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Elastic Waves at the Surface of Separation of Two Solids.

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§ 1. *Introduction:*

In considering how the energy of a seismic disturbance is dissipated one is led to enquire into the possibility of the existence of waves, analogous to Rayleigh waves and Love waves, that are propagated in the interior of the earth along the junction of strata, or chiefly within a certain stratum, so that the energy is dissipated by internal viscosity without the occurrence of any appreciable surface displacement.

Two surfaces of discontinuity of density and elastic properties are commonly believed to exist below that part of the earth's crust which is accessible to geologists, namely, the junction of the granitic layer with the basic rocks, and the surface of separation of the Wiechert metallic core from the rocky shell. It becomes of interest to examine whether a wave of the Rayleigh type can be propagated along such an interface ; an enquiry may also be made into the circumstances in which a wave of the Love type may exist if a stratum of uniform thickness is bounded on both sides by very deep layers of different materials.

It has been pointed out to me by Dr. Harold Jeffreys that the former problem is in some respects a particular case of Prof. Love's discussion* of the effect of a surface layer on the propagation of Rayleigh waves; the "layer" is here taken as of infinite thickness. Whereas, however, Prof. Love's problem is concerned with a disturbance confined chiefly to the free surface, the present paper deals with a wave motion that is greatest at the surface of separation of the two media, and is not restricted to the case of incompressible solids. Some simplification is effected, however, when the media are taken as incompressible, and several such particular cases have, accordingly, been solved in detail; these throw some light on the general problem, and, in fact, suggested in the first place the investigation of § 3.

Generalised Rayleigh Wave.

§ 2. *The Wave-Velocity Equation.*

The two media will be distinguished by suffixes 1 and 2, and will be supposed in "welded" contact along an infinite plane face, and otherwise

* 'Geodynamics,' p. 163.

extending to infinity, so that there is no slipping at the interface, in which an origin and a set of axes of x and y are taken; the axis of z is drawn positively into the medium 1. Let ρ denote the density and λ, μ the usual elastic constants.

If for a wave propagated in the x direction we assume for the displacement in medium 1 a solution of the type $(U_1, V_1, W_1)\, e^{i\kappa(x-ct)}$, where U_1, V_1, W_1 are functions of z only, we find in the usual way* that a solution tending to zero at infinite distance from $z = 0$ is given by

$$(U_1, V_1, W_1) = (U_1', V_1', W_1') + (U_1'', V_1'', W_1''), \tag{1}$$

where

$$(U_1', V_1', W_1') = -\frac{\lambda_1 + 2\mu_1}{\rho_1\kappa^2c^2}(i\kappa, 0, -r_1)\, D_1 e^{-r_1 z}, \tag{2}$$

$$(U_1'', V_1'', W_1'') = (s_1, a_1, i\kappa)\, Q_1 e^{-s_1 z}, \tag{3}$$

in which D_1, Q_1, a_1, are constants,

$$r_1^2 = \kappa^2\left(1 - \frac{\rho_1 c^2}{\lambda + 2\mu}\right). \tag{4}$$

$$s_1^2 = \kappa^2\left(1 - \frac{\rho_1 c^2}{\mu_1}\right). \tag{5}$$

In medium 2 we find in a similar manner, for a disturbance which becomes insensible at a great distance from $z = 0$,

$$(U_2, V_2, W_2) = (U_2', V_2', W_2') + (U_2'', V_2'', W_2''), \tag{6}$$

where

$$(U_2', V_2', W_2') = -\frac{\lambda_2 + 2\mu_2}{\rho_2\kappa^2c^2}(i\kappa, 0, r_2)\, E e^{r_2 z}, \tag{7}$$

$$(U_2'', V_2'', W_2'') = (s_2, a_2, i\kappa)\, Q_2 e^{s_2 z}, \tag{8}$$

in which E, Q_2, a_2 are constants,

$$r_2^2 = \kappa^2\left(1 - \frac{\rho_2 c^2}{\lambda_2 + 2\mu_2}\right) \tag{9}$$

and

$$s_2^2 = \kappa^2\left(1 - \frac{\rho_2 c^2}{\mu_2}\right). \tag{10}$$

At the bounding surface the displacements and stresses are supposed continuous. Thus, if (u_1, v_1, w_1), (u_2, v_2, w_2) are the displacements in the two media, we have

$$\mu_1\left(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x}\right) = \mu_2\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\right) \tag{11}$$

$$\mu_1\left(\frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial y}\right) = \mu_2\left(\frac{\partial v_2}{\partial z} + \frac{\partial w_2}{\partial y}\right) \tag{12}$$

$$\lambda_1\Delta_1 + 2\mu\frac{\partial w_1}{\partial z} = \lambda_2\Delta_2 + 2\mu_2\frac{\partial w_2}{\partial z}, \tag{13}$$

* Love: 'Elasticity,' 3rd Edition, p. 311, *et seq.* The notation is based on that of Jeffreys, 'The Earth,' p. 157.

where Δ is the dilatation,

$$u_1 = u_2 \tag{14}$$

$$v_1 = v_2 \tag{15}$$

$$w_1 = w_2 \tag{16}$$

at the boundary, which to the first order of small quantities may be taken as $z = 0$.

The equations (15) and (12) give respectively

$$a_1 Q_1 = a_2 Q_2\,; \quad -\mu_1 a_1 Q_1 s_1 = \mu_2 a_2 Q_2 s_2, \quad \text{whence} \quad a_1 = a_2 = 0,$$

and therefore

$$v_1 = v_2 = 0. \tag{17}$$

Write

$$\left.\begin{aligned}(\lambda_1 + 2\mu_1)/\rho_1 = \alpha_1^2\,; \ \mu_1/\rho_1 = \beta_1^2\\ (\lambda_2 + 2\mu_2)/\rho_2 = \alpha_2^2\,; \ \mu_2/\rho_2 = \beta_2^2\end{aligned}\right\}; \tag{18}$$

then, on substituting in (11), (13), (14), (16) the values of the displacements given by (1) and (6), we obtain

$$2\mu_1\left(1-\frac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}}\frac{\alpha_1^2}{c^2}D + 2\mu_2\left(1-\frac{c^2}{\alpha_2^2}\right)^{\frac{1}{2}}\frac{\alpha_2^2}{c^2}E + \mu_1\left(2-\frac{c^2}{\beta_1^2}\right)i\kappa^2 Q_1 + \mu_2\left(2-\frac{c^2}{\beta_2^2}\right)i\kappa^2 Q_2 = 0 \tag{19}$$

$$(\lambda_1 + 2\mu_1)\left(1-\frac{2\beta_1^2}{c^2}\right)D - (\lambda_2 + 2\mu_2)\left(1-\frac{2\beta_2^2}{c^2}\right)E - 2\mu_1\left(1-\frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}}i\kappa^2 Q_1 + 2\mu_2\left(1-\frac{c^2}{\beta_2^2}\right)^{\frac{1}{2}}i\kappa^2 Q_2 = 0. \tag{20}$$

$$\frac{\alpha_1^2}{c^2}D - \frac{\alpha_2^2}{c^2}E + \left(1-\frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}}i\kappa^2 Q_1 - \left(1-\frac{c^2}{\beta_2^2}\right)^{\frac{1}{2}}i\kappa^2 Q_2 = 0, \tag{21}$$

$$\frac{\alpha_1^2}{c^2}\left(1-\frac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}}D + \frac{\alpha^2}{c^2}\left(1-\frac{c^2}{\alpha_2^2}\right)^{\frac{1}{2}}E + i\kappa^2 Q_1 + i\kappa^2 Q_2 = 0. \tag{22}$$

Eliminating D, E, $i\kappa^2 Q_1$, $i\kappa^2 Q_2$ from these four equations, and substituting for μ_1, μ_2, $(\lambda_1 + 2\mu_1)$, $(\lambda_2 + 2\mu_2)$, we obtain as an equation for the wave-velocity c

$$\begin{vmatrix} 2\rho_1\beta_1^2\left(1-\frac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}}; & 2\rho_2\beta_2^2\left(1-\frac{c^2}{\alpha_2^2}\right)^{\frac{1}{2}}; & \rho_1\beta_1^2\left(2-\frac{c^2}{\beta_1^2}\right); & \rho_2\beta_2^2\left(2-\frac{c^2}{\beta_2^2}\right) \\ \rho_1(c^2-2\beta_1^2); & -\rho_2(c^2-2\beta_2^2); & -2\rho_1\beta_1^2\left(1-\frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}}; & 2\rho_2\beta_2^2\left(1-\frac{c^2}{\beta_2^2}\right)^{\frac{1}{2}} \\ 1; & -1; & \left(1-\frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}}; & -\left(1-\frac{c^2}{\beta_2^2}\right)^{\frac{1}{2}} \\ \left(1-\frac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}}; & \left(1-\frac{c^2}{\alpha_2^2}\right)^{\frac{1}{2}}; & 1; & 1 \end{vmatrix} = 0. \tag{23}$$

§3. *Compressible Media.*

On writing
$$A_1 = \left(1 - \frac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}}; \ A_2 = \left(1 - \frac{c^2}{\alpha_2^2}\right)^{\frac{1}{2}};$$
$$B_1 = \left(1 - \frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}}; \ B_2 = \left(1 - \frac{c^2}{\beta_2^2}\right)^{\frac{1}{2}};$$
$$K = 2\,(\rho_1\beta_1^2 - \rho_2\beta_2^2),$$
the equation (23) of the preceding section reduces to
$$c^4\{(\rho_1-\rho_2)^2 - (\rho_1 A_2 + \rho_2 A_1)(\rho_1 B_2 + \rho_2 B_1)\} + 2Kc^2\{\rho_1 A_2 B_2 - \rho_2 A_1 B_1 - \rho_1 + \rho_2\} + K^2(A_1B_1 - 1)(A_2B_2 - 1) = 0. \quad (1)$$

It is easy to show that when $\rho_2 = 0$ this equation reduces to the ordinary equation for Rayleigh waves
$$\left\{\frac{c^2}{\beta_1^2} - 2\right\}^2 = 4\left\{1 - \frac{c^2}{\alpha_1^2}\right\}^{\frac{1}{2}}\left\{1 - \frac{c^2}{\beta_1^2}\right\}^{\frac{1}{2}}. \quad (2)$$

On account of the ambiguity introduced by squaring both sides of an equation it is advisable, where practicable, to work with the equation (1) as it stands, not attempting to rationalise.

An important particular case of (1) arises when $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. These conditions appear to be satisfied at the Wiechert surface of discontinuity within the earth, as may be seen from Knott's table of wave-velocities.* If we write
$$\alpha_1 = \alpha_2 = \alpha; \quad A_1 = A_2 = A; \quad \beta_1 = \beta_2 = \beta; \quad B_1 = B_2 = B;$$
$$\gamma = \beta^2/\alpha^2; \quad c^2/\beta^2 = x,$$
we obtain
$$f(x) \equiv x^2\{(\rho_1-\rho_2)^2 - (\rho_1+\rho_2)^2(1-x)^{\frac{1}{2}}(1-\gamma x)^{\frac{1}{2}}\} + 4(\rho_1-\rho_2)^2\{(1-x)(2-\gamma x) + (1-\gamma x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}(x-2)\} = 0. \quad (3)$$

We see at once that $f(1) = (\rho_1 - \rho_2)^2$, which is positive. $f(0)$ is equal to zero, implying that some power of x is a factor of $f(x)$. If x is small, so that A and B may be expanded as power series in x, we find in fact $f(x)/x^2 = -(\rho_1+\rho_2)^2 + \gamma^2(\rho_1-\rho_2)^2 +$ terms containing x as a factor; thus the limiting value of $f(x)/x^2$ as x tends to zero is $-(\rho_1+\rho_2)^2 + \gamma(\rho_1-\rho_2)^2$. But $\rho_1+\rho_2 > |\rho_1-\rho_2|$, and $\gamma < 1$, so that $(\rho_1+\rho_2)^2 > \gamma^2(\rho_1-\rho_2)^2$, and accordingly $\{f(x)/x^2\}_{x=0}$ is negative, while $\{f(x)/x^2\}_{x=1}$ is positive. Thus

* 'Roy. Soc. Proc., Edin.,' vol. 39 (2), p. 168 (1918–1919).

a root of $f(x)/x^2 = 0$ always exists between $x = 0$ and $x = 1$, in other words, a wave can be propagated along the interface with a velocity less than the common velocity of distortional waves.

We can deduce that a Rayleigh wave must always exist at the free surface of an infinite solid. The direct proof is, however, very simple, for the ordinary equation, obtained by squaring (2) may be written

$$x^3 - 8x^2 + 24x - 16\gamma x - 16 + 16\gamma = 0. \qquad (4)$$

When $x = 0$ and 1 the left-hand side takes the values $16(\gamma - 1)$ and 1 respectively, so that there is a root between 0 and 1. The value so obtained is either a solution of (2) or else a solution of that equation with the right-hand side changed in sign; since the left-hand side is essentially positive the latter possibility is ruled out, and thus a Rayleigh wave always exists.

In Lord Rayleigh's original paper* the numerical solutions given are found from a rationalised equation, and the fact that these must satisfy an equation equivalent to (2) is not explicitly stated.

As a check on the preceding work we may put $\rho_1 = \rho_2$, so that the two media are the same; the equation (3) now gives $x = 1$ or $1/\gamma$, corresponding to $c = \beta$ or $c = \alpha$, as would be expected.

Analogy with the incompressible case (§ 4) suggests that in the general problem a wave does not necessarily exist.

It will be supposed that α_1 and β_1 do not differ greatly from α_2 and β_2 respectively; in these circumstances a wave motion of the assumed type may be shown to exist if the differences are small enough. Put

$$1/\alpha_2^2 = (1 - m)/\alpha_1^2; \quad 1/\beta_2^2 = (1 - n)/\beta_1^2; \quad \rho_2/\rho_1 = \sigma. \qquad (5)$$

Then equation (1) gives

$$\begin{aligned} f(x) \equiv x^2 [(1-\sigma)^2 - \{(1 - \gamma(1-m)x)^{\frac{1}{2}} + \sigma(1-\gamma x)^{\frac{1}{2}}\} \\ \{(1 - (1-n)x)^{\frac{1}{2}} + \sigma(1-x)^{\frac{1}{2}}\}] \\ + 4x\left(1 - \frac{\sigma}{1-n}\right)\{(1 - \gamma x(1-m))^{\frac{1}{2}}(1 - x(1-n))^{\frac{1}{2}} \\ - \sigma(1-\gamma x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}} + \sigma - 1\} \\ + 4\left(1 - \frac{\sigma}{1-n}\right)^2 \{(1 - \gamma x(1-m))^{\frac{1}{2}}(1 - x(1-n))^{\frac{1}{2}} - 1\} \\ \{(1-\gamma x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}} - 1\} = 0. \qquad (6) \end{aligned}$$

It is found that $f(1)$ is equal to

$$(1-\sigma)^2 - n^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}\{4\sigma^2 - 3\sigma + 1\},$$

together with terms of degree n, $n^{3/2}$, $mn^{1/2}$, and higher orders.

* 'Proc. Lond. Math. Soc.,' vol. 17, p. 7 (1885).

Whatever the value of σ, then, by taking n sufficiently small we can ensure that $f(1)$ is positive.

Taking now the case of x small, we have

$$\begin{aligned} f(x)/x^2 = &- 4\sigma - (1-\sigma)^2(1-\gamma^2) \\ &+ (1-\sigma)\left[n(1+\sigma-\gamma\sigma-\gamma-2\gamma^2\sigma) + m\gamma(1-\gamma+\sigma+\sigma\gamma)\right] \\ &+ \text{terms in } x;\ x^2 \ldots \text{etc.,} \end{aligned}$$

which can always be made negative by taking m and n sufficiently small. Thus, we can definitely assert that when the wave-velocities are not too widely different for the two media, a wave of the Rayleigh type can exist at the interface. If the wave velocity is not very different from β_1, we may use the value of $f(1)$ to obtain, for given values of σ and γ, the approximate limits of permissible variation of the quantity n; the variation of velocity of distortional waves thus appears to be of more importance than an equal variation in the velocity of compressional waves.

§ 4. *Incompressible Media.*

The results obtained in the preceding section are true in particular of incompressible media, when A_1 and A_2 both become unity. This assumption, however, considerably simplifies the numerical work in actual examples, several of which have been examined.

Equation (1) of § 3 now becomes, on squaring and rearranging,

$$2\left(1-\frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}}\left(1-\frac{c^2}{\beta_2^2}\right)^{\frac{1}{2}}\rho_1\rho_2(\rho_1+\rho_2)^2c^4 = Ac^6+Bc^4+Cc^2+D, \tag{1}$$

where

$$\left.\begin{aligned} A &= (\rho_1+\rho_2)^2\left\{\frac{\rho_2^2}{\beta_1^2}+\frac{\rho_1^2}{\beta_2^2}\right\}, \\ B &= 4(\rho_1+\rho_2)K\left\{\frac{\rho_2^2}{\beta_1^2}-\frac{\rho_1^2}{\beta_2^2}\right\} - 2\rho_1\rho_2(3\rho_1^2-2\rho_1\rho_2+3\rho_2^2), \\ C &= 2K^2\left\{\frac{\rho_2(\rho_1+3\rho_2)}{\beta_1^2}+\frac{\rho_1(3\rho_1+\rho_2)}{\beta_2^2}\right\} + 16K\rho_1\rho_2(\rho_1-\rho_2), \\ D &= 4K^2\left\{\frac{\rho_2K}{\beta_1^2}-\frac{\rho_1K}{\beta_2^2}-4\rho_1\rho_2\right\} + \frac{K^4}{\beta_1^2\beta_2^2}. \end{aligned}\right\} \tag{2}$$

This equation is rational when $\beta_1 = \beta_2 = \beta$, and it is easy to prove in this particular case the general result previously found, that a Rayleigh wave always exists. By putting $\rho_2 = 0$ we obtain the usual equation for the velocity of a Rayleigh wave at the surface of an incompressible solid. Writing $\rho_1 = 8 \cdot 2$ and $\rho_2 = 3 \cdot 2$, as in Wiechert's density law, we have

$$16889 \cdot 6 \left(\frac{c}{\beta}\right)^6 - 4225 \cdot 6 \left(\frac{c}{\beta}\right)^4 + 77976 \cdot 0 \left(\frac{c}{\beta}\right)^2 - 51984 \cdot 0 = 0, \tag{3}$$

which has a root corresponding to

$$c = 0 \cdot 99287 \beta, \tag{4}$$

so that a wave may be propagated along the bounding surface with a velocity very slightly less than that of distortional waves in either medium.

As pointed out by Rayleigh,* on general dynamical principles we shall not expect the complex roots to correspond to any real wave motion. We do, in fact, find for these complex roots

$$c^2/\beta^2 = 0 \cdot 75806 \pm 1 \cdot 57720 i. \tag{5}$$

On reintroducing s and κ, we may write the simplified equation (1) of § 3 in the form

$$\{(\rho_1 - \rho_2)(c^2 - 2\beta^2)\}^2 + 4 \frac{s^2}{\kappa^2} (\rho_1 - \rho_2)^2 \beta^4$$
$$= \frac{s}{\kappa} \{4 \rho_1 \rho_2 c^4 + (\rho_1 - \rho_2)^2 (c^2 - \beta^2)^2\}, \tag{6}$$

and substituting s^2/κ^2 from equation (5) of § 2 we have, on reduction

$$\frac{s}{\kappa} = \frac{0 \cdot 02261 \mp 5 \cdot 49478 i}{-10 \cdot 46019 \pm 9 \cdot 27612 i}. \tag{7}$$

On bringing this fraction to a real positive denominator it is found that the real part of the numerator is negative whether the upper or the lower sign be taken, and accordingly e^{-sz} and $e^{-\kappa z}$ cannot both tend to zero at infinity. Thus the complex roots are inadmissible.

It may be noted as a check that when $\rho_1 = \rho_2$, (6) can only be satisfied if $c = 0$, or if $s = 0$, the latter corresponding to $c = \beta$.

Equation (1) becomes on squaring

$$\left\{A^2 - \frac{4\rho_1^2\rho_2^2(\rho_1 + \rho_2)^4}{\beta_1^2\beta_2^2}\right\} c^{12} + \left\{2AB + \frac{4\rho_1^2\rho_2^2(\rho_1 + \rho_2)^4}{\beta_1^2\beta_2^2}(\beta_1^2 + \beta_2^2)\right\} c^{10}$$
$$+ \{B^2 + 2AC - 4\rho_1^2\rho_2^2(\rho_1 + \rho_2)^4\} c^8 + 2(BC + AD) c^6 + (C^2 + 2BD) c^4$$
$$+ 2CDc^2 + D^2 = 0. \tag{8}$$

From a consideration of particular cases it appears that solutions of this equation corresponding to real waves may not always exist. It is essential that c should be positive and less than both β_1 and β_2, otherwise s_1 and s_2

* *Loc. cit.*, p. 8; or 'Sci. Papers,' 2, p. 441.

will not both be real, and the disturbances will not be insensible at great distance from the surface of separation.

A very simple case arises when $\rho_1 = 2\rho_2$, $\beta_2{}^2 = 2\beta_1{}^2$, so that $K = 0$. Equation (8) then reduces to

$$81\,(c/\beta_1)^4 - 432\,(c/\beta_1)^2 + 640 = 0, \tag{9}$$

which has not real roots.

If $\rho_1 = 2$; $\rho_2 = 1$; $\beta_1{}^2 = 2\beta_2{}^2$, so that $\mu_1 = 4\mu_2$, equation (8) becomes

$$y^6 - 22\cdot1033y^5 + 199\cdot446y^4 - 970\cdot280y^3 + 2993\cdot20y^2 - 5006\cdot80y + 3265\cdot31 = 0, \tag{10}$$

where $$y = c^2/\beta_2{}^2.$$

This equation has no roots between $y = 0$ and $y = 1$.

By taking $\rho_1 = 2$; $\rho_2 = 1$; $\beta_2{}^2 = \frac{3}{2}\beta_1{}^2$, $x = c/\beta_1$, we obtain

$$x^{12} - 9\cdot17333x^{10} + 30\cdot24000x^8 - 44\cdot09481x^6 + 34\cdot94123x^4 - 17\cdot61580x^2 + 4\cdot74272 = 0, \tag{11}$$

which has two solutions between $x = 0$ and $x = 1$, viz.: $x^2 = 0\cdot9042$ and $x^2 = 0\cdot93615$, corresponding to the values $c = 0\cdot951\beta_1$ and $c = 0\cdot967\beta_1$. It would therefore appear at first sight that two waves, having different velocities, might exist. It is to be remembered, however, that the equation (1) of § 3 has been twice squared in reduction to the form (8). On actual substitution it is found that neither of these roots will satisfy (1), but that both satisfy the equation derived from (1) by changing the sign of one side.

That there should be an even number of roots of the wave-velocity equation for these values of ρ_1 and ρ_2, β_1/β_2, between $c = 0$ and $c = \beta_1$, can be seen by reverting to equation (1) of § 3, and writing $A_1 = A_2 = 1$. It is then found that $\{f(x)/x^2\}_{x=0}$ is equal to $-25/12$, in the notation of (3), § 3, and that $f(1)$ is $-(3/16)^{\frac{1}{2}}$.

It thus appears that a wave of the type under discussion will exist when the velocity of distortional waves in the two media is the same, but not if the wave-velocities are greatly different. The method of the preceding section may be applied to obtain a rough estimate of how widely β_1 and β_2 may differ for given values of ρ_1 and ρ_2. In the notation of § 3, (6), $f(1)$ is approximately $(1-\sigma)^2 - n^{\frac{1}{2}}\{4\sigma^2 - 3\sigma + 1\}$, and $\{f(x)/x^2\}_{x=0}$ is approximately $-(1+\sigma)^2 + n(1-\sigma^2)$; this may be regarded as a particular case of § 3, with $\gamma = 0$, or it may be obtained more simply from § 3, (1) by putting $A_1 = A_2 = 1$. It is seen that by making n sufficiently small, $f(x)/x^2$ can be made to change sign between $x = 0$ and $x = 1$. Moreover, if n is very small, $n^{\frac{1}{2}}$ is large compared with n, so that if $4\sigma^2 - 3\sigma + 1$ and $1 - \sigma^2$ are comparable, the range of variation of $f(x)/x^2$ is greater in the neighbourhood of $x = 1$ than at $x = 0$. If, then, we disregard the possibility that $\{f(x)/x^2\}_{x=0}$ may change its sign for a small variation of n, the condition that there shall be a root between $x = 0$ and $x = 1$ is that $n^{\frac{1}{2}}$ is not greater than

$$(1 - \sigma)^2/(4\sigma^2 - 3\sigma + 1),$$

for the denominator is always positive. In this way it is possible to obtain roughly the result of the preceding paragraph, but the approximation is too rough to be convincing. In other respects these waves conform to the Rayleigh type. When c is known, expressions may be written down for the displacements of any particle, and by corresponding reasoning it may be shown that particles describe ellipses about their mean positions.

Furthermore, this appears to be the only type of wave that can be propagated along the surface. It is easy to verify that transverse waves of the Love type cannot exist. The displacements in the two media would be

$$(0,\ A_1,\ 0)\, e^{-s_1 z}\, e^{i\kappa(x-ct)}$$

and

$$(0,\ A_2,\ 0)\, e^{+s_2 z}\, e^{i\kappa(x-ct)}\ ;$$

the conditions that the displacement and stress at $z = 0$ must be continuous would then give the incompatible relations

$$A_1 = A_2\ ; \quad -\mu_1 A_1 s_1 = \mu_2 A_2 s_2\ ; \quad \mu_1 A_1 = \mu_2 A_2.$$

The geophysical interest of this discussion is that in addition to that portion of the energy of an earthquake which is dissipated by solid friction before reaching the surface, a further fraction may be "trapped" by surfaces of discontinuity, and may involve a correction to estimates* of the energy involved in a seismic disturbance.

Transverse Waves in an Internal Stratum.

§ 5. *A Generalised Type of Love Wave.*

Suppose the medium (1) to extend from $z = \infty$ to $z = 0$, the medium (2) to extend from $z = 0$ to $z = -T$, and the medium (3) from $z = -T$ to $z = -\infty$. Then in any layer we may take as the components of displace-

* *e.g.*, see Jeffreys: 'M.N.R.A.S.,' *Geophys. Suppt.*, vol. I, 2, p. 22 (Jan. 1923).

ment (u, v, w) in a plane wave travelling in the direction of x increasing the real parts of

$$(0, \mathrm{V}, 0)\, e^{i\kappa(x-ct)} \tag{1}$$

where V is a function of z only.

It is at once verified that the dilatation is everywhere zero, and if ρ, μ denote respectively the density and the rigidity, the only equation to be satisfied is

$$\rho \frac{\partial^2 v}{\partial t^2} = \mu \nabla^2 v, \tag{2}$$

or, on substituting from (1),

$$\frac{d^2\mathrm{V}}{dz^2} - \left(1 - \frac{\rho c^2}{\mu}\right) \kappa^2 \mathrm{V} = 0. \tag{3}$$

For a type of wave in which V is periodic in medium (2) and exponential in (1) and (3) we have

$$c^2 > c_2{}^2; \quad c^2 < c_1{}^2; \quad c^2 < c_3{}^2, \tag{4}$$

where c_1, c_2, c_3 denote the velocities of distortional waves in the three media. We thus have

$$\left.\begin{aligned} \mathrm{V}_1 &= \mathrm{D}e^{-s_1 z} \\ \mathrm{V}_2 &= \mathrm{A} \cos s_2 z + \mathrm{B} \sin s_2 z \\ \mathrm{V}_3 &= \mathrm{E}e^{s_3 z} \end{aligned}\right\} \tag{5}$$

where A, B, D, E are constants,

$$s_1 = \kappa \left\{1 - \kappa \frac{c^2}{c_1{}^2}\right\}^{\frac{1}{2}}, \quad s_2 = \kappa \left\{\frac{c^2}{c_2{}^2} - 1\right\}^{\frac{1}{2}} \quad \text{and} \quad s_3 = \kappa \left\{1 - \frac{c^2}{c_3{}^2}\right\}^{\frac{1}{2}}.$$

The boundary conditions are that the displacement and stress are continuous at $z = 0$ and $z = -\mathrm{T}$. We thus obtain

$$\mathrm{A} = \mathrm{D}; \ \mathrm{A} \cos s_2\mathrm{T} - \mathrm{B} \sin s_2\mathrm{T} = \mathrm{E}e^{-s_3\mathrm{T}};$$

$$-s_1\mu_1\mathrm{D} = s_2\mu_2\mathrm{B}; \ s_2\mu_2 (\mathrm{A} \sin s_2\mathrm{T} + \mathrm{B} \cos s_2\mathrm{T}) = s_3\mu_3\mathrm{E}^{-s_3\mathrm{T}}. \tag{6}$$

Eliminating A, D, B, E from these equations we have the equation determining the wave velocity

$$\tan s_2\mathrm{T} = s_2\mu_2 (s_1\mu_1 + s_3\mu_3)/(s_2{}^2\mu_2{}^2 - s_1\mu_1 s_3\mu_3). \tag{7}$$

If $s_1 = \kappa\sigma_1$; $s_2 = \kappa\sigma_2$; $s_3 = \kappa\sigma_3$, the equation (7) may be written in the equivalent form

$$(\sigma_2{}^2\mu_2{}^2 - \sigma_1\mu_1\sigma_3\mu_3)(\tan \kappa\sigma_2\mathrm{T})/\sigma_2\mu_2 = \sigma_1\mu_1 + \sigma_3\mu_3. \tag{8}$$

Suppose now $$c_1 > c_3 > c > c_2; \tag{9}$$

then when $c = c_2$, $\sigma_2 = 0$, σ_1 and σ_3 are positive, so that the left-hand side of (8) is negative and the right-hand side positive when c is slightly greater than c_2. As c increases the right-hand side continually diminishes; $(\tan \kappa\sigma_2 T)/\sigma_2\mu_2$ increases. When $c = c_3$ the factor $(\sigma_2{}^2\mu_2{}^2 - \sigma_1\mu_1\sigma_2\mu_2)$ is positive, and therefore changes sign between c_2 and c_3; this will happen when

$$\mu_2{}^2\left(\frac{c^2}{c_2{}^2} - 1\right) = \mu_1\mu_3\left(1 - \frac{c^2}{c_1{}^2}\right)^{\frac{1}{2}} \tag{10}$$

denote the corresponding value of c by c'.

Then for $c_2 < c < c'$ the left-hand side is negative and the right-hand side positive unless $\kappa\sigma_2 T$ has passed the value $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, etc. (as, for example, when $\kappa\sigma_2 T$ varies from 0 to π in the range c_2 to c'), in which case the left-hand side will have diminished to $-\infty$, and then dwindled from $+\infty$ to zero, thus equalling at some point the value of the right-hand side. On reaching the value zero for $\kappa\sigma_2 T = \pi$, the left-hand side either begins to increase (as an exceptional case), or (in general) it continues to diminish. In the latter case, if for $c = c'$ (when the left-hand side vanishes) $\pi < \kappa\sigma_2 T < \frac{3}{2}\pi$, then in this interval the left-hand side must have decreased to a minimum and increased to zero; this is in accordance with the fact that at $c = c'$ the factor $\sigma_2{}^2\mu_2{}^2 - \sigma_1\mu_1\sigma_3\mu_3$ changes from negative to positive. If now $\kappa\sigma_2 T$ reaches the value $\frac{3}{2}\pi$ before $c = c_3$, there will be another root of the equation in this interval. There will thus be a root, or a series of roots, between c_2 and c' if κT is made sufficiently large, either by making κ or T large.

Between c' and c_3 there will be a root of the equation (8) if $\tan \kappa\sigma_2 T$ is greater than q at $c = c_3$, where

$$q = \mu_1\left\{1 - \frac{c_3{}^2}{c_1{}^2}\right\}^{\frac{1}{2}} \Big/ \mu_2\left\{\frac{c_3{}^2}{c_2{}^2} - 1\right\}^{\frac{1}{2}},$$

for in this range the factor $\sigma_2\mu_2 - \sigma_1\mu_1\sigma_3\mu_3/\sigma_2\mu_2$ continually increases from zero to $\mu_2(c_3{}^2 - c_2{}^2)^{\frac{1}{2}}/c_2$, so that the left-hand side takes the sign of $\tan \kappa\sigma_2 T$, and has always a positive gradient.

If κT is so small that, throughout the range $c_2 < c < c_3$, $\kappa\sigma_2 T$ remains less than $\frac{1}{2}\pi$, and at $c = c_3$, $\tan \kappa\sigma_2 T$ is less than q, there will be no root. Thus, if the wave length is sufficiently long or the middle layer too thin, no wave motion of the Love type is possible. This accords with the result found in § 4, that when T = 0, no Love wave exists.

If $c_2 < c_3 < c < c_1$, σ_3 is imaginary; the right-hand side of equation (7) is now complex and the left-hand side real. There are thus no real roots in this region.

When $c_1 < c_3$ we have as in the foregoing discussion that roots, if they occur (and roots will exist if κT is large enough), must occur for $c_2 < c_1$.

Within the layer (2) the displacement is $Ve^{i\kappa(x-ct)}$, where V is of the form

$$A\{s_2\mu_2 \cos s_2 z - s_1\mu_1 \sin s_2 z\}/s_2\mu_2,$$

which vanishes when $\tan s_2 z = s_2\mu_2/s_1\mu_1$. Now in this layer z is negative; if, then, $s_2 z$ includes the range $\frac{1}{2}\pi$ to π for $|z| < T$, the displacement must vanish for a certain value of z and a "nodal plane" will exist, as it may in the case of the ordinary Love waves in a surface layer. If $s_2 z$ includes the range from 0 to $n\pi$, while $|z|$ remains less than T, n nodal planes will exist.

A similar argument is applicable to the case $c_3 > c_1 > c > c_2$.

If we suppose that in medium (2) $c < c_2$, and therefore that

$$s_2{}^2 = \kappa^2\left(1 - \frac{c^2}{c_2{}^2}\right),$$

we have $V_2 = A \cosh s_2 z + B \sinh s_2 z$ in place of the harmonic terms. The boundary conditions now give

$$A = D;\ \ A\cosh s_2 T - B \sinh s_2 T = Ee^{-s_3 T};\ \ -s_1\mu_1 D = s_2\mu_2 B;$$
$$s_2\mu_2\{A \sinh(-s_2 T) + B\cosh(-s_2 T)\} = s_3\mu_3 Ee^{-s_3 T} \tag{11}$$

leading to

$$(s_1\mu_1 \cdot s_3\mu_3 + s_2{}^2\mu_2{}^2)\tanh s_2 T = -s_2\mu_2(s_3\mu_3 + s_1\mu_1) \tag{12}$$

which gives no relevant solutions.

In the other cases which might arise we should have either c_1 or c_3 less than c, so that in one of these regions the solution would be periodic, and therefore the displacement would not be inappreciable at great distances from $z = 0$, and, moreover, would require an infinite amount of energy to be present in a cylinder whose generators are perpendicular to $z = 0$. The physical interpretation of this result is similar to that given by Dr. Jeffreys to the maintenance of Love waves. The velocity of distortional waves in either of the media adjoining the layer (2) is greater than the velocity in the layer, and a wheeling of the wave-front, analogous to that which occurs in the phenomenon of total internal reflection, would cause the wave motion to be confined mainly to the central layer.

[*Added—July 3.*—The solution of a numerical example in the case where $\alpha_1 = \alpha_2 = \alpha$; $\beta_1 = \beta_2 = \beta$, illustrates one or two points of difference from the corresponding example worked out in § 4 for incompressible solids. The equations are numbered in continuation of those in §3.

If we write $M \equiv (\rho_1 + \rho_2)^2/(\rho_1 - \rho_2)^2$, equation becomes

$$x^2 + 4(1-x)(2-\gamma x) = \{Mx^2 + 4(2-x)\}(1-x)^{\frac{1}{2}}(1-\gamma x)^{\frac{1}{2}}, \tag{7}$$

where x, as previously, is c^2/β^2.

On squaring, it is found that the constant term and that in x vanish, so that the equation reduces to the quartic

$$\gamma M^2x^4 - \{(1+\gamma)M^2 + 8\gamma M\}x^3 + (M^2 + 8M - 1 + 8\gamma + 24M\gamma - 16\gamma^2)x^2 \\ - 8(\gamma - 4\gamma^2 + 3M + 2\gamma M)x + 16(M - \gamma^2) = 0. \tag{8}$$

The corresponding equation for incompressible media is a cubic, obtained by putting $\gamma = 0$; this is in agreement with § 4, equation (1).

As a check on this equation, it may be observed that (8) should reduce to the ordinary Rayleigh wave equation when $\rho_2 = 0$, *i.e.* when $M = 1$. It is, in fact, found that when $M = 1$, (8) becomes

$$\{\gamma x - 1 - \gamma\}\{x^3 - 8x^2 + 8x(3 - 2\gamma) - 16(1-\gamma)\} = 0, \tag{9}$$

where the second factor, equated to zero, is the same as (2), and the former factor gives the inadmissible solution $x = (1+\gamma)/\gamma$.

To bring the equation (8) into line with this result it is sufficient to observe that, whatever M and γ, the left-hand side of (8) reduces to -1 when x is unity. Since $M - \gamma^2$ is essentially positive, the equation must possess a root greater than unity.

At the Wiechert surface of discontinuity, $\rho_1 = 8{\cdot}2$, $\rho_2 = 3{\cdot}2$, $\gamma = 0{\cdot}2864$.*

The corresponding quartic equation has a root $x = 0{\cdot}9795\ldots$ corresponding to $c = 0{\cdot}9897\beta$, and by actual substitution it is verified that this value satisfies the unsquared equation (7). It may be noted that, as would be expected on general dynamical grounds, this velocity is less than that of a wave at the junction of incompressible media (see § 4, (4)).]

In conclusion, I wish to thank Dr. H. Jeffreys, who has kindly read through the manuscript and made several valuable suggestions.

* Knott: 'Roy. Soc. Proc., Edin.,' p. 170 (1918–1919).