rewritten, utilizing Eqs. 15 through 17, as

$$m_o^2\kappa^2\left[1 + 2\alpha^2 + \alpha^4\left(1 + \frac{1}{4}\kappa^2\right)\right] \quad (49)$$

where

$$\alpha = A_{\text{rms}}\bar{k} = (2m_o)^{1/2}\bar{k} \quad (50)$$

and represents a measure of wave steepness or nonlinearity. Typically, $0 < \alpha < 0.3$ (Huang et al. 1981a; Shum and Melville 1984). It can easily be shown that, to $O(\nu)$, α corresponds to the root-mean-square wave steepness. In the present context $\alpha \simeq \nu$, so that terms of $O(\alpha^4)$ are negligible. On this basis, the spectral density associated with the slowly varying part of η^2 is given by

$$2(1 + 2\alpha^2)\int_0^\infty S(u)S(u + \omega)du \quad (51)$$

Evidently, the effect of second-order nonlinearities is reflected in the constant factor $(1 + 2\alpha^2)$ which serves to amplify each spectral amplitude by the fractional amount $2\alpha^2$ as a nonlinear correction.

Envelope Spectral Density

It is worthwhile to explore further the nature of the spectral density $\Psi_{1,1}$ associated with A itself corresponding to some typical sea surface conditions. This is most conveniently done by specifying first the density S, from which ${}_1\Psi_1$ and thereby ${}_1\Psi_2$, and other terms of $\Psi_{1,1}$ would follow by numerical integration. The spectral density S is then utilized to generate via fast Fourier transform (FFT) time series of η, $\hat{\eta}$, and therefore a sample of the envelope function A from Eq. 3. The latter is processed further to remove its mean, and the usual FFT procedures are then employed to construct estimates of the underlying envelope spectral density $\Psi_{1,1}$. These can subsequently be compared with the theoretical quantities, e.g., ${}_1\Psi_1$ and ${}_1\Psi_1 + {}_1\Psi_2$, to which the following analysis will be confined.

Spectral Density of Surface Displacements

The spectral density of the sea surface displacements is assumed to be of the form

$$S(\omega) = \beta\left(\frac{\omega_p}{\omega}\right)^\gamma \exp\left[-\frac{\gamma}{4}\left(\frac{\omega_p}{\omega}\right)^4\right] \quad (52)$$

where ω_p = frequency of the spectral peak; γ = parameter which controls the spectral shape, particularly the bandwidth ν; and, β = parameter with dimensions m^2s. The above form is identical to the Wallops spectral density (Huang et al. 1981b) for which the parameter β depends on the equilibrium-

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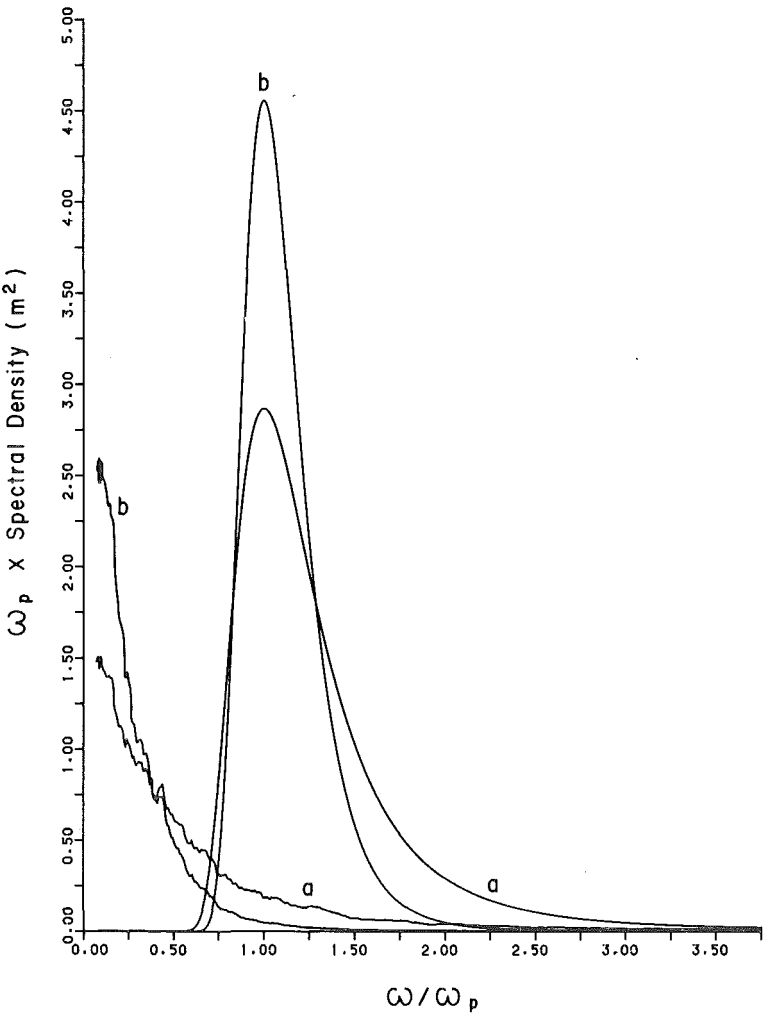

FIG. 1. Spectra of Surface Displacements and Corresponding Estimates of Envelope Spectra for (*a*) $\nu \simeq 0.4$, and (*b*) $\nu \simeq 0.2$

range constant, gravitational acceleration, and ω_p.

Two cases are considered: For the first case $\gamma = 5$, and $\beta = 10\ m^2s$; and, for the second $\gamma = 10$, and $\beta \simeq 56\ m^2s$. Correspondingly, Eq. 52 implies a constant value of $m_o \simeq 2\ m^2$ for both cases, but $\nu \simeq 0.4$ for the first case, and $\omega \simeq 0.2$ for the second, typical of (wide band) wind seas and (narrow band) swell conditions, respectively. The frequencies are scaled with respect to ω_p. Therefore, the results would be independent of ω_p. In both cases, simulated series of η, $\hat{\eta}$, and A comprise $2^{13} = 8{,}192$ points sampled at a rate of 0.5 s. The high-frequency cut-off of S is taken as $\omega/\omega_p = 2\pi$, ensuring that better than 99.9% of the total spectral area is accounted for in both cases.

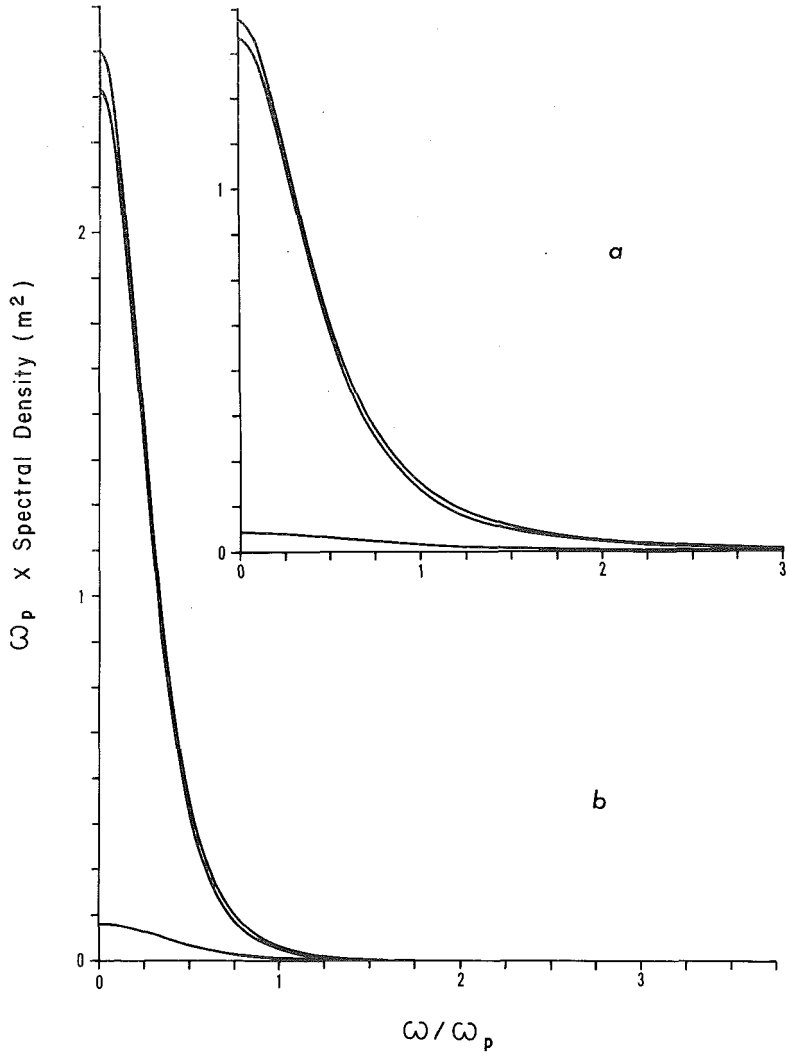

FIG. 2. Theoretical Forms of $_1\Psi_1 + {}_1\Psi_2$ (top), $_1\Psi_1$ (middle), and $_1\Psi_2$ (bottom) for (*a*) $\nu \simeq 0.4$, and (*b*)$\nu \simeq 0.2$

Results

The theoretical surface spectra and the corresponding estimates of envelope spectra derived from simulations are shown in Fig. 1. The estimates represent smooth values with 200 degrees of freedom, obtained by block-averaging each raw estimate symmetrically over the neighboring values. It is evident that, the narrower the surface spectral density, the more concentrated and peaked around the origin the envelope spectral density becomes.

The theoretical predictions of $_1\Psi_1$, $_1\Psi_2$, and $_1\Psi_1 + {}_1\Psi_2$ are shown in Fig. 2. The nature of these results seems self-evident and perhaps needs no elab-

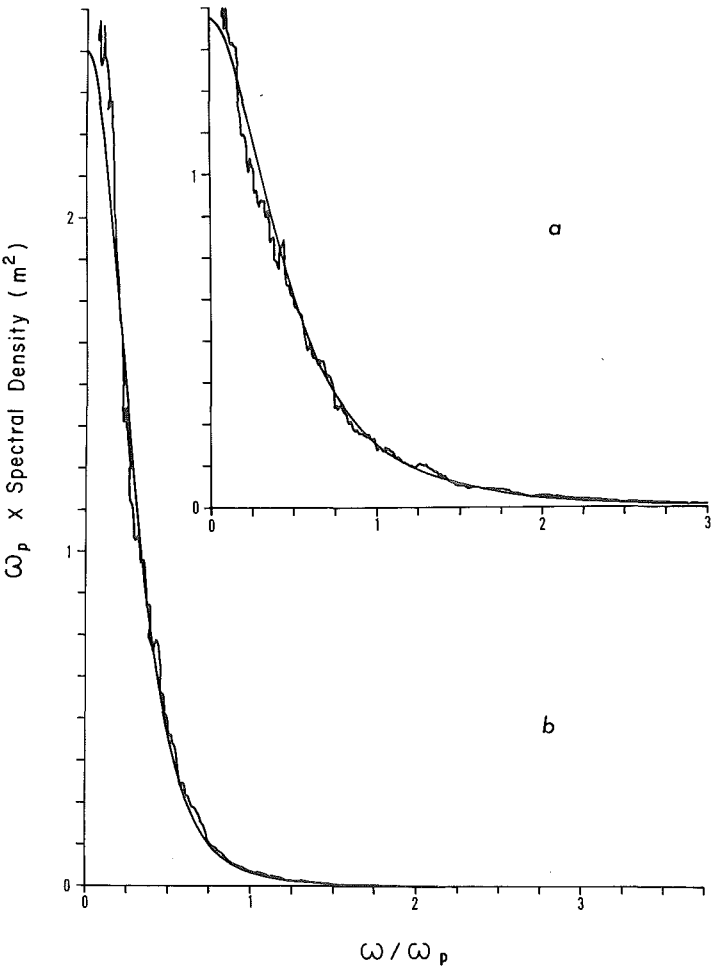

FIG. 3. Comparison of ${}_1\Psi_1 + {}_1\Psi_2$ with Estimates of Envelope Spectra for (*a*) $\nu \simeq$ 0.4, and (*b*) $\nu \simeq$ 0.2

oration. It should be emphasized that the spectral peak occurs at $\omega = 0$, and that the second approximation tends to sharpen this peak more as the band width of the surface density becomes narrower.

The comparisons between the estimates of envelope spectral density $\Psi_{1,1}$ derived from numerical simulations and the corresponding theoretical sums ${}_1\Psi_1 + {}_1\Psi_2$ are given in Fig. 3. In both cases, the estimates compare favorably with the theoretical predictions. It appears then that the envelope spectral density $\Psi_{1,1}$ is well approximated by the sum of the two leading terms, ${}_1\Psi_1$ and ${}_1\Psi_2$.

APPROXIMATIONS

The relatively complex analytic forms associated with envelope-related

spectra suggest that it may be worthwhile to consider the possibility of approximating them with more practical expressions. One such possibility presents itself immediately in view of the basic nature of envelope-related spectra. To be specific, any spectarl density associated with the wave envelope involves the typical term

$$\int_0^\infty S(u)S(u+\omega)du \quad \ldots\ldots (53)$$

either by itselt as the principal functional form or as the leading approximation. Therefore, it can be argued that spectral densities such as $\Psi_{n,n}$ ($n = 1, 3, 5, \ldots$) and $\Psi_{2n,2n}$ ($n = 2, 3, 4, \ldots$) associated with the odd and certain even powers of A can be approximated with sufficient accuracy for practical purposes by the above functional form, provided that it is normalized to have the same total area as the spectral density under consideration does. The total area under $\Psi_{n,n}$ ($n = 1, 2, 3, \ldots$) is $M_o^{n,n} = C_{n,n}(0)$ from Eq. 22, whereas the area under the approximating form can be verified to be $m_o^2/2$. Therefore,

$$\tilde{\Psi}_{n,n}(\omega) = 2m_o^{-2}M_o^{n,n}\int_0^\infty S(u)S(u+\omega)du \quad \ldots\ldots (54)$$

corresponds to the proposed approximation, where $n = 1, 2, 3, \ldots$; and,

$$M_o^{n,n} = \left[n! - \Gamma^2\left(\frac{n}{2}+1\right)\right](2m_o)^n \quad \ldots\ldots (55)$$

The 2nd-order moments of $\Psi_{n,n}$ and $\tilde{\Psi}_{n,n}$ will follow from Eqs. 22 and 54 as (see, Appendix II)

$$\tilde{M}_2^{n,n} = 2\bar{\omega}^2\nu^2 M_o^{n,n} \quad \ldots\ldots (56)$$

$$M_2^{n,n} = \frac{1}{2}\bar{\omega}^2\nu^2 n\Gamma(n+1)(2m_o)^n \quad \ldots\ldots (57)$$

where $n = 1, 2, 3, \ldots$ as before. Note that

$$r_n = \frac{M_2^{n,n}}{\tilde{M}_2^{n,n}} = \frac{n\Gamma(n+1)}{4\left[n! - \Gamma^2\left(\frac{n}{2}+1\right)\right]} \quad \ldots\ldots (58)$$

which can be used as a rough figure of merit to judge how $\tilde{\Psi}_{n,n}$ fares in approximating $\Psi_{n,n}$. When $n = 2$, $\tilde{\Psi}_{2,2} = \Psi_{2,2}$ identically so that $r_2 = 1$. For any other $n \neq 2$, $r_n > 1$, implying that the approximate form tends to underestimate the theoretical moments $M_j^{n,n}$ for $j = 1, 2, 3, \ldots$. This constitutes a point of some concern in theory, but can perhaps be tolerated in practical applications involving relatively small values of n.

As an explicit demonstration of the proposed approximations, consider the spectral density $\tilde{\Psi}_{1,1}$ of A. In this case, $n = 1$, and so $r_1 \simeq 1.16$, as compared to, e.g., $r_3 \simeq 1.06$, and $r_4 \simeq 1.09$ corresponding to $n = 3$ and 4, respectively. The approximating spectral density is

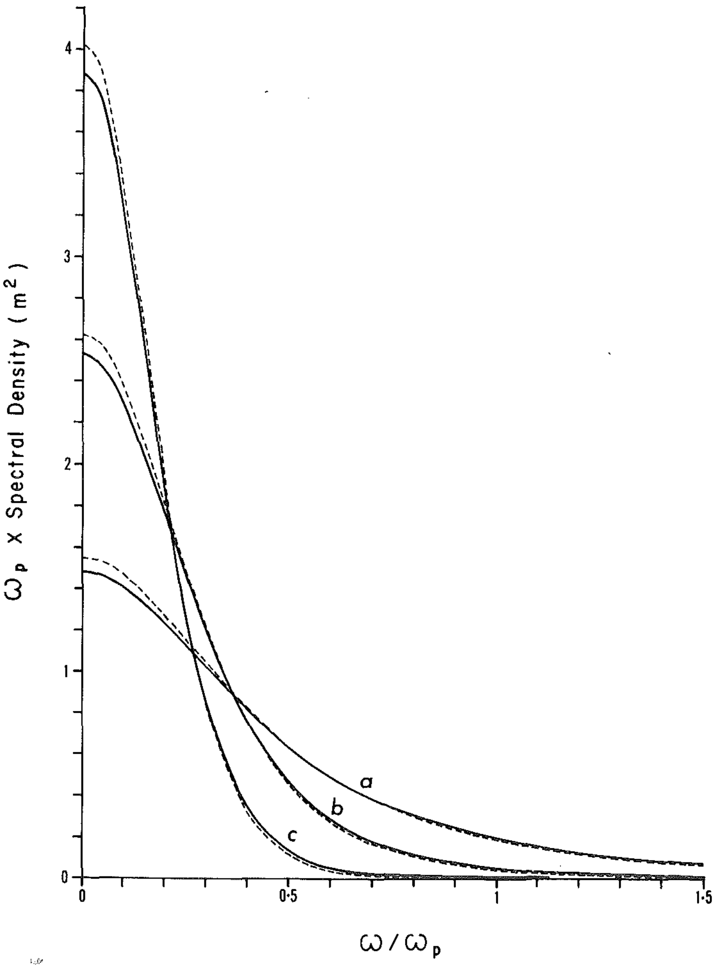

FIG. 4. Comparison of Approximation $\bar{\Psi}_{1,1}$ (dashed) with Theory $\Psi_{1,1} \simeq {}_1\Psi_1 + {}_1\Psi_2 + {}_1\Psi_3$ (continuous) for (*a*) $\nu \simeq 0.4$, (*b*) $\nu \simeq 0.2$, and (*c*) $\nu \simeq = 0.12$

$$\bar{\Psi}_{1,1}(\omega) = \frac{4 - \pi}{m_o} \int_0^{\infty} S(u)S(u + \omega)du \quad \text{.............................} \quad (59)$$

Again assume that S has the Wallops form given by Eq. 52, with the same high-frequency cut-off $\omega/\omega_p = 2\pi$. $\bar{\Psi}_{1,1}$ follows from Eq. 59 by numerical integration, and will be compared this time with the sum of the first three terms of $\Psi_{1,1}$, i.e., ${}_1\Psi_1 + {}_1\Psi_2 + {}_1\Psi_3$. Considering three cases corresponding to $\gamma = 5$, 10, and 20 ($\nu \simeq 0.4$, 0.2, and 0.12) with $m_o \simeq 2\ m^2$ as before, the comparison between $\bar{\Psi}_{1,1}$ and the respective theoretical sums will be as shown in Fig. 4. It is apparent that the approximations tend to overpredict the spectral amplitudes near the origin to some extent, and accordingly display a slight but persistent deficiency toward the high-frequency tail.

Summary and Conclusions

The theoretical spectral densities associated with various powers of the wave envelope were examined. It was shown that the spectral densities for the even powers of the envelope admit exact forms, whereas those for the odd powers lead to a series of terms each of which needs to be calculated numerically from a convolution type functional of the surface spectral density. Then, the spectral density of the envelope itself was examined in detail, showing that the first term is the most significant, and accounts for nearly 91.50% of the whole spectral content; the inclusion of the second term improves this further by 5.72%; and, all the remaining terms contribute less than 2.80%. Therefore, it can be argued that the representation of the envelope spectral density by the sum of the two leading terms, with the appropriate normalization to 100%, should be sufficiently accurate for most purposes.

The basic nature of envelope-related spectra also implies that some of these theoretical forms can functionally be approximated with sufficient accuracy for practical purposes by the leading term only, provided that the latter is properly normalized to have the same theoretical zeroth-order moment. An explicit application of this concept to the envelope spectral density itself gave reasonable results, which tend to suggest that the proposed approximation might be all that is needed in practice.

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Appendix I. Cosine Transform of κ^{2n}

By virtue of Eq. 23

$$\kappa^{2n}(\tau) = \int_0^\infty G_{2n}(\omega) \cos \omega\tau d\omega \quad \ldots\ldots (60)$$

In particular,

$$\kappa^2(\tau) = \int_0^\infty G_2(\omega) \cos \omega\tau d\omega \quad \ldots\ldots (61)$$

Setting $\kappa^{2n} = \kappa^{2n-2}\,\kappa^2$, and replacing κ^2 with Eq. 61, Eq. 23 can be rewritten as

$$G_{2n} = \frac{2}{\pi} \int_0^\infty \int_0^\infty G_2(u)\kappa^{2n-2}(\tau) \cos u\tau \cos \omega\tau du\, d\tau \quad \ldots\ldots (62)$$

where

$$\cos u\tau \cos \omega\tau = \frac{1}{2} \cos |u - \omega|\tau + \frac{1}{2} \cos (u + \omega)\tau \quad \ldots\ldots (63)$$

which is substituted in Eq. 62 to obtain Eq. 24.

Clearly, if G_2 is known, then G_{2n} ($n = 2, 3, \ldots$) will follow from the recursive use of Eq. 24. To obtain G_2, Eqs. 8 through 10 are combined to give

$$m_o^2\kappa^2(\tau) = \int_0^\infty \int_0^\infty S(u)S(v)\cos(u - v)\tau\, du\, dv \quad (64)$$

Let $\omega = u - v$ in integrating with respect to u to obtain

$$m_o^2\kappa^2(\tau) = \int_{v=0}^\infty \int_{\omega=-v}^\infty S(v)S(v + \omega)\cos\omega\tau\, d\omega\, dv = I_1 + I_2 \quad (65)$$

where

$$I_1 = \int_0^\infty \int_0^\infty S(u)S(u + \omega)\cos\omega\tau\, du\, d\omega \quad (66)$$

$$I_2 = \int_{v=0}^\infty \int_{\omega=-v}^\infty S(v)S(v + \omega)\cos\omega\tau\, d\omega\, dv \quad (67)$$

First, set $\omega = -\omega$ in I_2, then interchange the order of integration, and finally replace $v - \omega$ in the resulting expression with u to verify that $I_1 = I_2$. Therefore, $m_o^2\kappa^2 = 2I_1$, and so

$$\kappa^2(\tau) = \int_0^\infty G_2(\omega)\cos\omega\tau\, d\omega \quad (68)$$

where $G_2(\omega)$ is given by Eq. 25.

Appendix II. Second Moments of $\tilde{\Psi}_{n,n}$ and $\Psi_{n,n}$

By definition and using Eq. 54, the second moment of $\tilde{\Psi}_{n,n}$ is

$$\tilde{M}_2^{n,n} = 2m_o^{-2}M_o^{n,n}\int_0^\infty \int_0^\infty \omega^2 S(u)S(u + \omega)\, du\, d\omega \quad (69)$$

Let $u + \omega = v$, and $d\omega = dv$ to rewrite Eq. 69 as

$$\tilde{M}_2^{n,n} = 2m_o^{-2}M_o^{n,n}\int_0^\infty \int_{v=u}^\infty (v - u)^2 S(u)S(v)\, du\, dv$$

$$= m_o^{-2}M_o^{n,n}\int_0^\infty \int_0^\infty (v - u)^2 S(u)S(v)\, du\, dv \quad (70)$$

Now, note that $v - u = (v - \bar{\omega}) - (u - \bar{\omega})$, which can be substituted in the double integral above to show that it equals $2(m_o\bar{\omega}\nu)^2$ identically, and therefore Eq. 55 follows immediately.

The second moment of $\Psi_{n,n}$ follows from Eq. 22 as

$$M_2^{n,n} = \int_0^\infty \omega^2\Psi_{n,n}\, d\omega = -C_{n,n}^{(2)}(0) \quad (71)$$

where

$$C^{(2)}_{n,n}(\tau) = (2m_o)^n \Gamma^2\left(1 + \frac{n}{2}\right) \frac{d^2}{d\tau^2} F\left(-\frac{n}{2}, -\frac{n}{2}; 1; \kappa^2\right) \quad \ldots\ldots (72)$$

In general (Abramowitz and Stegun 1965

$$\frac{d^2}{d\tau^2} F(a, b; c; z) = \frac{a(a+1)b(b+1)}{c(c+1)} F(a+2, b+2; c+2; z)\left(\frac{dz}{d\tau}\right)^2$$

$$+ \frac{ab}{c} F(a+1, b+1; c+1; z) \frac{d^2 z}{d\tau^2} \quad \ldots\ldots (73)$$

As $\tau \to 0$, $\kappa^2(0) = 1$, and it can be verified from Eqs. 8 through 10 that

$$\frac{d\kappa^2(0)}{d\tau} = 0 \quad \ldots\ldots (74)$$

$$\frac{d^2\kappa^2(0)}{d\tau^2} = -2\bar{\omega}^2\nu^2 \quad \ldots\ldots (75)$$

The substitution of Eqs. 72 through 75 in Eqs. 71 now gives Eq. 57 for $M_2^{n,n}$.

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APPENDIX IV. NOTATION

The following symbols are used in this paper:

$A(t)$ = wave envelope or amplitude function;
$C_{m,n}(\tau)$ = cross-covariance kernel of $A^m(t)$ and $A^n(t + \tau)$;
c_j = amplitude of jth-frequency component;
F = $F(a, b; c; z)$, Gauss hypergeometric function with arguments a, b, c, and z;
$G_{2n}(\omega)$ = cosine transform of $\kappa^{2n}(\tau)$;
$H(\omega)$ = Fourier transform of $h(\tau)$;
$h(\tau)$ = window or weighting function;
$\bar{k}$ = spectral mean wave number;
$M_j^{m,n}$ = jth moment of $\Psi_{m,n}$ defined by Eq. 22;
m_j = jth moment of spectral density of sea surface displacement about origin;
$O(\ )$ = of order of ( );
$R(\tau)$ = correlation fucntion of sea surface displacements in time;
r_n = ratio defined by Eq. 58;
$S(\omega)$ = spectral density of sea surface displacements;
t = time;
u, v = dummy variables of integration;
W = complex envelope defined by Eq. 1;
α = steepness parameter;
β = one of parameters for Wallops spectral form;
Γ = gamma function;
γ = one of parameters for Wallops spectral form;
$\eta, \hat{\eta}$ = sea surface displacement, and its Hilbert transform, respectively;
$\kappa^2(\tau)$ = autocorrelation coefficient between $A^2(t)$ and $A^2(t + \tau)$;

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$\lambda(\tau)$ = function defined by Eq. 10;
μ_j = random phase of jth frequency component;
ν = spectral bandwidth parameter;
ϕ = argument of W (total wave phase function);
$\rho(\tau)$ = function defined by Eq. 9;
τ = time lag;
$\Psi_{m,n}(\omega)$ = (cross) spectral density associated with $A^m(t)$ and $A^n(t + \tau)$;
$\tilde{\Psi}_{n,n}(\omega)$ = approximation for $\Psi_{n,n}(\omega)$ ($n = 1, 2, 3, \ldots$);
${}_n\Psi_j(\omega)$ = jth term of $\Psi_{n,n}(\omega)$ ($n = 1, 3, 5, \ldots$);
$\Psi_E(\omega)$ = SIWEH spectral density;
ω = wave radian frequency;
ω_p = frequency of spectral maximum;
$\bar{\omega}$ = spectral mean frequency; and
$\Delta\omega_j$ = frequency increment associated with jth-frequency component.

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