High-Wave-Number/Frequency Attenuation of Wind-Wave Spectra

By M. Aziz Tayfun,[1] Member, ASCE

Abstract: Properties of surface-elevation spectra representative of nonlinear deep-water wind waves are examined theoretically. For a wave field characterized by second-order nonlinearities, expressions describing the spatial covariance and the directional wave-number spectrum of the surface geometry are derived. The nature of these quantities are then examined with emphasis on the high-wave-number attenuation of spectral amplitudes. It is found that, if the spectrum of the first-order linear wave field decays as k^{-p} toward the high-wave-number extreme, then the spectrum of the nonlinear wave field must decay as k^{-p+2}. This condition coupled with the saturation/equilibrium range concepts is shown to necessitate the existence of certain upper-limit asymptotes to the high-frequency attenuation of linear-wave spectra. Practical implications of this result are explored with reference to low-pass filtering of wave records, and the representation of Gaussian sea waves based on various empirical and/or theoretical forms of wind-wave spectra.

Introduction

An accurate description of the surface geometry and its statistics in a developed wind-wave field requires knowledge of the surface-elevation spectra and associated ordinary moments. The development and the eventual shape of the spectrum under a given set of wind and fetch conditions depend on various processes such as wave-wave interactions, energy input from the wind, wave-breaking, and so on. Despite the progress made in recent decades, the current knowledge of these processes is perhaps still insufficient to predict the precise functional form that a wave spectrum will assume under a given set of circumstances. The fact that there exists not one unique form but a variety of different representations of wind-wave spectra, theoretical or empirical, is apparently a result of this lack of complete understanding. Thus, such representations are normally derived from dimensional analyses and/or similarity laws, and involve certain parameters, e.g. gravity, wind velocity, wind fetch, and so on. These parameters and thereby the explicit functional form are obtained by fitting an "ensemble" average of observed spectra, typically over a restricted range of frequencies. A case in point is the JONSWAP spectrum (Hasselman et al. 1973), which is a generalized form of the Pierson-Moskowitz (P-M) spectrum, modified by a peak-enhancement factor. The enhanced peak appears to be typical of wind-wave spectra under fetch-limited growth conditions. Hasselman et al. (1973) attempt to explain the presence and evolution of such pronounced peaks as a self-stabilizing feature of the resonant wave-wave interaction process. However, the more accurate later calculations of Longuet-Higgins (1976) and Fox (1976) contradict this, and in fact suggest that such interactions should help attenuate rather than augment the spectral peak, making the spectrum broader.

[1]Prof., Civ. Engrg. Dept., Coll. of Engrg. and Petroleum, Kuwait Univ., P.O. Box 5969, 13060 Safat, Kuwait.

Note. Discussion open until October 1, 1990. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on October 15, 1989. This paper is part of the ***Journal of Waterway, Port, Coastal, and Ocean Engineering***, Vol. 116, No. 3, May/June, 1990. ©ASCE, ISSN 0733-950X/90/0003-0381/$1.00 + $.15 per page. Paper No. 24708.

Nevertheless, one frequently observed consistent property of wind-wave spectra is that their growth over frequencies and wave numbers sufficiently larger than those of the spectral peaks follows certain upper-limit asymptotes, which fall off as $\omega^{-(4\sim5)}$ and $k^{-(7/2\sim4)}$ in the gravity-wave range [see for example Forristall (1981), Kitaigorodski (1962, 1983), and Phillips (1958, 1985)]. Phillips (1958) originally explained these asymptotes as saturation limits above which any excursion of the spectral density is relieved by breaking. The additional theoretical insight developed since 1960, and the trend of the recent more accurate measurements forced Phillips (1985) to reformulate these upper-limit asymptotes to be consistent with a statistical equilibrium state determined by the balance among the various energy-transfer and dissipation processes operating in the gravity-wave range. The limiting shape of the sea surface implied by these types of saturation/equilibrium asymptotes is characterized by the occurrence of sharp wave crests that sporadically break and form patches of foam or whitecaps. Evidently, the presence of sharp crests renders the surface gradient discontinuous. Thus, the attenuation of the two-dimensional wave-number spectra in the form of $k^{-(7/2\sim4)}$ at large k is entirely consistent with this reasoning (Phillips 1977; Glazman 1986). Not surprisingly, a variety of empirical or theoretical representations of wind-wave spectra, viz, the P-M, JONSWAP, and other forms of one-dimensional frequency spectra have been formulated to fall off as $\omega^{-(4\sim5)}$ at large ω.

The physics of gravity waves are rather difficult because of the nonlinear kinematic and dynamic conditions to be satisfied on the moving air-water interface, which is unknown a priori. The action of turbulent wind and the resulting surface stresses complicate the situation further. Thus, the mathematical models describing the propagation of deep-water wind waves are based on the perturbational solutions of the equations of free wave motion (Tick 1959; Hasselman 1962; Longuet-Higgins 1963; Weber and Barrick 1977). So, surface stresses are ignored, and the displacements from the mean surface level are described in the form of a series. The leading term of the series, viz, the first-order solution, provides the bulk of the description, which is assumed to be linear Gaussian. Nonlinearities are viewed as small departures or perturbations from the Gaussian state. In theory, all nonlinear terms depend on, and thus must be solved in, terms of the first-order solution. But, the algebra is so prohibitive that explicit expressions exist only for the second-order solution, representing the nonresonant forced waves generated by the first-order field. The resonant wave-wave interactions arise in the higher orders, and are time-dependent. Fortunately, the temporal and spatial scales over which the resonant interactions and other energy-transfer/dissipation processes assume importance are sufficiently large [see for example Phillips (1977), Barrick and Weber (1977)] so that the perturbational solutions to the second-order can provide a valid physical description for wind-generated waves under conditions of statistical equilibrium, for example, in the sense of Phillips (1985) and/or locally, namely, over spatial and time scales relatively small compared to those for which the aforementioned processes become dynamically significant.

Oceanic observations of a variety of surface-wave properties, inclusive of those gathered under hurricane conditions, bear out the preceding viewpoint remarkably well. For instance, the departure of certain key surface statistics from the Gaussian norms can be predicted quite accurately by way of the second-order solutions (Longuet-Higgins 1963; Tayfun and Lo 1989, 1990). Observed spectra often display secondary peaks at frequencies and wave numbers coincident with the second harmonics of those of the primary peaks. The second-order perturbational calculations by Tick (1959), Barrick and Weber (1977), and others also explain such features well, provided that the spectrum representative of the first-order wavefield has a sufficiently sharp maximum.

One aspect that would be of interest but has so far remained essentially unexplored is the high-wave-number/frequency behavior of wind-wave spectra within the context of second-order pertubational theory. That is the principal purpose of the present study, and requires first the derivation of theoretical expressions describing the spatial covariance and thereby the two-dimensional wave-number spectrum of the surface geometry. Then, the attenuation of second-order spectral amplitudes toward the high-wave-number extreme must be determined, assuming initially that the wave-number spectrum relevant to the first-order Gaussian sea surface falls off as k^{-p}. Once the latter objective is achieved, a simple reverse argument based on Phillips' (1958, 1985) saturation- and equilibrium-range concepts can be used to establish the appropriate upper-limit asymptotes that first-order spectra should follow. The eventual goal would be to explore the practical implications of such asymptotes, for example, with reference to the low-pass filtering of wave records and to the representation of Gaussian sea waves based on certain empirical and/or theoretical forms of wind-wave spectra.

Perturbational Model

The surface elevation from the mean level is given by

$$\eta = \eta(\mathbf{x}, t) = \eta_1 + \eta_2 \quad (1)$$

where $\mathbf{x} = (x_1, x_2)$ = horizontal coordinates fixed on the mean surface level; t = time; η_1 and η_2 = the first- and second-order solutions, respectively. The first-order solution is linear Gaussian, and so can be described in the form

$$\eta_1 = \lim_{N \to \infty} \sum_{n=1}^{N} a_n \cos \chi_n \quad (2)$$

where $\chi_n = \mathbf{k}_n \cdot \mathbf{x} - \omega_n t + \mu_n$; $\mathbf{k}_n$ = horizontal wave-number vector with the modulus $k_n = |\mathbf{k}_n|$; ω_n = frequency in rad/s; μ_n = random phases, uniformly distributed over the interval $(0, 2\pi)$; and a_n = amplitude of the nth spectral component. Evidently

$$\omega_n^2 = g k_n \quad (3)$$

where g = gravitational acceleration.

The amplitude a_n can be related to various forms of the surface-elevation spectrum. For example, in the simplest case when η_1 is observed at fixed $\mathbf{x}$ as a function of t only

$$\frac{1}{2} a_n^2 = S_1(\omega_n) \Delta\omega_n \quad (4)$$

J. Waterway, Port, Coastal, Ocean Eng. 1990.116:381-398.

where $\Delta\omega_n$ = discrete intervals of frequency such that $\Delta\omega_n \rightarrow d\omega$ as $N \rightarrow \infty$; and S_1 = one-dimensional frequency spectrum (density). If, on the other hand, t is fixed while η_1 is viewed as a function of $\mathbf{x}$, then

$$\frac{1}{2} a_n^2 = \psi_1(\mathbf{k}_n)\Delta\mathbf{k}_n \quad (5)$$

where $\Delta\mathbf{k}_n$ = discrete elements of area over the horizontal wave-number plane such that $\Delta\mathbf{k}_n \rightarrow d\mathbf{k} = dk_1 dk_2$ as $N \rightarrow \infty$; $\mathbf{k} = (k_1, k_2)$ with k_1 and k_2 designating the components of $\mathbf{k}$ in the x_1 and x_2 directions, respectively; and, ψ_1 = two-dimensional wave-number spectrum (density). It is often necessary to employ a polar description for the wave-number plane, in which case

$$\psi_1(\mathbf{k})d\mathbf{k} = \psi_1(\mathbf{k})k\,dk\,d\theta \quad (6)$$

with $k_1 = k \cos \theta$; $k_2 = k \sin \theta$; and $\theta = \tan^{-1}(k_2/k_1)$ measured positive counterclockwise from the x_1-axis. Finally

$$S_1(\omega) = k \frac{dk}{d\omega} \int \psi_1(\mathbf{k})d\theta \quad (7)$$

The second-order solution is given by [see, for example, Longuet-Higgins (1963), Tayfun and Lo (1990)]

$$\eta_2 = \eta_2^+ + \eta_2^- \quad (8)$$

where

$$\eta_2^{\pm} = \lim_{N\rightarrow\infty} \frac{1}{4} \sum_{m=1}^{N} \sum_{n=1}^{N} a_m a_n K_{m,n}^{\pm} \cos(\chi_m \pm \chi_n) \quad (9)$$

$$K_{m,n}^{\pm} = K^{\pm}(\mathbf{k}_m, \mathbf{k}_n) = (k_m k_n)^{-1/2}[2B_{m,n}^{\pm} - (\mathbf{k}_m \cdot \mathbf{k}_n \mp k_m k_n)] + (k_m + k_n) \quad (10)$$

$$B_{m,n}^{\pm} = \frac{(k_m^{1/2} \pm k_n^{1/2})^2(\mathbf{k}_m \cdot \mathbf{k}_n \mp k_m k_n)}{(k_m^{1/2} \pm k_n^{1/2})^2 - |\mathbf{k}_m \pm \mathbf{k}_n|} \quad (11)$$

Covariance Kernels and Spectra

Directional Waves

Assume that η represents a statistically homogeneous wave field generated by steady winds blowing in the direction of the positive x_1-axis. The spatial covariance of η at points separated by a distance $\mathbf{r}$ is defined by

$$Z(\mathbf{r}) = \langle \eta(\mathbf{x} + \mathbf{r}, t)\eta(\mathbf{x}, t) \rangle \quad (12)$$

where $\langle\ \rangle$ = expected value operator. Equivalently

$$Z(\mathbf{r}) = \int \psi(\mathbf{k}) \cos(\mathbf{k} \cdot \mathbf{r})d\mathbf{k} \quad (13)$$

where ψ = two-dimensional wave-number spectrum of η. Since η_1, η_2^+, and η_2^- are mutually uncorrelated, Z can be rewritten

$$Z(\mathbf{r}) = Z_1 + Z_2 = Z_1 + Z_2^+ + Z_2^- \quad (14)$$

where Z_1, Z_2, and $Z_2^{\pm}$ = covariance of η_1, η_2, and $\eta_2^{\pm}$, respectively. Accordingly

$$\psi(\mathbf{k}) = \psi_1 + \psi_2 = \psi_1 + \psi_2^+ + \psi_2^- \quad (15)$$

Evidently, $\psi(\mathbf{k}) = 0$ unless $\mathbf{k}$ lies in the right half-plane in accord with the assumed wind direction.

For the first-order field

$$Z_1 = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2} a_n^2 \cos(\mathbf{k}_n \cdot \mathbf{r}) = \int \psi_1(\mathbf{k}) \cos(\mathbf{k}\cdot\mathbf{r}) d\mathbf{k} \quad (16)$$

which is consistent with the definition of ψ_1, as was previously given by Eq. 5.

The covariances of $\eta_2^\pm$ have the form

$$Z_2^\pm = \lim_{N\to\infty} \sum_{m=1}^{N} \sum_{n=1}^{N} \left(\frac{1}{4} a_m a_n K_{m,n}^\pm\right)^2 \cos[(\mathbf{k}_m \pm \mathbf{k}_n)\cdot\mathbf{r}] \quad (17)$$

The derivation of $\psi_2^\pm$ from Eq. 17 is fairly straightforward, but requires some care. For example, $k_n > 0$ for any n since $\langle\eta\rangle = 0$, so that setting $\mathbf{k}_j = \mathbf{k}_m + \mathbf{k}_n$ allows Z_2^+ to be rewritten

$$Z_2^+ = \lim_{N\to\infty} \sum_{j=1}^{N} \left[\sum_{1\le n<j} \left(\frac{1}{4} a_n a_{j-n} K_{j-n,n}^+\right)^2\right] \cos(\mathbf{k}_j\cdot\mathbf{r}) \quad (18)$$

A comparison of the latter to Eq. 16 immediately leads to

$$\psi_2^+ = \frac{1}{4} \int_I [K^+(\mathbf{k} - \mathbf{k}', \mathbf{k}')]^2 \psi_1(\mathbf{k} - \mathbf{k}')\psi_1(\mathbf{k}') d\mathbf{k}' \quad (19)$$

where it is to be understood that $\mathbf{k}_j \to \mathbf{k} = (k_1, k_2)$ and $\mathbf{k}_n \to \mathbf{k}' = (k_1', k_2')$ as $N \to \infty$; and $I = \{\mathbf{k}': 0 < k_1' \le k_1, \infty < k_2' < \infty\}$.

In considering Z_2^-, let $\mathbf{k}_j = \mathbf{k}_m - \mathbf{k}_n$ or $\mathbf{k}_j = \mathbf{k}_n - \mathbf{k}_m$ depending on whether $k_m \cos\theta_m > k_n \cos\theta_n$ or $k_m \cos\theta_m \le k_n \cos\theta_n$, respectively. Thus

$$Z_2^- = \lim_{N\to\infty} \sum_{j=1}^{N} \left[\sum_{n\le N-j} \left(\frac{1}{4} a_n a_{j+n} K_{j+n,n}^-\right)^2 + \sum_{n>j} \left(\frac{1}{4} a_n a_{n-j} K_{n-j,n}^-\right)^2\right] \cos(\mathbf{k}_j\cdot\mathbf{r}) \quad (20)$$

Now, set $i = n - j \ge 1$ so that $n = i + j$ in the second sum within the square brackets, and then make use of the symmetry $K_{i,i+j}^- = K_{i+j,i}^-$ to show that the two sums within the square brackets are identical. Consequently

$$Z_2^- = \lim_{N\to\infty} \sum_{j=1}^{N} \left[2 \sum_{n\le N-j} \left(\frac{1}{4} a_n a_{j+n} K_{j+n,n}^-\right)^2\right] \cos(\mathbf{k}_j\cdot\mathbf{r}) \quad (21)$$

and so

$$\psi_2^- = \frac{1}{2} \int [K^-(\mathbf{k} + \mathbf{k}', \mathbf{k}')]^2 \psi_1(\mathbf{k} + \mathbf{k}')\psi_1(\mathbf{k}') d\mathbf{k}' \quad (22)$$

where the region of integration with respect to $\mathbf{k}' = (k_1', k_2')$ is the right half-plane on which $0 < k_1' < \infty$ and $-\infty < k_2' < \infty$.

J. Waterway, Port, Coastal, Ocean Eng. 1990.116:381-398.

Bounds

Following Longuet-Higgins (1963), it is convenient to define

$$\rho = \frac{k_m + k_n}{2(k_m k_n)^{1/2}} \quad (23a)$$

$$c = \cos\gamma = \cos(\theta_m - \theta_n) \quad (23b)$$

such that $\rho \geq 1$ in general, and $-\pi < \gamma < \pi$ in the present case. Thus, Eq. 10 can be rewritten

$$\frac{K^{\pm}_{m,n}}{2(k_m k_n)^{1/2}} = f_{\pm} = \rho \pm \frac{1}{2}(1 \mp c) \mp \frac{(\rho \pm 1)(1 \mp c)}{(\rho \pm 1) - \left[\rho^2 - \frac{1}{2}(1 \mp c)\right]^{1/2}} \quad (24)$$

It is noticed that $f_+ = \rho$ and $f_- = -(\rho^2 - 1)^{1/2}$ as $\gamma \rightarrow 0$. These particular limits together with straightforward numerical evaluations of $f^2_{\pm}$ for varied values of γ will yield

$$\max_{-\pi<\gamma<\pi} (K^{\pm}_{m,n})^2 = (K^{\pm}_{m,n})^2|_{\gamma=0} = (k_m \pm k_n)^2 \quad (25)$$

and so

$$[K^{\pm}(\mathbf{k} \mp \mathbf{k}', \mathbf{k}')]^2 \leq k^2 \quad (26)$$

Further, by way of Schwarz's inequality

$$\int \psi_1(\mathbf{k} \pm \mathbf{k}')\psi_1(\mathbf{k}')d\mathbf{k}' \leq \int \psi_1^2(\mathbf{k})d\mathbf{k} \quad (27)$$

Thus, the preceding upper bounds are substituted into Eqs. 19, 22, and $\psi_2 = \psi_2^+ + \psi_2^-$ to obtain

$$0 \leq \psi_2(\mathbf{k}) \leq \frac{3}{4} k^2 \int \psi_1^2(\mathbf{k})d\mathbf{k} \quad (28)$$

The variance of η is given by

$$\langle \eta^2 \rangle = Z(0) = \int \psi(\mathbf{k})d\mathbf{k} \quad (29)$$

Equivalently

$$\langle \eta^2 \rangle = \langle \eta_1^2 \rangle + \langle \eta_2^2 \rangle = \langle \eta_1^2 \rangle + \langle (\eta_2^+)^2 \rangle + \langle (\eta_2^-)^2 \rangle \quad (30)$$

By virtue of Eqs. 17 and 25

$$0 < Z_2^{\pm}(0) \leq \lim_{N\rightarrow\infty} \sum_{m=1}^{N} \sum_{n=1}^{N} \left(\frac{1}{4} a_m a_n |k_m \pm k_n|\right)^2 \quad (31)$$

Utilizing the dispersion relation $gk_j = \omega_j^2$ for $j = m$ and n, and expanding the right-hand side of Eq. 31 will give

$$\langle (\eta_2^{\pm})^2 \rangle = Z_2^{\pm}(0) \leq \frac{1}{2g^2}(m_0 m_4 \pm m_2^2) \quad (32)$$

where m_j = the jth ordinary moment of S_1. Thus

$$0 < \langle \eta_2^2 \rangle \leq \frac{m_0 m_4}{g^2} \quad (33)$$

This result and $\langle \eta_1^2 \rangle = m_0$ can now be substituted into Eq. 30 to conclude that

$$m_0 < \langle\eta^2\rangle \le m_0\left(1 + \frac{m_4}{g^2}\right) \quad (34)$$

which will be of practical use later.

Unidirectional Waves

When all the spectral components propagate, say, in the positive $x = x_1$ direction, then $k = k_1$ and $K^{\pm} = \pm k$ in Eqs. 19 and 22. Thus, $\psi_2^{\pm}$ reduces to

$$\psi_2^{+}(k) = \frac{k^2}{4}\int_0^k \psi_1(k - k')\psi_1(k')dk' \quad (35a)$$

$$\psi_2^{-}(k) = \frac{k^2}{2}\int_0^\infty \psi_1(k + k')\psi_1(k')dk' \quad (35b)$$

Further

$$\int_0^k \psi_1(k - k')\psi_1(k')dk' = \int_0^\infty [\psi_1(|k - k'|) - \psi_1(k + k')]\psi_1(k')dk' \quad (36)$$

which can be used together with Eqs. 35*a* and 35*b* to obtain

$$\psi_2(k) = \frac{k^2}{4}\int_0^\infty [\psi_1(k + k') + \psi_1(|k - k'|)]\psi_1(k')dk' \quad (37)$$

In the present case

$$Z_1(r) = \int_0^\infty \psi_1(k)\cos krdk \quad (38)$$

One can now make use of the convolution properties of cosine transforms [see, for example, Hildebrand (1976)] to show that the integral part of Eq. 37 is the cosine transform of $2Z_1^2$. Clearly, then, ψ_2 is the cosine transform of $Z_2 = -(1/2)d^2(Z_1^2)/dr^2$. Thus, the spatial covariance of η is given by

$$Z(r) = Z_1 - \frac{1}{2}\frac{d^2}{dr^2}(Z_1^2) \quad (39)$$

Finally, the variance of η follows from a combination of Eqs. 3, 7, 38, and 39 as

$$\langle\eta^2\rangle = Z(0) = m_0\left(1 + \frac{m_4}{g^2}\right) \quad (40)$$

which is identical with the upper bound in the directional case (see Eq. 32).

The frequency spectrum is of the form

$$S(\omega) = S_1 + S_2 = S_1 + S_2^{+} + S_2^{-} \quad (41)$$

where S_1, S_2, and $S_2^{\pm}$ = frequency spectra associated with η_1, η_2, and $\eta_2^{\pm}$, respectively. In contrast to the linear case, where S_1 and ψ_1 are related to one another in a unique way by virtue of the dispersion equation $\omega^2 = gk$, there is no such relationship that can be used to relate $S_2^{\pm}$ to $\psi_2^{\pm}$. Thus, $S_2^{\pm}$

J. Waterway, Port, Coastal, Ocean Eng. 1990.116:381-398.

must be derived from the cosine transforms of the temporal covariances of $\eta_2^{\pm}$. That has already been done elsewhere [see for example Tick (1959) and Tayfun (1986)] and so need not be repeated here. For $\omega \geq 0$, the second-order spectra are given by (Tayfun 1986)

$$S_2^+ = \frac{1}{4g^2}\int_0^{\omega} [(\omega')^2 + (\omega - \omega')^2]^2 S_1(\omega - \omega')S_1(\omega')d\omega' \quad \text{(42a)}$$

$$S_2^- = \frac{1}{2g^2}\int_0^{\infty} [(\omega')^2 - (\omega + \omega')^2]^2 S_1(\omega + \omega')S_1(\omega')d\omega' \quad \text{(42b)}$$

The temporal covariance of η at an arbitrary time lag τ is

$$R(\tau) = \langle \eta(x, t + \tau)\eta(x, t)\rangle = \int_0^{\infty} S(\omega)\cos\omega\tau d\omega \quad \text{(43)}$$

It is understood that $R = R_1 + R_2$, $R_2 = R_2^+ + R_2^-$, and that the temporal covariances R_1 and $R_2^{\pm}$ are given by the cosine transforms of S_1 and $S_2^{\pm}$, respectively. Now, let

$$\hat{R}_1(\tau) = \int_0^{\infty} S_1(\omega)\sin\omega\tau d\omega \quad \text{(44)}$$

and apply the convolution properties associated with the sine and cosine transforms (Hildebrand 1976) to Eqs. 42*a* and 42*b* to obtain

$$R(\tau) = R_1 + \frac{1}{g^2}\left[R_1\frac{d^4}{d\tau^4}R_1 - \left(\frac{d^2}{d\tau^2}\hat{R}_1\right)^2\right] \quad \text{(45)}$$

This is the correct form of a relation given by Tick (1959) some years ago. Notice that $R(0)$ leads to Eq. 40 exactly, as is to be expected.

High-Wave-Number/Frequency Attenuation

Preliminaries

In view of Eq. 28, assume that

$$\int \psi_1^2(\mathbf{k})d\mathbf{k} < \infty \quad \text{(46)}$$

It can then be shown that for an arbitrary small $\delta > 0$, there exists a wave number k_0 such that

$$\int_{\bar{J}} \psi_1^2 d\mathbf{k} = \int \psi_1^2 d\mathbf{k} - \int_J \psi_1^2 d\mathbf{k} \leq \delta \quad \text{(47)}$$

where $J = \{\mathbf{k}: 0 < k \leq k_0, |\theta| < \pi/2\}$ and $\bar{J} = \{\mathbf{k}: k > k_0, |\theta| < \pi/2\}$. To be more specific, let

$$\psi_1(\mathbf{k}) = \beta s(\cos\theta)k^{-p} \qquad (k >> k_*) \quad \text{(48)}$$

where β = a dimensional constant; k_* = wave number of the spectrum peak; and $s(\ )$ = a directional spreading function. For $p = 7/2$, ψ_1 reduces to Phillips' (1985) equilibrium-range spectrum, for which β is proportional to the wind-friction speed. The substitution of Eq. 48 into the left-hand side of Eq. 47 gives

$$k_0^{2(p-1)} \geq \frac{\beta^2}{2(p-1)\delta} \int_{-\pi/2}^{\pi/2} s^2 d\theta \quad (49)$$

Thus, provided that $p > 1$, it would be sufficient to choose k_0 so as to satisfy Eq. 49 for a given δ.

When $\mathbf{k}' \in J$ and $\mathbf{k} \in \bar{J}$ so that $k > k_0 > k'$ invariably

$$|\mathbf{k} \pm \mathbf{k}'| = [k^2 \pm 2kk' \cos(\theta - \theta') + (k')^2]^{1/2} = k\left[1 \pm O\left(\frac{k'}{k}\right)\right] \quad (50)$$

where $O(\ )$ = of order of ( ). Similarly, the cosine of the angle between $\mathbf{k} \pm \mathbf{k}'$ and $\mathbf{k}'$ is given by

$$c^{\pm} = \frac{(\mathbf{k} \pm \mathbf{k}')\cdot\mathbf{k}'}{|\mathbf{k} \pm \mathbf{k}'|k'} = \cos(\theta - \theta')\left[1 \pm O\left(\frac{k'}{k}\right)\right] \quad (51)$$

Further, if k_m and k_n are replaced respectively with $|\mathbf{k} \pm \mathbf{k}'|$ and k' in Eq. 23*a*, then

$$\rho = \frac{1}{2}\left(\frac{k'}{k}\right)^{1/2}\left[1 \pm O\left(\frac{k'}{k}\right)\right] \quad (52)$$

Thus, it can be verified, after some algebra, from Eqs. 24, 51, and 52 that

$$[K^{\pm}(\mathbf{k} \pm \mathbf{k}', \mathbf{k}')]^2 = k^2 \cos^2(\theta - \theta')\left[1 \pm O\left(\frac{k'}{k}\right)^{1/2}\right] \quad (53)$$

If Eq. 48 is valid, then

$$\psi_1(\mathbf{k} \pm \mathbf{k}') = \beta s(\cos\phi^{\pm})|\mathbf{k} \pm \mathbf{k}'|^{-p} \quad (54)$$

where it is understood that $p > 1$, $|\mathbf{k} \pm \mathbf{k}'| >> k_*$, and $\phi^{\pm}$ = the direction of $\mathbf{k} \pm \mathbf{k}'$. Evidently,

$$\cos\phi^{\pm} = \frac{k\cos\theta \pm k'\cos\theta'}{|\mathbf{k} \pm \mathbf{k}'|} = \cos\theta\left[1 \pm O\left(\frac{k'}{k}\right)\right] \quad (55)$$

The substitution of Eqs. 50 and 55 into Eq. 54 gives

$$\psi_1(\mathbf{k} \pm \mathbf{k}') = \beta s(\cos\theta)k^{-p}\left[1 \pm O\left(\frac{k'}{k}\right)\right] \quad (56)$$

Result

Examine now $\psi_2 = \psi_2^+ + \psi_2^-$ for $k > k_0 >> k_*$. By virtue of Eqs. 53 and 56

$$\psi_2 = \frac{3}{4}\beta s(\cos\theta)k^{-p+2}\int_{\mathbf{k}'\in J}\cos^2(\theta - \theta')\psi_1(\mathbf{k}')\left[1 \pm O\left(\frac{k'}{k}\right)^{1/2}\right]d\mathbf{k}' \quad (57)$$

Thus, if ψ_1 attenuates as k^{-p}, then ψ_2 and so ψ must attenuate as k^{-p+2}.

Now, let ψ stand for the total spectrum actually observed to that it describes the joint contribution of the linear and nonlinear spectral components

J. Waterway, Port, Coastal, Ocean Eng. 1990.116:381-398.

of *all* orders. Evidently, if ψ follows an upper-limit asymptote that attenuates as k^{-p}, then ψ_2 must fall off at least just as fast. Accordingly, the first-order spectrum ψ_1 must attenuate at least as fast as $k^{-(p+2)}$. This in turn would immediately imply, by virtue of Eq. 7, that the first-order frequency spectrum S_1 also must attenuate at least as fast as $\omega^{-(2p+1)}$ for $\omega >> \omega_* = (gk_*)^{1/2}$. In particular, if the wave number and frequency spectra representative of a developed wind-wave field follow the saturation (equilibrium) type upper limits $k^{-4}(k^{-7/2})$ and $\omega^{-5}(\omega^{-4})$, then the corresponding first-order spectra must fall off *at least* as fast as $k^{-6}(k^{-11/2})$ and $\omega^{-9}(\omega^{-8})$, respectively.

Example

As a simple illustrative case, consider a unidirectional wave field and assume that the first-order spectrum is given, in a dimensionless and normalized form, by

$$\tilde{S}_1(u) = \frac{\omega_*}{m_0} S_1(u\omega_*) = C_p u^{-p} \exp\left(-\frac{p}{4u^4}\right) \quad (58)$$

where $u = \omega/\omega_*$; $C_p = 4(p/4)^{(p-1)/4}/\Gamma[(p-1)/4]$; and Γ = gamma function. For $p = 5$, Eq. 58 reduces to the Pierson-Moskowitz spectrum; and, for p arbitrary, it gives the Wallops spectrum of Huang et al. (1981). However, in the present case, $p \gtrsim 8$ necessarily.

The second-order spectra will follow from Eqs. 42*a* and 42*b* simply as

$$\tilde{S}_2^+ = \frac{\alpha^2}{8}\int_0^u [(u')^2 + (u-u')^2]^2 \tilde{S}_1(u-u')\tilde{S}_1(u')du' \quad (59a)$$

$$\tilde{S}_2^- = \frac{\alpha^2}{4}\int_0^\infty [(u')^2 - (u+u')^2]^2 \tilde{S}_1(u+u')\tilde{S}_1(u')du' \quad (59b)$$

where $\alpha = k_*(2m_0)^{1/2}$ and, $\tilde{S}_2^{\pm} = (\omega_*/m_0)S_2^{\pm}$. In the most general case, the parameter α can be shown to be proportional to the rms surface slope $\langle|\nabla\eta_1|^2\rangle^{1/2} = m_4^{1/2}/g$, where ∇ = horizontal gradient operator.

The moments of S_1 have the form

$$m_j = m_0\omega_*^j\left(\frac{p}{4}\right)^{j/4}\frac{\Gamma\left(\frac{p-j-1}{4}\right)}{\Gamma\left(\frac{p-1}{4}\right)} \quad (60)$$

It can be shown [see for example Tayfun (1986)] that the total areas under $S_2^{\pm}$ are given exactly by the upper bounds in Eq. 32. Thus

$$\frac{\langle(\eta_2^{\pm})^2\rangle}{\langle\eta_1^2\rangle} = \frac{1}{2g^2}\left(m_4 \pm \frac{m_2^2}{m_0}\right) = \frac{p}{4}\alpha^2\left[\frac{1}{(p-5)} \pm \frac{\Gamma^2\left(\frac{p-3}{4}\right)}{4\Gamma^2\left(\frac{p-1}{4}\right)}\right] \quad (61)$$

and so,

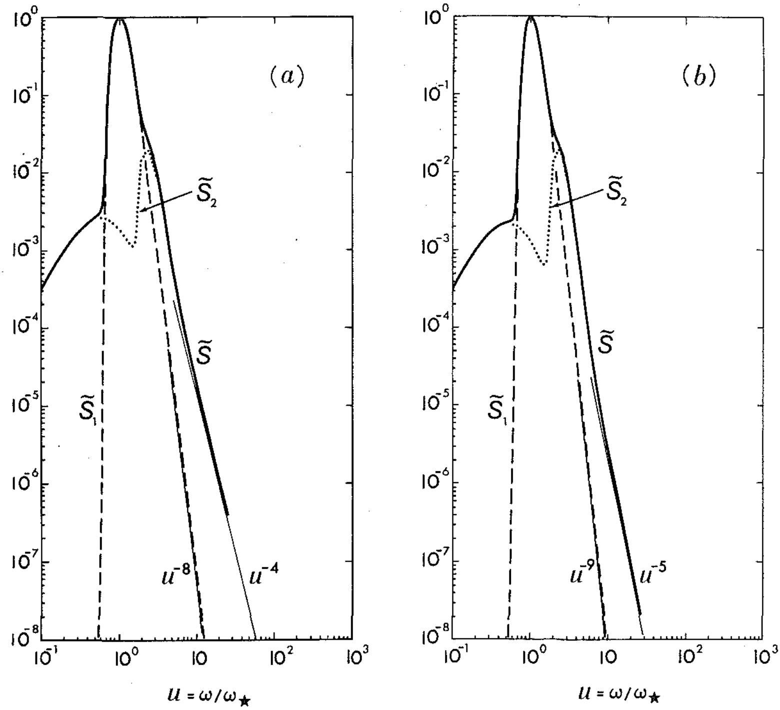

FIG. 1. Illustrative Examples for High-Frequency Attenuation of Spectra S_1 and $S_1 + S_2$ in a Dimensionless Form: (*a*) $S_1 \propto \omega^{-8}$; (*b*) $S_1 \propto \omega^{-9}$

$$\frac{\langle\eta_2^2\rangle}{\langle\eta_1^2\rangle} = \frac{1}{2}\left(\frac{p}{p-5}\right)\alpha^2 \quad (62)$$

which is the specific form assumed by $\langle|\nabla\eta_1|^2\rangle$ in the unidirectional case here.

Now, consider two specific cases corresponding to $p = 8$ and $p = 9$, respectively. It is assumed that $m_0 = 10\ m^2$ and $\omega_* \simeq 0.66$ rad/s in either case. Thus, $\alpha \simeq 0.2$, which is typical of extreme seas generated by intense hurricanes (Tayfun and Lo 1989, 1990). The results computed numerically using Eqs. 58, 59*a*, and 59*b* are given in Fig. 1, which shows $\tilde{S}_1$, $\tilde{S}_2 = \tilde{S}_2^+ + \tilde{S}_2^-$ and $\tilde{S} = \tilde{S}_1 + \tilde{S}_2$ for both cases. It is seen that when $\tilde{S}_1 \propto u^{-8}(u^{-9})$, then $\tilde{S} \propto u^{-4}(u^{-5})$ for $u > 10$, approximately. Clearly, the energy-containing part of $\tilde{S}$ located over frequencies near $u_* = \omega/\omega_* = 1$ is primarily determined by $\tilde{S}_1$. Except for the relatively minor increase of the spectral area in the low-frequency range, which is largely due to $\tilde{S}_2^-$, the influence of $\tilde{S}_2^\pm$ on $\tilde{S}_1$ is negligible up to about $u = 2$. For $u > 2$, the spectral area does increase noticeably, in particular, due to the $\tilde{S}_2^+$ contribution. However, despite the extreme case implied by $\alpha \simeq 0.2$, it can be verified from Eq. 62 with $p = 8(9)$ that the total area under $\tilde{S}_2$ is only 5.33 (4.50%) of the unit area under $\tilde{S}_1$. Further, Eq. 61 indicates that more than 87% of the area under $\tilde{S}_2$ in either case is due to the $\tilde{S}^+$ contribution. Thus, $\tilde{S}_2^+$ is the predominant component of $\tilde{S}_2$ for deep-water waves.

J. Waterway, Port, Coastal, Ocean Eng. 1990.116:381-398.

IMPLICATIONS

The most obvious physical effect of nonlinearities manifests itself in the form of sharp wave crests on the sea surface. Evidently, these features are an inherent property of developed wind-wave fields whose spectra display the saturation-equilibrium-type upper-limit asymptotes in the gravity-wave range. Thus, empirical/theoretical spectra, for example, the JONSWAP, P-M, and other forms of one-dimensional frequency spectra that fall off as $\omega^{-(4\sim5)}$ at large ω, are in essence representative of nonlinear seas and so can not be used to describe linear Gaussian waves without due modification. For such spectra, high-order ordinary moments tend to be divergent and cause complications in the interpretation of surface-wave characteristics [see for example Glazman (1986), Nath and Yeh (1987)].

It appears that sharp corners and crests are relatively small-scale high-frequency features of a developed wind-wave field. The large-scale or predominant structure of the surface geometry is determined mostly by the linear spectral components that make up the energy-containing part of the spectrum over frequencies clustered around the spectral peak. Thus, if attention is to be focused on the first-order surface structure, it may be necessary to modify actual spectra accordingly. In theory, this requires an iterative solution of certain integral equations, viz, Eqs. 19 and 22 for $\psi_1(\mathbf{k})$, assuming initially that the actual spectrum can be substituted for $\psi(\mathbf{k})$. Obviously, this approach is neither practical nor often feasible, for the simple reason that most actual observations and theoretical forms of spectra are restricted to the one-dimensional frequency domain. Thus, one is forced to resort to the use of low-pass filters as a practical alternative.

Filters that attenuate high frequencies and pass low frequencies relatively unchanged are commonly referred to as low-pass, or smoothing, filters. The so-called ideal low-pass filter truncates all frequencies above a prescribed frequency ω_c and leaves untouched the frequencies below that. Typically, the suppression of high-frequency components yields a surface profile with smoother, less erratic appearance. Of course, the sharper crests are smoothed out more than the rounded troughs, so that a first consequence of low-passing is to reduce the deviation of surface statistics from the Gaussian norms. To the leading first-order of accuracy in the rms surface slope, the key measure of this deviation is the skewness coefficient $\lambda_3 = \langle\eta^3\rangle/\langle\eta^2\rangle^{3/2}$. It is known (Longuet-Higgins 1963) that in general

$$0.44\Omega(\infty) \leq \lambda_3 \leq 1.01\Omega(\infty) \quad (63)$$

where

$$\Omega(\omega_c) = \frac{6}{g}(\bar{M}_0)^{-3/2}\int_0^{\omega_c}\int_0^{\omega'}\omega^2 S(\omega)S(\omega')d\omega d\omega' \quad (64a)$$

$$\bar{M}_j = \int_0^{\omega_c}\omega^j S(\omega)d\omega \quad (64b)$$

such that, as $\omega_c \rightarrow \infty$, $\bar{M}_j \rightarrow M_j$ = jth ordinary moment of S; and, $\Omega(\infty) = \lambda_3$ exactly in the unidirectional case. Further, $S = S_1$, correct to $O(\alpha)$. Thus, S and S_1 are interchangeable, and that would leave the preceding relations unchanged to $O(\alpha)$. Now, note that

$$\int_0^{\omega_c}\int_0^{\omega'} \omega^{2j}S(\omega)S(\omega')d\omega d\omega' \leq \frac{1}{2}\bar{M}_0\bar{M}_{2j} \quad (65)$$

which can be used to verify that for any $\omega_c > 0$

$$\frac{d}{d\omega_c}\Omega(\omega_c) \geq \frac{3}{2g}S(\omega_c)\bar{M}_2(\bar{M}_0)^{-3/2} \geq 0 \quad (66)$$

Thus, Ω is a monotonously increasing function of ω_c, suggesting that the skewness can be reduced significantly if ω_c is chosen sufficiently small, for example, as close to ω_* as possible without affecting the energy-containing part S_1 of S appreciably.

The basic premise of low-pass filtering is thus to modify the actual spectrum and the associated moments so that various statistics of the large-scale surface geometry can be interpreted within the Gaussian norms. The practical support and applications of this concept are found in the studies of Nolte and Hsu (1979), Longuet-Higgins (1984), and Glazman (1986). Evidently, low-passing can be done either directly in the frequency domain or, somewhat indirectly, in the time domain by partial averaging, depending to a certain extent on the nature of the eventual objective. In either case, the concept has immediate physical appeal and seems to work well. However, its major premise remains largely intuitive. Various decisions as to the explicit functional forms of frequency, or time-domain, filters, their effective widths, upper cutoff frequencies, and so on are mainly based on heuristic formalism rather than guided by the physics of nonlinear random waves. Thus, it may be worthwhile to explore if the perturbational results can be used to guide some of these decisions.

So far, the nature of third-order moments did imply that the more S is low-passed, the better it would be in terms of reducing $\langle\eta^3\rangle$ and so the skewness coefficient. Evidently, the present results provide two more specific criteria that should be satisfied. One of these implies that if the actual spectrum falls off as $\omega^{-4}(\omega^{-5})$, then the spectrum representative of the large-scale Gaussian structure must fall off *at least* as fast as $\omega^{-8}(\omega^{-9})$. The second criterion is embedded in Eq. 34, which can be rewritten in the equivalent form

$$0 < \frac{\langle\eta^2\rangle - \langle\eta_1^2\rangle}{\langle\eta_1^2\rangle} \leq \langle|\nabla\eta_1|^2\rangle \quad (67)$$

where $\langle|\nabla\eta_1|^2\rangle = m_4/g^2$, as was noted previously. In the most general case $\langle|\nabla\eta|^2\rangle \geq \langle|\nabla\eta_1|^2\rangle$, and since $\langle\eta^2\rangle \geq \langle\eta_1^2\rangle$, the preceding criterion becomes

$$0 < \frac{\langle\eta^2\rangle - \langle\eta_1^2\rangle}{\langle\eta^2\rangle} \leq \langle|\nabla\eta|^2\rangle \quad (68)$$

which requires that the fractional reduction in the total spectral area due to filtering lie within the indicated bounds. Unless low-pass filtering is performed improperly, the lower bound is a foregone conclusion. Thus, the upper bound is of relevance here, but it requires knowledge of max $\langle|\nabla\eta|^2\rangle$ in a developed wind-wave field. Theoretical considerations and the trend of field observations at wind speeds greater than 10 m/s enable Phillips (1985) to suggest 0.06 as a possible upper-limit asymptote to $\langle|\nabla\eta|^2\rangle$. If this is indeed so, then the second criterion will simplify to

$$\frac{\langle\eta^2\rangle - \langle\eta_1^2\rangle}{\langle\eta^2\rangle} \leq 0.06 \qquad (69)$$

Suppose now that the actual spectrum S is approximated by the same dimensionless form, say $\tilde{S}$, as that given by the right-hand side of Eq. 58 with $p = 4$ or $p = 5$. Further, let $h(t)$ = a low-pass filter, and $H(\omega)$ = frequency response of h. Thus, the filtered η, its spectrum, and mean-square value will be given respectively by

$$\tilde{\eta}(t) = \int_{-\infty}^{\infty} h(t - t')\eta(t')dt' \qquad (70a)$$

$$\tilde{S}(\omega) = |H|^2 S(\omega) \qquad (70b)$$

$$\langle\tilde{\eta}^2\rangle = \int_0^{\infty} \tilde{S}(\omega)d\omega \qquad (70c)$$

If low-passing is to be performed directly on S, the simplest filter forms that will lead to a modified $\tilde{S}$ consistent with the nature of nonlinear corrections and the first criterion are given by

$$|H|^2 = 1, \qquad (0 \leq u \leq u_c) \qquad (71a)$$

$$|H|^2 = \left(\frac{u_c}{u}\right)^q, \qquad (u > u_c) \qquad (71b)$$

where $u = \omega/\omega_*$ as before; $q \geq 4$; and the transitional value $u_c = \omega/\omega_c$ is chosen so as to satisfy the second criterion and, perhaps, the condition $u_c \gtrsim 2$. Thus, with $\langle\tilde{\eta}^2\rangle$ replacing $\langle\eta_1^2\rangle$ in Eq. 69, the second criterion reduces to

$$\int_{u_c}^{\infty} \left[1 - \left(\frac{u_c}{u}\right)^q\right]\tilde{S}(u)du \leq 0.06 \qquad (72)$$

Because $\tilde{S} \leq C_p u^{-p}$ in the present case, Eq. 72 yields

$$u_c^{p-1} \geq \frac{qC_p}{0.06(p - 1)(q + p - 1)} \qquad (73)$$

For $q = 4$, Eq. 73 requires that $u_c \geq 2.18$ for $p = 4$, and that $u_c \geq 1.80$ for $p = 5$. If the additional condition $u_c \geq 2$ is to be satisfied, then it would be necessary to set $u_c \geq 2$ in the latter case. For the limit $q \to \infty$ representing an ideal low-pass filter, it is required that $u_c \geq 2.62$ for $p = 4$, and that $u_c \gtrsim 2.14$ for $p = 5$. Thus, the lower bound in each case is in fact the optimal value of u_c for that particular case.

The same criteria may also be of relevance in the real-time smoothing of wave records, for which the functional form and the effective width of h need be specified. For example, the partial-averaging filter used by Glazman (1986)

$$h(t) = \frac{1}{T}, \qquad \left(|t| < \frac{T}{2}\right) \qquad (74)$$

with the frequency response $H = \sin(\omega T/2)/(\omega T/2)$ clearly violates the first criterion, since H^2 does not attenuate sufficiently fast. In contrast, the triangular form

$$h(t) = \frac{2}{T}\left(1 - 2\frac{|t|}{T}\right), \qquad \left(|t| < \frac{T}{2}\right) \quad \text{(75)}$$

with $H = [\sin(\omega T/4)/(\omega T/4)]^2$ is consistent with the first criterion, because H^2 falls off as ω^{-4}. It will conform to the second criterion also, if the filter width T is chosen so that

$$\int_0^{\infty} \tilde{S}(u)du \geq 0.94 \quad \text{(76)}$$

The optimal T is that for which the equality holds. That solution may in general require a trial-and-error type numerical evaluation, which will not be pursued here.

Summary and Conclusions

The high-wave-number/frequency properties of wind-wave spectra were theoretically examined within the context of second-order perturbational solutions. The results suggests that if observed spectra follow certain saturation (equilibrium) type of upper-limit asymptotes that attenuate as $\omega^{-5}(\omega^{-4})$, then the first-order Gaussian structure of the sea surface must be characterized by spectra that fall off at least as fast as $\omega^{-9}(\omega^{-8})$.

Physically, nonlinearities manifest themselves in the form of sharp crests on the sea surface. Such features are an inherent property of developed wind-wave fields whose spectra tend to display the saturation-equilibrium-type upper-limit asymptotes in the gravity-wave range. It can thus be concluded further that empirical and/or theoretical spectral forms that fall off as $\omega^{-(4\sim 5)}$ at large ω are in essence representative of nonlinear wind waves, and so cannot directly be used to describe Gaussian wave fields.

An accurate interpretation of predominant or large-scale surface-wave properties and associated statistics within the Gaussian norms may require the actual spectra to be modified in the high-frequency range. In practice, this often involves low-passing of observed spectra, but it also raises awkward questions as to the nature of filters that can be employed in such a process. In this respect, the present results provide two fairly specific criteria. One of these requires the low-pass frequency response to fall off as ω^{-2} or faster, and the other restricts the filtered spectral area to remain within certain bounds.

Appendix I. References

Barrick, D. E., and Weber, B. L. (1977). "On the nonlinear theory for gravity waves on the ocean's surface: 2. Interpretation and applications." *J. Physical Oceanography,* 7(1), 11–21.

Forristall, G. Z. (1981). "Measurements of a saturated range in ocean wave spectra." *J. Geophysical Res.,* 86(C9), 8075–8084.

Fox, M. J. H. (1976). "On the nonlinear transfer of energy in the peak of a gravity-wave spectrum II." *Proc., Royal Soc. of London,* London, England, 348, 467–483.

Glazman, R. E. (1986). "Statistical characterization of sea surface geometry for a wave slope field discontinuous in the mean square." *J. Geophysical Res.*, 91(C5), 6629–6641.
Hasselmann, K. (1962). "On the nonlinear energy transfer in a gravity-wave spectrum: 1. General theory." *J. Fluid Mech.*, 12, 481–500.
Hasseman, K., et al. (1973). "Measurements of wind wave growth and swell decay during the Joint North Sea Wave Project (JONSWAP)." *Deutschen Hydrographischen Zeitschrift,* Hamburg, Germany, Supplement A, 8(12), 1–95.
Hildebrand, F. B. (1976). *Advanced calculus for applications,* 2nd Ed., Prentice-Hall, Inc., Englewood Cliffs, N.J., 264.
Huang, N. E., et al. (1981). "A unified two-parameter wave spectral model for a general sea state." *J. Fluid Mech.*, 112, 203–224.
Kitaigorodski, S. A. (1962). "Applications of the theory of similarity to the analysis of wind-generated wave motions as a stochastic process." *Bulletin Acad. Sci. USSR,* Geophysical Series, 73–80.
Kitaigorodski, S. A. (1983). "On the theory of the equilibrium range in the spectrum of wind-generated gravity waves." *J. Physical Oceanography,* 13(May), 810–827.
Longuet-Higgins, M. S. (1963). "The effect of nonlinearities on statistical distributions in the theory of sea waves." *J. Fluid Mech.*, 17, 459–480.
Longuet-Higgins, M. S. (1976). "On the nonlinear transfer of energy in the peak of a gravity-wave spectrum: A simplified model." *Proc., Royal Soc. London,* London, England, 347, 311–328.
Longuet-Higgins, M. S. (1984). "Statistical properties of wave groups in a random sea state." *Philosophical Transactions, Royal Society of London,* 312, 219–250.
Nath, J. H., and Yeh, R. C. (1987). "Some time and frequency relations in random waves." *J. Waterway, Port, Coastal and Ocean Engrg.*, ASCE, 113(6), 672–683.
Nolte, K. G., and Hsu, F. H. (1979). "Statistics of larger waves in a sea state." *J. Waterway, Port, Coastal and Ocean Div.*, ASCE, 105(4), 389–404.
Phillips, O. M. (1958). "The equilibrium range in the spectrum of wind-generated ocean waves." *J. Fluid Mech.*, 4, 426–434.
Phillips, O. M. (1977). *The dynamics of the upper ocean.* Cambridge University Press, London, England.
Phillips, O. M. (1985). "Spectral and statistical properties of the equilibrium range in wind-generated gravity waves." *J. Fluid Mech.*, 156, 505–531.
Tayfun, M. A. (1986). "On narrow-band representation of ocean waves: 1. Theory." *J. Geophysical Res.*, 91(C6), 7743–7752.
Tayfun, M. A., and Lo, J.-M. (1989). "Envelope, phase, and narrow-band models of sea waves." *J. Waterway, Port, Coastal and Ocean Engrg.*, ASCE, 115(5), 594–613.
Tayfun, M. A., and Lo, J.-M. (1990). "Nonlinear effects on wave envelope and phase." *J. Waterway, Port, Coastal, and Ocean Engrg.*, ASCE, 116(1), 79–100.
Tick, L. J. (1959). "A nonlinear random model of gravity wave I." *J. Mathematics Mech.*, 8, 643–652.
Weber, B. L., and Barrick, D. E. (1977). "On the nonlinear theory for gravity waves on the ocean's surface: 1. Derivations." *J. Physical Oceanography,* 7(1), 3–10.

APPENDIX II. NOTATION

The following symbols are used in this paper:

a_n = amplitude of nth spectral component;
$B^{\pm}_{m,n}$ = coefficient in second-order solutions, defined by Eq. 11;
C_p = proportionality constant in frequency spectra;
$c, c^{\pm}$ = cosines of angles between $\mathbf{k}_m$ and $\mathbf{k}_n$, and $\mathbf{k} \pm \mathbf{k}'$ and $\mathbf{k}'$, respectively;
$f_{\pm}$ = function defined by Eq. 24;
g = gravitational acceleration;

H, h = low-pass filters in frequency and time domain;
$I, J, \bar{J}$ = regions of integration in Eqs. 19 and 47;
i, j = summation indices or subscripts;
$K^{\pm}_{m,n}$ = $K^{\pm}(\mathbf{k}_m, \mathbf{k}_n)$, coefficient in second-order solutions, given by Eq. 10;
k = modulus of $\mathbf{k}$;
k_* = spectral-peak wave number;
k_0 = wave number defined by Eq. 49;
$\mathbf{k}$ = (k_1, k_2), horizontal wave-number vector;
$\mathbf{k}_m, \mathbf{k}_n$ = wave number of mth and nth spectral components;
$\bar{M}_j, M_j$ = jth ordinary moments of $\bar{S}$ and S;
m = summation index or subscript;
m_j = jth ordinary moment of S_1;
N = total number of frequency components in η_1;
$O(\ )$ = of order of ( );
p, q = decay exponents for spectra and filters, respectively;
R, R_1, R_2 = temporal covariance functions of η, η_1, and η_2;
R_2^+, R_2^- = components of R_2;
$\hat{R}_1$ = Hilbert transform of R_1;
$\mathbf{r}, r$ = horizontal displacement vector and its modulus, respectively;
$S, \tilde{S}$ = frequency spectrum of η, and its dimensionless form, respectively;
S_1, S_2 = frequency spectra of η_1 and η_2;
$\tilde{S}_1, \tilde{S}_2$ = dimensionless forms of S_1 and S_2;
S_2^+, S_2^- = components of S_2;
$\tilde{S}_2^+, \tilde{S}_2^-$ = dimensionless forms of $S_2^{\pm}$;
$\bar{S}$ = low-passed spectrum S;
$s\ (\cos\theta)$ = directional spreading function;
T = time-domain width of low-pass filter;
t = time;
u = frequency scaled with respect to spectral-peak frequency;
u_c = scaled transitional or upper-cutoff frequency;
u_* = scaled spectral-peak frequency (= 1);
$\mathbf{x}$ = (x_1, x_2), horizontal position vector;
Z, Z_1, Z_2 = spatial covariance functions of η, η_1, and η_2;
Z_2^+, Z_2^- = components of Z_2;
α = steepness parameter;
β = a dimensional constant in frequency spectra;
Γ = gamma function;
γ = $\theta_m - \theta_n$, angle between $\mathbf{k}_m$ and $\mathbf{k}_n$;
δ = an arbitrary small constant;
$\eta, \bar{\eta}$ = sea surface elevation and its filtered form, respectively;
η_1, η_2 = first- and second-order solutions for η;
η_2^+, η_2^- = components of η_2;
$\theta, \theta_m, \theta_n$ = directions of $\mathbf{k}$, $\mathbf{k}_m$, $\mathbf{k}_n$, respectively;
μ_n = random phase of nth spectral component in η_1;
λ_3 = skewness coefficient of η;
ρ = ratio defined by Eq. 23a;
τ = time lag;
$\phi^{\pm}$ = direction of $\mathbf{k} \pm \mathbf{k}'$;
χ_n = total phase of nth spectral component in η_1;

J. Waterway, Port, Coastal, Ocean Eng. 1990.116:381-398.

ψ, ψ_1, ψ_2	=	wave-number spectra of η, η_1, and η_2;
ψ_2^+, ψ_2^-	=	wave-number spectra of η_2^+ and η_2^-, respectively;
Ω	=	skewness coefficient of unidirectional η as a function of ω_c;
ω	=	circular frequency in rad/s;
ω_c	=	upper-cutoff frequency;
ω_m, ω_n	=	frequencies of *m*th and *n*th spectral components;
ω_*	=	$\sqrt{gk_*}$, spectral-peak frequency; and
$\in$	=	belongs to or is an element of.