# Excitation of normal modes by non-linear interaction of ocean waves

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## SUMMARY

Recent development in seismology has shown that seismic noise is a useful source of Earth structure study. Non-linear interaction of ocean waves is considered to be one of the important causes for seismic noise, although source locations and detailed mechanisms are not clear yet. In order to improve our understanding of this mechanism quantitatively, we derive a normal-mode excitation formula due to this effect. Longuet-Higgins' pressure formula naturally comes out from the vertical forcing term in the excitation coefficient. A novel aspect of our formula is the horizontal forcing term that becomes comparable to the vertical forcing term for frequencies below about 5 mHz (millihertz), especially in shallow oceans. This term is not likely to be important for microseisms because the main frequency range of microseisms is about 0.05–0.4 Hz but it may make significant contributions to the excitation of the continuous background oscillations (the hum) whose frequency range is 2–7 mHz.

Key words: normal modes, oceans, seismic noise, wave propagation.

## **1 INTRODUCTION**

Seismic noise makes up a significant portion of seismograms and has been regarded as unwanted noise until recently. However, recent development in seismology has shown that useful information about the Earth structure can be garnered from its analysis (e.g. Campillo & Paul 2003; Shapiro *et al.* 2005; Sabra *et al.* 2005). Its source of excitation is not well understood, however, and sources may vary depending on the frequency band of seismic noise.

Longuet-Higgins (1950) pointed out that non-linear interaction of ocean waves explain some features in microseisms for frequencies about 0.1 Hz. The most attractive aspect of his theory was the way that the double-frequency microseisms (about 0.15 Hz) were naturally explained by this mechanism. This mechanism became widely accepted after observational support came out in the 1960s (e.g. Haubrich *et al.* 1963). An alternative theoretical approach by Hasselmann (1963), that incorporated statistical behaviour of ocean waves, also lent support for this mechanism.

The continuous background oscillations, a seismic noise in a much lower frequency band about 2–7 mHz (millihertz), were reported in 1998 (Kobayashi & Nishida 1998; Nawa *et al.* 1998; Suda *et al.* 1998; Tanimoto *et al.* 1998). They are often referred as the hum in recent literatures and we follow this convention in this paper. The hum consists of a sequence of fundamental-mode spheroidal modes that are excited continuously either by the atmosphere or by the oceans. Recent papers tend to claim an oceanic mechanism as its cause (e.g. Rhie & Romanowicz 2004; Tanimoto 2005), especially because of good correlation with satellite ocean-wave data (Ekström & Ekström 2005; Tanimoto 2005). Some claim that a similar non-linear mechanism to the microseisms (Webb 2005) plays a major role for its excitation.

In this paper, in order to extend our capability to quantitatively evaluate this non-linear ocean-wave effect, we derive formulae for the excitation of seismic normal modes. While the forcing pressure term was derived by Longuet-Higgins (1950), formulae for seismic displacement that show explicit dependence on the non-linear source (ocean waves) have not been shown. Our main goal is to present such formulae using the normal mode excitation theory (Gilbert 1970) and to discuss some characteristics of the solution.

We confirm that the pressure term derived by Longuet-Higgins (1950) naturally appears in the solution in the excitation coefficient. Our solution also contains an interesting horizontal forcing term which does not seem to have been paid much attention so far. This term becomes comparable to the vertical forcing (pressure) term for frequencies about 1–5 mHz. Therefore, this horizontal term is probably not important for the microseisms (0.05–0.4 Hz) but may play a critical role for the excitation of the hum.

We state our problem in Section 2 and derive the formal normal-mode solution in Section 3. In Section 4, we derive the leading order terms in the solution using approximations suitable for the Earth's normal modes and ocean waves. The final formula in Section 4 is our main result in this paper. Relationship to the Longuet-Higgins pressure term (Section 5), the importance of the horizontal forcing term (Section 6), and the conclusions (Section 7) will follow. In Appendix A, we present the equivalent body/surface force formula for non-linearly interacting ocean waves that we use in the main text.

GJI Marine geoscience

## 2 STATEMENT OF THE PROBLEM

We consider a spherically symmetric Earth with the oceanic layer on top. This oceanic layer is assumed to be filled with linear ocean waves (surface waves) that are generated by interactions between the atmosphere and the oceans.

Linear ocean waves are a part of the elastic eigenfunctions of the Earth and thus are orthogonal to other solid-earth modes in the Earth. Therefore, in a spherically symmetric Earth, they do not couple to solid-earth modes. In the real 3-D Earth, linear ocean waves and solid-earth modes may couple through topographic variations, other boundary deformations and 3-D velocity variations. However, differences in phase velocity between ocean waves and solid-earth surface waves  $(100-200 \text{ m s}^{-1} \text{ for long-period ocean waves and } 3-5 \text{ km s}^{-1} \text{ for surface waves in the solid earth) seem to suggest that strong coupling is not likely, although a more careful analysis should be done on this problem.$ 

These linear ocean waves may interact among themselves through the non-linear term in the Navier–Stokes equation and can become a source of excitation for seismic waves in the solid Earth. In this non-linear process, generation of long-wavelength waves from short-wavelength waves (or vice versa) is possible and thus phase-velocity mismatch problem disappears.

Denoting seismic displacement by  $\mathbf{u}$  and ocean-wave velocity by  $\mathbf{v}_{oc}$ , the basic equations become

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathcal{H}(\mathbf{u}) - \rho(\mathbf{v}_{oc} \cdot \nabla)\mathbf{v}_{oc} - \rho \zeta \left[\frac{\partial \mathbf{v}_{oc}}{\partial t}\right] \delta(z^-),\tag{1}$$

where  $\rho$  is density and  $\mathcal{H}$  is the elastic-gravitational operator that appears in the normal mode problem for the Earth. The second and third terms on the right-hand side are the source terms due to non-linear interaction of ocean waves. The second term is the volumetric term and the third term is the surface contribution as the delta function indicates. These source terms are correct up to the order of  $O(\zeta^2)$  where  $\zeta$  is the surface vertical displacement of ocean waves. The minus sign in  $z^-$  denotes the condition that integration of this term over the ocean depth, from ocean bottom (z = -d) to the surface (z = 0), yields the surface boundary condition. Alternatively, this term may be treated as a surface boundary condition. Details in the derivation of these source terms are given in Appendix A.

The principal aim of this paper is to analyse the solutions to eq. (1) with the boundary conditions that are now well known and documented (e.g. Gilbert 1970; Dahlen & Tromp 1998).

## **3 DERIVATION**

#### 3.1 Framework

The Earth's normal modes satisfy the equations

$$-\rho\omega_n^2 \mathbf{u}_n = \mathcal{H}(\mathbf{u}_n) \tag{2}$$

with appropriate boundary conditions (e.g. Dahlen & Tromp 1998). Here, the index *n* refers to the *n*th set of normal modes, its eigenfrequency is  $\omega_n$  and the eigenfunctions are  $\mathbf{u}_n$ . The eigenfunctions are normalized in this paper by

$$\int_{E} \rho \, \mathbf{u}_{m}^{*} \mathbf{u}_{n} \, dV = \delta_{mn}, \tag{3}$$

where E is the volume of the Earth and the star denotes complex conjugation. Orthogonality between linear ocean-wave modes and solid-earth modes is defined through this relation.

Following Gilbert (1970), we apply the Laplace transformation to (1):

$$\rho p^2 \overline{\mathbf{u}} = \mathcal{H}(\overline{\mathbf{u}}) - \rho \overline{(\mathbf{v}_{oc} \cdot \nabla)\mathbf{v}_{oc}} - \rho \overline{\zeta \left[\frac{\partial \mathbf{v}_{oc}}{\partial t}\right]} \delta(z^-), \tag{4}$$

where a bar indicates a transformed quantity and p is the Laplace transform parameter as defined in

$$\overline{\mathbf{u}}(p) = \int_0^\infty \mathbf{u}(t) e^{-pt} \, dt. \tag{5}$$

Expanding the transformed displacement by a set of eigenfunctions

$$\overline{\mathbf{u}} = \sum_{n} c_n \mathbf{u}_n \tag{6}$$

and substituting it in eq. (4), we have

$$\rho p^{2} \sum_{n} c_{n} \mathbf{u}_{n} = \sum_{n} c_{n} \mathcal{H}(\mathbf{u}_{n}) - \rho \overline{\zeta \left[\frac{\partial \mathbf{v}_{oc}}{\partial t} - \rho \overline{\zeta \left[\frac{\partial \mathbf{v}_{oc}}{\partial t}\right]} \delta(z^{-}).$$
<sup>(7)</sup>

Multiplying  $\mathbf{u}_m^*$  from left and integrating over the volume of the Earth, we get

$$c_m = -\frac{1}{p^2 + \omega_m^2} \int_E \rho \mathbf{u_m}^* \cdot \overline{(\mathbf{v}_{oc} \cdot \nabla) \mathbf{v}_{oc}} \, dV - \frac{1}{p^2 + \omega_m^2} \int_A \rho \mathbf{u_m}^* \cdot \overline{\zeta} \left[ \frac{\partial \mathbf{v}_{oc}}{\partial t} \right] dS,\tag{8}$$

where the first term is the integration over the volume of the Earth and the second term is the integration over the surface of the Earth. In either term, the integration range eventually reduces to oceanic regions where Non-linear interaction of ocean waves are significant.

For convenience, mainly because ocean waves are handled in the Cartesian coordinate in the following sections, we adopt the Cartesian coordinate and replace the modal summation in eq. (6) by the wavenumber integral. In detail, we replace eq. (6) by

$$\overline{\mathbf{u}} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \ c(k_x, k_y) \ \mathbf{R}(k_x, k_y; x, y, z),$$
(9)
where  $c(k_x, k_y)$  is the coefficient.

$$\mathbf{P}(k, k; x, y, z) = \begin{bmatrix} U(k_x, k_y; z) \\ V(k, k; z)^{ik_x} \end{bmatrix} e^{i(k_x x + ik_y y)}$$

 $\mathbf{R}(k_x, k_y; x, y, z) = \begin{bmatrix} V(k_x, k_y; z) \frac{ik_x}{k} \\ V(k_x, k_y; z) \frac{ik_y}{k} \end{bmatrix} e^{i(k_x x + ik_y y)},$ (10)
where k and k are the horizontal components of the wavenumber (of seismic waves)  $k = \sqrt{k^2 + k^2}$  and U and V are the vertical and

where  $k_x$  and  $k_y$  are the horizontal components of the wavenumber (of seismic waves),  $k = \sqrt{k_x^2 + k_y^2}$ , and U and V are the vertical and horizontal eigenfunctions. We take the order of components z, x and y in order to align the terms with the convention in spherical coordinates r (radius),  $\theta$  (co-latitude) and  $\phi$  (longitude). x is positive southward and y is positive eastward.

The normalization is done through the relation

$$\int_{E} \rho \mathbf{R}(\overline{k}_{x}, \overline{k}_{y})^{*} \mathbf{R}(k_{x}, k_{y}) dV = (2\pi)^{2} \delta(\overline{k}_{x} - k_{x}) \delta(\overline{k}_{y} - k_{y}) I$$
and
(11)

$$I = \int \rho \left\{ U(k_x, k_y; z)^2 + V(k_x, k_y; z)^2 \right\} dz.$$
(12)

It is important to note that, if we include modes from fundamental-mode and overtone branches (as eq. 6 includes all modes), the wavenumber integral cannot be unique. We need to evaluate the wavenumber integrals like eq. (9) for all branches and sum up the contributions. However hereafter, we focus only on a single branch, assuming that the fundamental-mode branch dominates the observation. We can then proceed with eq. (9). If observation suggests that there are contributions from multiple branches, we must introduce the summation over overtone branches and re-analyse the problem.

Along with the change for eq. (6), eq. (8) should now be replaced by

$$c(\mathbf{k}) = -\frac{1}{p^2 + \omega^2} \left\{ \int_E \rho \mathbf{R}(\mathbf{k}; x, y, z)^* \cdot \overline{(\mathbf{v}_{oc} \cdot \nabla) \mathbf{v}_{oc}} \, dV + \int_A \rho \mathbf{R}(\mathbf{k}; x, y, 0)^* \cdot \overline{\zeta \left[\frac{\partial \mathbf{v}_{oc}}{\partial t}\right]} \, dS \right\},\tag{13}$$

where  $\mathbf{k} = (k_x, k_y)$  and  $\omega = \omega$  (k) is the dispersion relation of seismic (Rayleigh) waves.

The problem is now basically reduced to evaluation of (13). After its evaluation, substitution in (9) will give us the solution in the Laplace transformed domain and its back-transformation will give us the solution in the time domain.

#### 3.2 Ocean waves

In order to evaluate the excitation coefficient in (13), we will use analytical formulae for ocean waves (e.g. Stoker *et al.* 1957; Phillips 1977). These analytical solutions were developed for the rigid ocean-bottom boundary conditions but a very good match with elastic solutions (ocean over an elastic half-space) was shown, for example, in Tanimoto (2005). Strictly speaking, there are discrepancies in the dispersion curves and in the eigenfunctions but differences are about the order of a per cent. This is sufficiently accurate for the purpose of this paper.

We take the *z*-axis positive upward. Ocean surface is z = 0 and ocean bottom is z = -d. When the vertical ocean-surface displacement is given by

$$\zeta(x, y, t) = a(k_x, k_y)\sin(\omega t - k_x x - k_y y), \tag{14}$$

where *a* is an amplitude,  $\omega$  is the angular frequency and  $k_x$  and  $k_y$  are the x- and y- components of a wavenumber of ocean waves. The velocity potential for ocean waves is given by

$$\phi(x, y, z) = \frac{\omega \cosh\{k(z+d)\}}{k \sinh(kd)} a(k_x, k_y) \cos(\omega t - k_x x - k_y y)$$
(15)

with  $\nabla \phi = \mathbf{v}_{oc}$ . This solution satisfies the boundary conditions both at the surface  $v_z = \partial \zeta / \partial t$  (z = 0) and at the ocean bottom  $v_z = 0$  (z = -d).

Velocity wavefields of linear ocean waves can then be written by a linear superposition of these harmonic waves:

$$\mathbf{v}_{oc}(x, y, z, t) = \int d\mathbf{k} \, a(\mathbf{k}) \vec{\xi}(\mathbf{k}),\tag{16}$$

where  $d\mathbf{k} = dk_x dk_y$ ,  $a(\mathbf{k})$  is an amplitude for a harmonic component with the dispersion relation given by  $\omega^2 = gk \tanh(kd)$ , and  $\tilde{\xi}(\mathbf{k}) = (\xi_z(\mathbf{k}), \xi_x(\mathbf{k}), \xi_y(\mathbf{k}))$  with

$$\begin{aligned} \xi_z(\mathbf{k}) &= A_k \cos(\omega t - k_x x - k_y y), \\ \xi_x(\mathbf{k}) &= B_k \frac{k_x}{k} \sin(\omega t - k_x x - k_y y), \\ \xi_y(\mathbf{k}) &= B_k \frac{k_y}{k} \sin(\omega t - k_x x - k_y y). \end{aligned}$$
(17)

Here we defined

$$A_k = \frac{\omega \sinh\{k(z+d)\}}{\sinh(kd)},\tag{18}$$

$$B_k = \frac{\omega \cosh\{k(z+d)\}}{\sinh(kd)}.$$
(19)

The non-linear excitation terms in eq. (1) can now be expressed as

$$-\rho[(\mathbf{v}_{oc}\cdot\nabla)\mathbf{v}_{oc}]_{i} = -\int d\mathbf{k}' \int d\mathbf{k}''\rho \ a(\mathbf{k}')a(\mathbf{k}'')\cdot[\xi_{j}(\mathbf{k}')\partial_{j}]\xi_{i}(\mathbf{k}'')$$
(20)

and

$$-\rho\xi \left[\frac{\partial \mathbf{v}_{oc}}{\partial t}\right]\delta(z^{-}) = -\int d\mathbf{k}' \int d\mathbf{k}''\rho \ a(\mathbf{k}')a(\mathbf{k}'') \cdot \sin(\omega't - \mathbf{k}' \cdot \mathbf{x}) \frac{\partial\xi_i(\mathbf{k}'')}{\partial t} \ \delta(z^{-}), \tag{21}$$

where i = x, y, z and the summation over x, y and z is assumed for a repeated index j. This summation convention is followed throughout this paper.

#### 3.3 Formal solution

First, we substitute eq. (17) in eqs (20) and (21) and carry out the algebra in the time domain. Then we apply the Laplace transformation and obtain:

$$c(\mathbf{k}) = -\frac{1}{p^2 + \omega^2} \int d\mathbf{k}' \int d\mathbf{k}'' \int_E dV \rho a(\mathbf{k}') a(\mathbf{k}'') [\mathbf{R}(\mathbf{k})]_i^* \cdot \overline{[\xi_j(\mathbf{k}') \cdot \partial_j]\xi_i(\mathbf{k}'')} - \frac{1}{p^2 + \omega^2} \int d\mathbf{k}' \int d\mathbf{k}'' \int_A dS \rho a(\mathbf{k}') a(\mathbf{k}'') [\mathbf{R}(\mathbf{k})]_i^* \cdot \overline{\sin(\omega't - \mathbf{k}' \cdot \mathbf{x})} \frac{\partial \xi_i(\mathbf{k}'')}{\partial t}.$$
(22)

Hereafter, quantities with primes, either a prime or two primes, are those for ocean waves and satisfy the dispersion relations given by  $\omega'^2 = gk'$  tanh (k'd) and  $\omega''^2 = gk''$  tanh (k''d). Wavenumbers and frequencies of seismic modes are denoted without primes and their dispersion relation can only be written generally as  $\omega = \omega$  (**k**).

The *z*, *x* and *y* components of a quantity with a bar in (20) and (21) are given by

$$\overline{[\xi_{j}(\mathbf{k}') \cdot \partial_{j}]\xi_{z}(\mathbf{k}'')} = \frac{1}{2} \left( A_{k'} B_{k''} k'' + B_{k'} A_{k''} k''_{x'} \frac{k'_{y}}{k'} + B_{k'} A_{k''} k''_{x'} \frac{k'_{y}}{k'} \right) \\
\times \left[ \frac{p}{p^{2} + (\omega' - \omega'')^{2}} \cos\left\{ (k'_{x} - k''_{x})x + (k'_{y} - k''_{y})y \right\} \right] \\
+ \frac{\omega' - \omega''}{p^{2} + (\omega' - \omega'')^{2}} \sin\left\{ (k'_{x} - k''_{x})x + (k'_{y} - k''_{y})y \right\} \right] \\
+ \frac{1}{2} \left( A_{k'} B_{k'} k'' - B_{k'} A_{k''} \frac{k'_{x}}{k'} - B_{k'} A_{k''} \frac{k'_{y}}{k'} \right) \\
\times \left[ \frac{p}{p^{2} + (\omega' + \omega'')^{2}} \cos\left\{ (k'_{x} + k''_{x})x + (k'_{y} + k''_{y})y \right\} \right] \\
+ \frac{\omega' + \omega''}{p^{2} + (\omega' + \omega'')^{2}} \sin\left\{ (k'_{x} + k''_{x})x + (k'_{y} + k''_{y})y \right\} \right] \\
\overline{[\xi_{j}(\mathbf{k}') \cdot \partial_{j}]\xi_{x}(\mathbf{k}'')} = \frac{1}{2} \left( A_{k'} A_{k''} k''_{x} + B_{k'} B_{k''} \frac{k'_{x}}{k'} \frac{k''_{x}}{k''} + B_{k'} B_{k''} \frac{k'_{x}}{k''} \frac{k''_{x}}{k''} + B_{k'} B_{k''} \frac{k''_{x}}{k''} \left( k''_{x} - k''_{y})y \right\} \right] \\
\frac{1}{k'_{j}(\mathbf{k}') \cdot \partial_{j}]\xi_{x}(\mathbf{k}'')} = \frac{1}{2} \left( A_{k'} A_{k''} k''_{x} + B_{k'} B_{k''} \frac{k'_{x}}{k''} \frac{k''_{x}}{k''} + B_{k'} B_{k''} \frac{k''_{x}}{k''} + B_{k'} B_{k''} \frac{k''_{x}}{k''} \left( k''_{x} - k''_{y})y \right\} \right] \\
+ \frac{1}{p^{2} + (\omega' - \omega'')^{2}} \sin\left\{ (k'_{x} - k''_{x})x + (k'_{y} - k''_{y})y \right\} \\
+ \frac{1}{p^{2} + (\omega' - \omega'')^{2}} \cos\left\{ (k'_{x} - k''_{x})x + (k'_{y} - k''_{y})y \right\} \\
+ \frac{1}{2} \left( A_{k'} A_{k''} k''_{x} - B_{k'} B_{k''} \frac{k''_{x}}{k''_{x}}^{2} - B_{k'} B_{k''} \frac{k''_{x}}{k''_{x}} k''_{x}} \right) \\
\times \left[ \frac{\omega' + \omega''}{p^{2} + (\omega' - \omega'')^{2}} \cos\left\{ (k'_{x} + k''_{x})x + (k'_{y} - k''_{y})y \right\} \right]$$
(24)

$$\overline{[\xi_{j}(\mathbf{k}') \cdot \partial_{j}]\xi_{y}(\mathbf{k}'')} = \frac{1}{2} \left( A_{k'}A_{k''}k''_{y} + B_{k'}B_{k''}\frac{k'_{x}k''_{y}}{k''} + B_{k'}B_{k''}\frac{k'_{y}k''_{y}}{k''} \right) \\ \times \left[ -\frac{\omega' - \omega''}{p^{2} + (\omega' - \omega'')^{2}} \cos\left\{ (k'_{x} - k''_{x})x + (k'_{y} - k''_{y})y \right\} \right] \\ + \frac{p}{p^{2} + (\omega' - \omega'')^{2}} \sin\left\{ (k'_{x} - k''_{x})x + (k'_{y} - k''_{y})y \right\} \right] \\ + \frac{1}{2} \left( A_{k'}A_{k''}k''_{y} - B_{k'}B_{k''}\frac{k'_{x}k''_{y}}{k''} - B_{k'}B_{k''}\frac{k'_{y}k''^{2}}{k''} \right) \\ \times \left[ \frac{\omega' + \omega''}{p^{2} + (\omega' + \omega'')^{2}} \cos\left\{ (k'_{x} + k''_{x})x + (k'_{y} + k''_{y})y \right\} \right]$$

$$(25)$$

$$\overline{\sin(\omega' t - \mathbf{k}' \cdot \mathbf{x})} \frac{\partial \xi_z(\mathbf{k}'')}{\partial t} = \frac{\omega'' A_{k''}}{2} \\ \times \left[ -\frac{p}{p^2 + (\omega' - \omega'')^2} \cos\left\{ (k'_x - k''_x) x + (k'_y - k''_y) y \right\} - \frac{\omega' - \omega''}{p^2 + (\omega' - \omega'')^2} \sin\left\{ (k'_x - k''_x) x + (k'_y - k''_y) y \right\} \\ + \frac{p}{p^2 + (\omega' + \omega'')^2} \cos\left\{ (k'_x + k''_x) x + (k'_y + k''_y) y \right\} \\ + \frac{\omega' + \omega''}{p^2 + (\omega' + \omega'')^2} \sin\left\{ (k'_x + k''_x) x + (k'_y + k''_y) y \right\} \right]$$
(26)

$$\overline{\sin(\omega't - \mathbf{k}' \cdot \mathbf{x})} \frac{\partial \xi_x(\mathbf{k}'')}{\partial t} = \frac{\omega'' B_{k''}}{2} \frac{k_x''}{k''} \\ \times \left[ \frac{\omega' - \omega''}{p^2 + (\omega' - \omega'')^2} \cos\left\{ (k_x' - k_x'')x + (k_y' - k_y'')y \right\} - \frac{p}{p^2 + (\omega' - \omega'')^2} \sin\left\{ (k_x' - k_x'')x + (k_y' - k_y'')y \right\} + \frac{\omega' + \omega''}{p^2 + (\omega' + \omega'')^2} \cos\left\{ (k_x' + k_x'')x + (k_y' + k_y'')y \right\} - \frac{p}{p^2 + (\omega' + \omega'')^2} \sin\left\{ (k_x' + k_x'')x + (k_y' + k_y'')y \right\} \right]$$
(27)

$$\overline{\sin(\omega' t - \mathbf{k}' \cdot \mathbf{x})} \frac{\partial \xi_{y}(\mathbf{k}'')}{\partial t} = \frac{\omega'' B_{k''}}{2} \frac{k_{y}''}{k''} \\ \times \left[ \frac{\omega' - \omega''}{p^2 + (\omega' - \omega'')^2} \cos\left\{ (k_x' - k_x'')x + (k_y' - k_y'')y \right\} - \frac{p}{p^2 + (\omega' - \omega'')^2} \sin\left\{ (k_x' - k_x'')x + (k_y' - k_y'')y \right\} + \frac{\omega' + \omega''}{p^2 + (\omega' + \omega'')^2} \cos\left\{ (k_x' + k_x'')x + (k_y' + k_y'')y \right\} - \frac{p}{p^2 + (\omega' + \omega'')^2} \sin\left\{ (k_x' + k_x'')x + (k_y' + k_y'')y \right\} \right].$$
(28)

Substitution of eqs (23)–(28) in eq. (22) and evaluation of four wavenumber integrals  $(dk'_x, dk'_y, dk''_x, dk''_y)$  and three space integrals (dV = dx dy dz) over a source region constitute the formal solution.

## 4 LEADING ORDER TERMS

#### 4.1 Method and solution

We now embark on evaluation of leading order terms in (22) using approximations suitable for Earth's ocean waves and seismic modes. We present the formulae in a self-contained manner in this paper but the main ideas of approximations are essentially the same with those in Hasselmann (1963).

We first recognize that the integrals with respect to horizontal coordinates (x and y) contain terms like

$$\int dx \int dy e^{-ik_x x - ik_y y} \cos\left\{ \left( k'_x \pm k''_x \right) x + \left( k'_y \pm k''_y \right) y \right\}$$
  
=  $\frac{1}{2} \int dx \ e^{i(-k_x + k'_x \pm k''_x)x} \int dy \ e^{i(-k_y + k'_y \pm k''_y)y} + \frac{1}{2} \int dx \ e^{-i(k_x + k'_x \pm k''_x)x} \int dy \ e^{-i(k_y + k'_y \pm k''_y)y}$  (29)

which we approximate as

$$\approx \frac{L_{x}L_{y}}{2} \left[ \delta \left( -k_{x} + k'_{x} \pm k''_{x} \right) \delta \left( -k_{y} + k'_{y} \pm k''_{y} \right) + \delta \left( k_{x} + k'_{x} \pm k''_{x} \right) \delta \left( k_{y} + k'_{y} \pm k''_{y} \right) \right], \tag{30}$$

where  $L_x$  and  $L_y$  are approximate linear dimensions of a source region. We are assuming here that wavenumbers are sufficiently large and the integrand is highly oscillatory such that the only significant contributions for these integrals come from the case when combined phase of seismic mode and two ocean waves is zero.

We also note that there is an order of magnitude difference between wavenumbers of seismic modes and wavenumbers of ocean waves, that is,

$$|k| \ll |k'|, \ |k''|. \tag{31}$$

For example, for microseisms at period T = 14 seconds, seismic waves (Rayleigh waves) have wavelengths about 42 km (phase velocity about 3 km s<sup>-1</sup>), whereas ocean waves typically have wavelengths about 300 m. For the hum at a period T = 300 seconds, wavelength of seismic waves is about 1000 km whereas (infragravity) ocean waves have wavelengths about 60 km (phase velocity about 200 m s<sup>-1</sup>). Therefore, wavelengths of two ocean waves k' and k'' are about one to two orders of magnitude larger than wavelengths of seismic modes k.

Relations in eq. (31) mean that, when delta functions in above formulae are integrated, the main contributions approximately come from two cases, one from  $\mathbf{k}' = \mathbf{k}''$  and the other from  $\mathbf{k}' = -\mathbf{k}''$ . In terms of components, they are

$$k'_{x} = k''_{x}, \ k'_{y} = k''_{y}, \tag{32}$$

or

$$k'_{x} = -k''_{x}, \ k'_{y} = -k''_{y}, \tag{33}$$

and we put  $k_x = k_y \approx 0$  whenever they appear with primed quantities. These relations help simplify the formulae greatly.

Using these approximations, we carry out the integrations in the formal solution, first with respect to x and y. This procedure results in many terms with delta functions. These delta functions are removed by integrations over one set of the wavenumbers (we integrate over  $k''_x$  and  $k''_y$ ). The resulting formula has three integrals with respect to  $k'_x$ ,  $k'_y$  and z. We write it as

$$c(\mathbf{k}) = \frac{L_x L_y}{p^2 + \omega^2} \int d\mathbf{k}' \left[ \left\{ \frac{\rho \omega'^2 U(0)}{2} - \Phi_Z(\mathbf{k}') \right\} \frac{1}{p} a(\mathbf{k}')^2 - \left\{ \frac{\rho \omega'^2 U(0)}{2} + \Phi_Z(\mathbf{k}') \right\} \frac{p}{p^2 + 4\omega'^2} a(\mathbf{k}') a(-\mathbf{k}') \right\} - i \left\{ \frac{\rho g k' V(0)}{2} + \Phi_H(\mathbf{k}') \right\} \left( \frac{k_x}{k} \frac{k'_x}{k'} + \frac{k_y}{k} \frac{k'_y}{k'} \right) \frac{2\omega'}{p^2 + 4\omega'^2} a(\mathbf{k}') a(-\mathbf{k}') \right\}$$
(34)

where

$$\Phi_{Z}(\mathbf{k}') = \int_{-d}^{0} \rho U(z) A_{k'} B_{k'} k' dz$$
(35)

$$\Phi_H(\mathbf{k}') = \int_{-d}^0 \rho V(z) \frac{A_{k'}^2 + B_{k'}^2}{2} k' dz$$
(36)

and  $A_k$  and  $B_k$  are defined in (18) and (19).

**Table 1.** Value of U(z) for four spheroidal modes for the Earth model PREM. From the top row, values at ocean surface, at midpoint and at ocean bottom are shown. Normalization of eigenfunctions is done with certain units and thus only relative variations with depth should be examined. The ocean depth of PREM is 3 km, but if the oceans are shallower, the constancy of U in the ocean extends to higher frequencies.

Radius (km)	${}_{0}S_{6}$ (1.06 mHz)	$_0S_{44}$ (5.07 mHz)	$_0S_{98}$ (10.05 mHz)	<sub>0</sub> S <sub>202</sub> (20 mHz)
6371.0 6369.5	$1.34e^{-6}$ $1.34e^{-6}$	$4.76e^{-6}$ $4.76e^{-6}$	$6.38e^{-6}$ $6.37e^{-6}$	$8.73e^{-6}$ $8.66e^{-6}$
6368.0	$1.34e^{-6}$	$4.76e^{-6}$	$6.37e^{-6}$	$8.47e^{-6}$

#### 4.2 Removal of step function source term

Because we used the Laplace transformation that starts at t = 0, there is an effect of turning-on the ocean wavefield at this time. The first term in (34) corresponds to this effect. As one can see in its time domain form

$$\mathbf{u}(t) = \frac{1}{(2\pi)^2} \int d\mathbf{k} \, R(k) \, \int d\mathbf{k}' \frac{1 - \cos \omega t}{\omega^2} \, L_x L_y \, \left\{ \frac{\rho \omega'^2 U(0)}{2} - \Phi_Z(\mathbf{k}') \right\} a(\mathbf{k}')^2, \tag{37}$$

this is a case of a step function source, similar to the case shown in Gilbert (1970).

Another peculiar aspect of this term is that it is generated by ocean waves propagating in the same direction, as  $a(\mathbf{k}')^2$  implies. Other terms in (34) are caused by colliding ocean waves as they are proportional to  $a(\mathbf{k}') a(-\mathbf{k}')$ .

For low frequency modes, the vertical eigenfunction U(z) tends to become constant within the ocean (Table 1). In this limit, the quantity in the braces { $\rho \omega'^2 U(0)/2 - \Phi_Z(\mathbf{k}')$ } approaches zero. Therefore, we may drop this term for low frequency modes, but in general it is not zero.

Nonetheless, we exclude this term from the following discussion, because eq. (37) is not related to continuously interacting ocean waves but is the result of turning on the system at t = 0.

#### 4.3 Convolution form and radiation pattern

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After dropping the first term, the time domain solution for eq. (34) may be written in the form of convolution:

$$\mathbf{u}(t) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\mathbf{k} \ R(k) \int_{-\infty}^{t} d\tau \int_{-\infty}^{\infty} d\mathbf{k}' \frac{\sin \omega (t-\tau)}{\omega} \\ L_x L_y \left[ \left\{ \frac{\rho \omega'^2 U(0)}{2} + \Phi_Z(\mathbf{k}') \right\} \cos(2\omega'\tau) + i \left\{ \frac{\rho g k' V(0)}{2} + \Phi_H(\mathbf{k}') \right\} \cos \Psi \sin(2\omega'\tau) \right] a(\mathbf{k}') a(-\mathbf{k}'),$$
(38)

where the lower limit of integration for  $\tau$  is extended to  $-\infty$  from 0. We also defined in the second term

$$\cos \Psi = \frac{k_x k'_x}{k k'} + \frac{k_y k_y}{k k'}.$$
(39)

This term means the existence of directionality in radiated wavefields (radiation pattern) due to horizontal forcing terms. This radiation pattern is sketched in Fig. 1. The angle  $\Psi$  is measured from the direction of colliding ocean waves and seismic amplitudes reach their maximum in this direction and become zero in the perpendicular direction. In real data, contributions from vertical forcing term, which is isotropic, are superimposed on this pattern.

While we can write the solution in a different form, eq. (38) seems to depict the physics of our problem quite well. It shows that, of all the ocean waves, the main contribution to seismic wavefield comes from colliding ocean waves as expressed in  $a(\mathbf{k}') a(-\mathbf{k}')$ . Each collision of ocean waves at  $\tau$  makes contribution to  $\mathbf{u}(t)$  through the form of convolution as any continuously acting sources should show. And the excitation occurs at twice the frequency of colliding ocean waves as  $2\omega'$  indicates. Source of excitation consists of two different forcing terms, one associated with the vertical forcing term, proportional to  $\rho \omega'^2 U(0)/2 + \Phi_Z(\mathbf{k}')$  and the other with the horizontal forcing term, proportional to  $\rho gk' V(0)/2 + \Phi_H(\mathbf{k}')$ . The first term is related to the celebrated Longuet-Higgins pressure term and the second is related to the horizontal forcing term which has not been paid much attention so far. These two terms are discussed in the next two sections.

#### **5 RELATION TO LONGUET-HIGGINS' FORMULA**

Longuet-Higgins (1950) showed that, even if ocean waves are confined to near-surface depth range, the interaction of such waves can generate pressure fluctuation at the ocean bottom. In our notation, it is given by

$$P_L(t) = \frac{1}{2}\rho \frac{\partial^2}{\partial t^2} \overline{\zeta^2}.$$

## **Radiation Pattern**



Figure 1. Leading order terms for the excitation come from colliding ocean waves. Their directions are shown by arrows. Radiation pattern with the cosine dependence arises in the wavefield due to the horizontal forcing term. The maximum direction is along the direction of ocean-wave collision. In reality, contributions from vertical forcing term, which is isotropic, is superimposed on this pattern.

Here the bar indicates an averaging process over a wavelength.

To first order in O(a), we have

$$\zeta(x, y, t) = \int d\mathbf{k} \, a(\mathbf{k}) \sin(\omega t - \mathbf{k} \cdot \mathbf{x}), \tag{41}$$

and this is sufficient to evaluate (40) up to  $O(a^2)$ . Analysis on  $\overline{\zeta^2}$  leads to

$$P_{L}(t) = \frac{\rho}{4} \frac{\partial^{2}}{\partial t^{2}} \int d\mathbf{k} \left[ a(\mathbf{k})^{2} - a(\mathbf{k})a(-\mathbf{k})\cos(2\omega t) \right]$$
  
= 
$$\int d\mathbf{k} \, \rho \omega^{2} a(\mathbf{k})a(-\mathbf{k})\cos(2\omega t).$$
(42)

Let us examine the vertical forcing term in (38). For most seismic modes in the frequency band of our interest (Rayleigh waves frequencies up to about 100 mHz), values of vertical eigenfunction U(z) in the ocean are close to a constant. Table 1 shows examples from the Preliminary Reference Earth Model (Dziewonski & Anderson 1981). Values of U at three depths in the ocean, the ocean surface, the midpoint and the ocean bottom, are shown in this table. For frequencies below 10 mHz, U is basically constant throughout the ocean. Deviations from a constant U occur at higher frequencies but they remain relatively small. For shallower oceans, this constancy of U extends to even higher frequencies.

Let us assume this constancy of U in the ocean and denote U by  $\overline{U}$ . Obviously,  $\overline{U}$  is also equal to U(0). We can carry out the integration in (35):

$$\Phi_Z = \int_{-d}^0 \rho \overline{U} A_{k'} B_{k'} k' dz = \frac{\rho {\omega'}^2}{2} \overline{U},\tag{43}$$

where we used the dispersion relation  $\omega'^2 = gk' \tanh(k'd)$ .

Vertical component term in eq. (38) can then be written

$$\mathbf{u}(t) = -\frac{1}{(2\pi)^2} \int d\mathbf{k} \, R(k) \, \int_{-\infty}^t d\tau \int d\mathbf{k}' \, \frac{\sin \omega(t-\tau)}{\omega} \, L_x L_y \rho \omega'^2 \, \overline{U} \cos(2\omega'\tau) \, a(\mathbf{k}') a(-\mathbf{k}')$$

$$= -\frac{1}{(2\pi)^2} \int d\mathbf{k} \, R(k) \, \int_{-\infty}^t d\tau \, \frac{\sin \omega(t-\tau)}{\omega} L_x L_y \, P_L(\tau) \, \overline{U}.$$
(44)

This is nothing but the normal mode excitation formula by a (negative) vertical force when the force is given by  $L_x L_y P_L(\tau)$ , that is the area  $(L_x L_y)$  times the Longuet-Higgins pressure term.

## 6 IMPORTANCE OF HORIZONTAL FORCING TERM

Confirmation of the Longuet-Higgins pressure term is reassuring about the results of our results. However, our solution contains a novel aspect: the horizontal forcing term in eq. (38). A natural question is, how important is this term?

We numerically evaluated the ratio  $(\rho g k' V(0)/2 + \Phi_H)/(\rho \omega'^2 U(0)/2 + \Phi_Z)$  for various ocean depths. Below the ocean bottom, we assumed the same structure with PREM. Fig. 2 shows the numerical results for three different ocean depths, 0.5, 1 and 3 km. Abscissa is frequency and ordinate is the relative size. For a given angular frequency  $\omega$ , we picked  $\omega' = \omega/2$  as the frequency of ocean waves and evaluated the above ratio, because  $2\omega'$  is the frequency of generated noise by colliding ocean waves.



Figure 2. Relative importance of the horizontal to vertical terms. Three different cases of ocean depths, 0.5, 1.0 and 3.0 km, are shown. The ratio increases for lower frequencies below about 5 mHz. General patterns in the figure move to higher frequencies (right) for shallower oceans.

The importance of this horizontal term arises only at low frequencies, typically starting at about 10 mHz and becoming more and more significant below 5 mHz. For shallow oceans with 0.5 km depth, this term becomes more important than the vertical term at 3–4 mHz (note that the left end of this figure is 1 mHz).

The results in Fig. 2 suggest that the importance of this horizontal forcing term is probably negligible for microseisms whose frequency range is about 0.05–0.4 Hz. There is still a slight possibility that it may become important at extremely shallow ocean depths.

On the other hand, the horizontal forcing term is most likely quite important for the excitation of the hum whose frequency range is 2-7 mHz. In fact, the results in Fig. 2 imply that quantitative analysis of the hum may not be complete unless this term is included. We will examine this point in our future contribution.

#### 7 CONCLUSION

Excitation of seismic noise by non-linear interaction of ocean waves was formulated using the normal mode theory. Equivalent body/surface force terms that represent non-linearly interacting ocean waves were derived and used for evaluating normal-mode excitation process. Leading order terms were derived by using approximations suitable for the Earth's normal modes and ocean waves.

We found two notable features in our solution; first, the celebrated Longuet-Higgins pressure term naturally emerges from the vertical forcing term in the excitation coefficient. Second, the excitation coefficient contains an additional term related to horizontal forcing. This term introduces the radiation pattern with the cosine dependence; the maximum amplitude occurs in the direction of ocean-wave collision and the minimum (zero) in the perpendicular direction. The importance of this horizontal forcing term with respect to the vertical forcing term changes with ocean depths but generally this horizontal term becomes important for frequencies below 5 mHz. Inclusion of this term may be important for quantitative understanding of the hum.

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## APPENDIX A: APPENDIX

Stoker, J.J., 1957. Water Waves, Wiley Interscience, New York.

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The aim of this appendix is to provide a basis for the source terms in eq. (1). Our derivation is based on the analysis of the vertical momentum equation. The critical point is that the Longuet-Higgins pressure term consists of contributions from the non-linear volumetric term in the Navier–Stokes equation and the near-surface acceleration term. As it turns out, both contribute equally to the double-frequency term.

In-depth analyses of similar problems were performed by Longuet-Higgins (1950), Hasselmann (1963) and Phillips (1977), and our derivation closely follows that of Phillips (1976) which contains the analysis for the vertical momentum equation and some clever tricks in manipulating equations. However, we do not know of any specific statement on the body/surface force equivalent that we identified and used in this paper. The formula may be useful for quantitative evaluation of body waves, surface waves, and free oscillations generated by non-linear interactions of ocean waves.

#### A1 Vertical momentum equation

The equations of motion in the Cartesian coordinate are

$$\rho \frac{\partial v_i}{\partial t} + \rho \left( v_j \frac{\partial}{\partial x_j} \right) v_i = -\frac{\partial p}{\partial x_i} - \rho g \delta_{iz},\tag{A1}$$

where *i* is *x*, *y* or *z* and the summation convention for a repeated index *j* is assumed. We take the *x*- and *y*-axes in the horizontal plane and the vertical *z*-axis positive upward. *g* is the gravitational acceleration and  $\delta_{iz}$  is the Kronecker's delta.

We focus on the vertical component. The ocean bottom is z = -d and the ocean surface is  $z = \zeta(x, y, t)$ . The latter changes with time and space, although its spatial average is zero ( $\overline{\zeta} = 0$ ). Writing three components of velocity  $\mathbf{v}_{oc} = (u, v, w)$ , we can write

$$\rho \frac{\partial w}{\partial t} + \rho \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} - \rho g.$$
(A2)

For simplicity, we assume that density ( $\rho$ ) is constant. By integrating (A2) from the ocean bottom to the ocean surface, we get

$$p_{b} = p_{a} + \rho g(\zeta + d) + \rho \int_{-d}^{\zeta} \frac{\partial w}{\partial t} dz + \rho \int_{-d}^{\zeta} \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) dz,$$
(A3)

where  $p_b$  is pressure at the ocean bottom  $[p_b = p(z = -d)]$  and  $p_a$  is the atmospheric pressure.

The boundary condition at the surface is the kinematic boundary condition

$$w = \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y}$$
(A4)

and the boundary condition at the ocean bottom is w = 0 (the rigid boundary condition).

We first examine the formula for average pressure at ocean bottom and evaluate the contributions from each term. For this purpose, we take a horizontal average (in the x - y plane) of (A3).

$$\overline{p_b} = p_a + \rho_0 g d + \rho \overline{\int_{-d}^{\zeta} \frac{\partial w}{\partial t} dz} + \rho \overline{\int_{-d}^{\zeta} \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) dz}$$
(A5)

where bars denote spatial averaging in the horizontal (x - y) plane and  $\overline{\zeta} = 0$ . We next analyse the third and fourth terms in (A5) using formulae (16)–(19). They are referred to as the vertical acceleration term and the non-linear advection term.

#### A2 Vertical acceleration term

Before the averaging, the vertical acceleration term is

$$\rho \int_{-d}^{\zeta} \frac{\partial w}{\partial t} dz = -\rho \int d\mathbf{k} \frac{\omega^2}{k} \frac{\cosh\{k(\zeta+d)\} - 1}{\sinh(kd)} a(\mathbf{k}) \sin(\omega t - \mathbf{k} \cdot \mathbf{x}).$$
(A6)

Here we simply substituted relations from (16)-(19) and performed vertical integration. Noting that

$$\cosh\{k(\zeta+d)\} \approx \cosh(kd) + \sinh(kd) \cdot k\zeta + O(a^2),\tag{A7}$$

we can write terms of order O(a) and  $O(a^2)$  in (A6) as

$$O(a): -\rho \int d\mathbf{k} \frac{\omega^2}{k} \frac{\cosh(kd) - 1}{\sinh(kd)} a(\mathbf{k}) \sin(\omega t - \mathbf{k} \cdot \mathbf{x})$$
(A8)

$$O(a^{2}) : -\rho \int d\mathbf{k} \int d\mathbf{k}' \omega^{2} a(\mathbf{k}) a(\mathbf{k}') \sin(\omega t - \mathbf{k} \cdot \mathbf{x}) \sin(\omega' t - \mathbf{k}' \cdot \mathbf{x})$$

$$= -\rho \int d\mathbf{k} \int d\mathbf{k}' \omega^{2} a(\mathbf{k}) a(\mathbf{k}')$$

$$\times \frac{1}{2} \left[ \cos \left\{ (\omega - \omega')t - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{x} \right\} - \cos \{ (\omega + \omega')t - (\mathbf{k} + \mathbf{k}') \cdot \mathbf{x} \right\} \right].$$
(A9)

The term with O(a) disappears when the horizontal averaging is applied because of its dependence on a single sinusoidal function. The lowest order contributions to pressure come from terms of  $O(a^2)$  and they occur either when  $\mathbf{k}' = \mathbf{k}$  or when  $\mathbf{k}' = -\mathbf{k}$  (in both cases  $\omega = \omega'$ ). Averaging (A6) over horizontal spatial coordinates, we get

$$\rho \int_{-d}^{\zeta} \frac{\partial w}{\partial t} dz = \int d\mathbf{k} \, \frac{\rho \omega^2}{2} \left\{ -a(\mathbf{k})^2 + a(\mathbf{k})a(-\mathbf{k})\cos(2\omega t) \right\}. \tag{A10}$$

#### A3 Non-linear advection term

Using a similar approach, the average integral for the non-linear advection term in (A5) can be evaluated. There are some minor differences, however; for example, it is permitted to use the integration range from z = -d to z = 0 instead of up to  $z = \zeta$  for these terms, because the integrand is already  $O(a^2)$ . The difference between z = 0 and  $z = \zeta$  only causes differences of higher order. From the first two terms, we get

$$\rho \overline{\int_{-d}^{\zeta} \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} \right) dz} = \int d\mathbf{k} \frac{\rho \omega^2}{4} \left\{ a(\mathbf{k})^2 + a(\mathbf{k})a(-\mathbf{k})\cos(2\omega t) \right\}$$
(A11)

and from the third term, we get

$$\rho \int_{-d}^{\zeta} w \frac{\partial w}{\partial z} dz = \int d\mathbf{k} \, \frac{\rho \omega^2}{4} \left\{ a(\mathbf{k})^2 + a(\mathbf{k})a(-\mathbf{k})\cos(2\omega t) \right\}. \tag{A12}$$

There are equal contributions from (A11) and (A12). The sum is

$$\rho \overline{\int_{-d}^{\zeta} \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) dz} = \int d\mathbf{k} \frac{\rho \omega^2}{2} \left\{ a(\mathbf{k})^2 + a(\mathbf{k})a(-\mathbf{k})\cos(2\omega t) \right\}.$$
(A13)

#### A4 Sum

Substitution of (A10) and (A13) in (A5) yields

$$\overline{p_b} = p_a + \rho_0 g d + \int d\mathbf{k} \,\rho \omega^2 \,a(\mathbf{k})a(-\mathbf{k})\cos(2\omega t). \tag{A14}$$

The third term in (A14) is equivalent to the Longuet-Higgins pressure formula as we saw in Section 5. We note that there are equal contributions from the vertical acceleration term and the non-linear advection term to the Longuet-Higgins pressure term.

#### A5 Body/surface force equivalent term

We note from the derivations in (A2) that, up to  $O(a^2)$ ,

$$\rho \overline{\int_{-d}^{\zeta} \frac{\partial w}{\partial t} dz} \approx \rho \overline{\zeta \left[\frac{\partial w}{\partial t}\right]_{z=0}}.$$
(A15)

Even though the integration on the left-hand side is over the entire depth range of ocean, the main contribution can be approximated as the surface contribution. This is under the proviso that only the terms up to  $O(a^2)$  are kept. This was pointed out by Phillips (1976) and was used extensively in his derivations.

One can then conjecture from (A15) and write, by removing the horizontal averaging process and introducing the delta function at the (average) surface, that the body/surface force equivalent formula for non-linearly interacting ocean waves may be written as

$$\mathbf{f} = -\rho \zeta \left[ \frac{\partial \mathbf{v}_{oc}}{\partial t} \right] \delta(z) - (\mathbf{v}_{oc} \cdot \nabla) \mathbf{v}_{oc}, \tag{A16}$$

where the first term is basically the surface force and the second term is the volumetric force spread over the depth range of ocean.

In (A16), we have extended the formula from a single vertical component to all three components. This is permissible because horizontal terms have the same form. Eq. (A16) is the main conclusion of this appendix. We particularly note the importance of the surface acceleration term, without which one cannot recover the average pressure (the Longuet-Higgins pressure) at ocean bottom correctly.

By using this body/surface force equivalent formula, excitation of seismic waves can be evaluated by solving

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathcal{H}(\mathbf{v}) + \mathbf{f},\tag{A17}$$

where v is velocity of seismic waves and  $\mathcal{H}$  is the elastic-gravitational operator (e.g. Dahlen & Tromp 1998).

One may question the removal of first-order terms (O(a)) in deriving (A15) as the final formula (A16) does not contain horizontal averaging process. Justification comes from the fact that first-order terms consist of linear ocean waves that are part of the eigenfunctions of the Earth. For a spherically layered Earth model, they are orthogonal to seismic normal modes and thus should not contribute to the excitation of seismic signals in the solid Earth. Any deviation from this layered-structure assumption, such as topographic and seismic velocity variations, may invalidate this claim. However, for long-period seismic waves, the layered earth model is a very good assumption and removal of first-order terms should be justified.

Lastly, we remind readers that our argument is valid only when non-linearity of ocean-wave effects are sufficiently small that collection of terms up to  $O(a^2)$  is enough to represent a solution. Within this limitation, the body/surface force equivalent is given by eq. (A16).