The Effect of the Ocean on Rayleigh Waves. By R. Stoneley, M.A.

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1. *Introduction.*

In view of Gutenberg's * results concerning the velocities of Rayleigh waves below continents and below the oceans, it becomes of interest to examine the effect of the ocean on the propagation of surface waves over a homogeneous elastic solid. In this discussion Love waves will not be considered, since the only effect would arise from the viscous drag of oceanic waters.

The effect of an incompressible ocean has already been studied by Bromwich,† who finds that for waves of ordinary period the change in velocity is small, and that the wave-velocity c is equal to $c_R\{1-0{\cdot}522jh/\lambda\}$, where j is the ratio of the densities of water and rock, h the depth of the sea, λ the wave-length, and c_R the velocity of R-waves.‡ For $h=3$ km., and waves of 15 sec. period, the reduction in velocity is about $c_R/90$.

At the same time some dispersion is introduced. Using the formula $C=c-\lambda dc/d\lambda$ we obtain for the group-velocity $C=c_R\{1-1{\cdot}044jh/\lambda\}$, so that the group-velocity in the foregoing example exceeds c_R by about $c_R/45$. This correction is far too small to affect the rough measures at present available.

The formula for C shows that for actual wave-lengths the group-velocity, which is the velocity observed in practice,§ decreases linearly with $1/\lambda$.

The essential difference between the effects of incompressible and compressible fluid is summed up in the statement that in the former the velocity c_0 of a compressional wave is treated as very large compared with c_R, whereas in the latter case (and this corresponds with what actually occurs) c_0 is much less than c_R.

When compressibility is taken into account, therefore, a new feature may arise; planes may exist resembling the nodal planes of Love waves, where the vertical velocity is zero. As an illustration, one may consider the motion of the air inside a tube, open at one end and fitted with a movable piston at the other; the piston may be supposed massive and controlled by a strong spring. The effect of the air will be to introduce a small modification in the natural period of oscillation of the piston. If the period is long compared with the time taken for a pulse to travel along the tube, we have the analogue of an almost incompressible ocean. If the time taken to travel along the tube is long compared with the natural period, a number of nodal planes may be set up within the tube; the actual motion will be made up by the superposition of motions corresponding to the existence of 0, 1, 2, . . . nodes up to the greatest possible number.

It is not difficult to show that for such a system the velocity potential takes the form $\phi=\Sigma c_n \sin\frac{n}{c}(l-x)\cdot e^{int}$, where n satisfies a frequency

* *Der Aufbau der Erde*, 1925, 109.

† *Lond. Math. Soc. Proc.*, **30**, p. 107.

‡ *I.e.* Rayleigh waves.

§ *M.N.R.A.S., Geophys. Suppl.*, **1**, *6*, 281.

equation $\rho nc/(\mathrm{M}n^2-\mu)=\cot(nl/c)$, in which ρ is the density, c the velocity of sound, l the length of the pipe, M the mass of the piston, μ the strength of the spring.* For a nodal plane $\partial\phi/\partial x=0$, *i.e.* $n(l-x)/c=\frac{1}{2}\pi, \frac{3}{2}\pi \ldots$ Putting $d=2\pi c/n$, the distance which the sound-wave advances per period, x, becomes $l-(2s+1)d/4$, so that if $l<\frac{1}{4}d$ there will be no node; if $\frac{3}{4}d>l>\frac{1}{4}d$, one node; if $\frac{5}{4}d>l>\frac{3}{4}d$, two nodes, and so on. The summation sign in ϕ extends over all the admissible values of n.

When the depth of the water is small compared with the wave-length, both forms ($c_0>c_R$ and $c_R>c_0$) tend to the same result, which is, in fact, the approximate form obtained by Bromwich.

2. *Motion in the Compressible Fluid.*

Consider an ocean of uniform depth h, of density ρ_0 when at rest, and of incompressibility k. Let the axis of z be drawn vertically upwards so that the equations of the undisturbed free surface and ocean floor are $z=h$ and $z=0$ respectively. If a wave of wave-length $2\pi/\kappa$ is propagated in the direction x with velocity c, we will suppose all displacements small (so that their squares may be neglected) and proportional to $\exp i\kappa(x-ct)$. More strictly, ρ_0 should be treated as a function of z, given in terms of the surface value ρ_s by $\rho_0=\rho_s \exp g\rho_s(h-z)/k$; for an ocean of depth 3 km. the variation does not exceed about 1 per cent.

To the first order of small quantities the equations of motion,

$$\frac{\mathrm{DU}}{\mathrm{D}t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}; \quad \frac{\mathrm{DW}}{\mathrm{D}t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \quad . \quad . \quad . \quad (1)$$

reduce to

$$\left.\begin{aligned}\frac{\partial \mathrm{U}}{\partial t}&=-\frac{1}{\rho_0}\frac{\partial p_1}{\partial x}\\ \frac{\partial \mathrm{W}}{\partial t}&=-\frac{1}{\rho_0}\frac{\partial p_1}{\partial z}-\frac{gp_1}{k}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which

$$p=p_0+p_1;\ \rho=\rho_0+\rho_1=\rho_0(1+s);\ dp_0/dz=-g\rho_0 \quad . \quad (3)$$

Suppose that

$$(\mathrm{U},\mathrm{W})=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right)\phi \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Then the equations (2) become equivalent to a single equation, as may be immediately shown by cross-differentiation, so verifying that the motion is irrotational. The first equation of (2) is now

$$\frac{\partial^2\phi}{\partial t\partial x}=-\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The equation of continuity,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla^2\phi=0,$$

* *Cf.* Ramsey, *Hydromechanics*, part ii., p. 340, in which the corresponding problem is solved for a closed pipe.

becomes, to the first order,

$$\frac{\partial\rho_1}{\partial t}+\frac{\partial\phi}{\partial z}\cdot\frac{d\rho_0}{dz}+\rho_0\nabla^2\phi=0 \quad . \quad . \quad . \quad . \quad (6)$$

or, by (3),

$$\frac{\partial p_1}{\partial t}=-k\nabla^2\phi+g\rho_0\frac{\partial\phi}{\partial z} \quad . \quad . \quad . \quad . \quad (7)$$

Now assume

$$\left.\begin{aligned}\phi&=\Phi\exp i\kappa(x-ct)\\ p_1&=\Pi\exp i\kappa(x-ct)\end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

where Φ and Π are functions of z only.

Then (5) and (7) are respectively equivalent to

$$\kappa^2c\Phi=-i\kappa\Pi/\rho_0 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$-i\kappa c\Pi=-k\left(\frac{d^2\Phi}{dz^2}-\kappa^2\Phi\right)+\rho_0 g\frac{d\Phi}{dz} \quad . \quad . \quad . \quad (10)$$

leading to

$$\frac{d^2\Phi}{dz^2}-\frac{g}{c_0^2}\cdot\frac{d\Phi}{dz}+\kappa^2\left(\frac{c^2}{c_0^2}-1\right)\Phi=0 \quad . \quad . \quad . \quad (11)$$

Thus

$$\Phi=e^{mz}(\text{P}\cos\kappa_1 z+\text{Q}\sin\kappa_1 z) \quad . \quad . \quad . \quad (12)$$

where

$$2m=g/c_0^2\ ;\ \kappa_1^2=\kappa^2\left\{\frac{c^2}{c_0^2}-1\right\}-\frac{g^2}{4c_0^4},$$

and the variation of c_0 with z is neglected.

For an ocean 3 km. deep, mz is at most 8×10^{-3}, so that we may suppress the exponential term in (12). Since for waves of about 20 sec. period κ is about 10^{-6} cm.$^{-1}$, and we may take $c_0^2=2{\cdot}2\times10^{10}$, $c^2=10^{11}$ C.G.S. units, the ratio of $g^2/4c_0^4$ to $\kappa^2(c^2/c_0^2-1)$ is of the order of 2×10^{-4}, so that for waves of the periods prevailing in actual earthquakes we may write $\kappa_1^2=\kappa^2(c^2/c_0^2-1)$. The approximation will not hold for very long waves (κ small), and (11) shows that gravity then plays an increasingly preponderating rôle compared with compressibility.

It is at this point that the distinction between compressible and incompressible fluids is seen. As long as c is greater than c_0, Φ is a periodic function of z. For incompressible fluids, however, c/c_0 vanishes, and Φ is exponential in respect of z, as in Bromwich's solution.

3. *Boundary Conditions.*

At the free-surface we must have

$$p=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

At the ocean bottom the vertical displacement and vertical stress are continuous, while the horizontal traction on the ocean floor is zero.

(13) is equivalent to $s=0$; this may likewise be seen from the dynamical surface condition $Dp/Dt=0$, which, to the first order, is

$$\frac{\partial p}{\partial t}+\frac{\partial\phi}{\partial z}\cdot\frac{dp_0}{dz}=0\,;$$

this reduces to

$$\frac{\partial s}{\partial t}-\frac{g}{k}\cdot\frac{\partial\phi}{\partial z}=0,$$

while (6) is equivalent to

$$\frac{\partial s}{\partial t}-\frac{g}{k}\cdot\frac{\partial\phi}{\partial z}+\frac{\rho_0}{k}\nabla^2\phi=0,$$

so that $\nabla^2\phi=0$, or $s=0$. This equation may be evaluated at the surface $z=h$, so that $P\cos\kappa_1 h+Q\sin\kappa_1 h=0$, and ϕ may be written in the form, suitable for subsequent calculations,

$$\{A\sin\kappa_1(z-h)\exp i\kappa(x-ct)\}/\kappa_1\cos\kappa_1 h.$$

The displacements (u, w) in the R-wave motion in the earth must tend to zero as z tends to $-\infty$, and the appropriate values are:

$$(u,w)=\left[-\frac{\lambda+2\mu}{\rho_2\kappa^2c^2}(i\kappa, r)E\exp rz+(-s, i\kappa)Q\exp sz\right]\exp i\kappa(x-ct) \quad (14)$$

where λ, μ are the elastic constants in the usual notation, ρ_2 is the density, E, Q are constants,

$$r^2=\kappa^2(1-\rho_2c^2/(\lambda+2\mu)) \quad \text{and} \quad s^2=\kappa^2(1-\rho_2c^2/\mu) \quad . \quad (15)$$

To the first order of small quantities we have at $z=0$, with the usual notation for stresses,

$$\frac{\partial\phi}{\partial z}=\frac{\partial w}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$-p=p_{zz},$$

and hence

$$-\frac{\partial p_1}{\partial t}=k\nabla^2\phi-g\rho_0\frac{\partial\phi}{\partial z}=\frac{\partial p_{zz}}{\partial t}=\frac{\partial}{\partial t}\left\{(\lambda+2\mu)\frac{\partial w}{\partial z}+\lambda\frac{\partial u}{\partial x}\right\} \quad . \quad (17)$$

and

$$p_{zx}=0, \quad \text{or} \quad \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=0 \quad . \quad . \quad . \quad (18)$$

(16), (17), (18) give on substituting

$$A=-i\kappa c(-a^2rE/c^2\kappa^2+i\kappa Q) \quad . \quad . \quad . \quad (19)$$

$$kA\tan\kappa_1 h\,.\,(\kappa^2+\kappa_1^2)/\kappa_1-A\rho_0 g=i\kappa c\mu\left\{\frac{a^2E}{c^2}\left(2-\frac{c^2}{\beta^2}\right)-2i\kappa sQ\right\} \quad (20)$$

$$\frac{2a^2E}{\kappa^2c^2}=-\frac{Q\kappa^2}{i\kappa r}\left(2-\frac{c^2}{\beta^2}\right) \quad . \quad . \quad . \quad . \quad (21)$$

in which α, β are respectively the velocities of compressional and distortional waves in the earth, and are given by

$$\alpha^2=(\lambda+2\mu)/\rho_2\,;\ \beta^2=\mu/\rho_2 \quad . \quad . \quad . \quad (22)$$

After some reduction the elimination of A, Q, E from (19), (20), (21) gives, as an equation to determine c,

$$\frac{\kappa\beta^2 k\tan\kappa_1 h}{\kappa_1\mu c_0{}^2}-\frac{\beta^2\rho_0 g}{\mu c^2\kappa}=\left[4\left(1-\frac{c^2}{\beta^2}\right)^{\frac{1}{2}}-\frac{\kappa}{r}\left(2-\frac{c^2}{\beta^2}\right)^2\right]\Big/\frac{c^4}{\beta^4} \quad . \quad (23)$$

or writing $c^2/\beta^2=\xi$; $\kappa_1/\kappa=(c^2/c_0{}^2-1)^{\frac{1}{2}}=\epsilon$; $\beta^2/\alpha^2=\gamma$, and neglecting the term $\beta^2 c_0 g/\mu c^2\kappa$,

$$(\rho_0\tan\epsilon\kappa h)/\rho_2\epsilon=\{4(1-\xi)^{\frac{1}{2}}(1-\gamma\xi)^{\frac{1}{2}}-(2-\xi)^2\}/\xi^2(1-\gamma\xi)^{\frac{1}{2}} \quad . \quad (24)$$

As a check on this work we may note that when $h=0$, the left-hand side of this equation vanishes, and the numerator of the right-hand side, when equated to zero, gives the ordinary R-wave equation.

4. *Discussion of the Wave-Velocity Equation.*

We can further obtain the equation corresponding to $c_0>c$, and as a special case the equation for an incompressible fluid; $\kappa_1{}^2$ is now negative, but the right-hand side of (24) is independent of κ. The left-hand side passes over into $(\rho_0\kappa\tanh\kappa_1 h)/\rho_2\kappa_1$, where now $\kappa_1{}^2=\kappa^2(1-c^2/c_0{}^2)$. If c_0 tends to infinity, $\kappa_1=\kappa$, and the left-hand side of (24) becomes simply $(\rho_0/\rho_2)\tanh\kappa_1 h$. These two cases reduce to one only when κh is small, namely, the left-hand side is approximately $\rho_0\kappa h/\rho_2$, in accordance with Bromwich's result, which, as would be expected, is independent of the compressibility of the water.

For the purposes of illustration it will suffice to consider the earth as incompressible, so that $\gamma=0$. Then a rough graph of $F(\xi)\equiv\{4(1-\xi)^{\frac{1}{2}}-(2-\xi)^2\}/\xi^2$ shows that $F(\xi)$, which has the value -1 when $\xi=1$, increases to zero as ξ diminishes to 0·91275 . . . (*i.e.* $c=c_R$), and increases continually to infinity as ξ diminishes to zero; $\xi^2F(\xi)$ has a maximum within this range. It can be proved that $F(\xi)$ increases continually within this range.* For $\xi>1$ the left-hand side of (24) is real, while the right-hand side is complex. When κh is rather small, so that the left-hand side is positive but small, the value of c that will satisfy the equation must be slightly less than c_R, so that if gravity is neglected, the effect of a shallow ocean is to diminish the wave-velocity of R-waves.

Call the left-hand side of (24) $T(\xi)$. Then as ξ decreases from 1 to $c_0{}^2/\beta^2$, $T(\xi)$ decreases from $T(1)$ to $+\rho_0\kappa h/\rho_2$; this value of ξ for the earth is about 0·2. At this point the tangent becomes a hyperbolic tangent, and $T(\xi)$ will decrease continually to the finite value $(\rho_0\tanh\kappa h)/\rho_2$ when $\xi=0$, at which point $F(\xi)$ is infinite. Thus there must always be at least one root within this range.

* My proof, which is rather cumbrous, is not of sufficient interest to reproduce here.

We may consider the possibilities in order. However small κh, $T(1)$ is greater than $\kappa h\rho_0/\rho_2$, provided $\epsilon\kappa h$ is positive, as is the case. If $\kappa_1 h < \frac{1}{2}\pi$ when $\xi=1$, $T(\xi)$ will decrease as ξ decreases, remaining positive all the while. In these circumstances there will be a root c less than c_R. If $\frac{1}{2}\pi<(\kappa_1 h)_{\xi=1}<\pi$, $T(1)$ is negative, in which case $T(\xi)$ diminishes to $-\infty$ with decreasing ξ, and then subsequently diminishes in the same way as before from $+\infty$; thus again there will be a root, or, in fact, two roots if $0>T(1)>-1$. If $\pi<(\kappa_1 h)_{\xi=1}<\frac{3}{2}\pi$ there will certainly be two roots, and proceeding in this way it is seen that the number of roots increases as κh increases, *i.e.* as the wave-length diminishes.

Further, when more roots than one exist, between any two consecutive roots $\tan \kappa_1 h$ becomes infinite, so that $\cos \kappa_1 h$ vanishes. When we consider the motion corresponding to the larger root c_l, we note that $\kappa_1 h \equiv (c^2/c_0{}^2-1)^{\frac{1}{2}}\kappa h$ takes the value $\frac{1}{2}\pi$ between $(c_l{}^2/c_0{}^2-1)^{\frac{1}{2}}\kappa h$ and zero. Now $\partial\phi/\partial z$ contains a factor $\cos \kappa_1(h-z)$, and between the surface and the ocean floor $\kappa_1(h-z)$ varies continuously from zero to $(c_l{}^2/c_0{}^2-1)^{\frac{1}{2}}\kappa h$, and therefore must take the value $\frac{1}{2}\pi$ somewhere between these limits. At such a point the vertical displacement vanishes, and there exists what may be called a "quasi-nodal" plane, where the motion is everywhere horizontal.*

5. *Numerical Examples.*

To illustrate the order of magnitude of the effect, we may consider waves of period about 15 seconds, and write accordingly $\kappa=2/15$ km.$^{-1}$; for an ocean of depth 3 km., then, $\kappa h=\frac{2}{5}$. For water $k=2{\cdot}2\times 10^{10}$ dynes cm.$^{-2}$, so that $c_0{}^2=2{\cdot}2\times10^{10}$ (cm./sec.)^2. Putting $\rho_0/\rho_2=\frac{1}{3}$; $\beta^2=10^{11}$ (cm./sec.)^2 and solving by successive approximation $c^2=8{\cdot}82\times10^{10}$, or $c=2{\cdot}97$ km./sec., while $c_R=3{\cdot}02$ km./sec., a difference of $1{\cdot}7$ per cent.

The effect of dispersion can be illustrated roughly by solving for the case $\kappa=0{\cdot}44$, say, when it is found that δc is about $-0{\cdot}005$ km./sec., so that $\kappa\,.\,dc/d\kappa$ is about $-0{\cdot}05$ km./sec. Thus the group-velocity is about $0{\cdot}1$ km./sec. less than the velocity of R-waves (without dispersion, as in the classical treatment).

It may be mentioned here that this twofold effect of a small dispersing cause, namely, in altering both wave-velocity and group-velocity, leads to a curious result when gravity is the cause considered. Bromwich's formula † is $c=c_R(1+0{\cdot}213\lambda/a)$, where a is the radius of the earth, from which it follows that the group-velocity is simply c_R, so that so far as a seismograph record is concerned, on account of the cancelling of the two corrections referred to, gravity does not alter the time of transit of R-waves.

For a more detailed numerical treatment, recourse may be had to the method used by Jeffreys.‡ By assuming a series of values of ξ, corresponding values of κh may be calculated, and other values, and, in addition, the group-velocity, may be obtained by interpolation.

* When the earth is treated as compressible, it can be shown in like manner that a real root exists, but the question of quasi-nodal planes would require a more detailed examination.

† *Loc. cit.* ‡ *Loc. cit.*

When κ is small the gravity term in (23) becomes important, and an approximate value of ϕ has been used throughout, neglecting terms depending on g. When κ is large, *i.e.* for very short waves, the number of possible quasi-nodal planes becomes large, and $\tan \kappa_1 h$ goes through its range of values in a vanishingly small interval δc. By computation the following values were found :—

$\xi = c^2/\beta^2$.	κh.	C/β.
0·91	0·04303	
0·90	0·19534	
0·89	0·32001	0·928
0·88	0·41686	0·912
0·87	0·49149	0·894
0·86	0·54960	0·872
0·85	0·59698	0·848
0·84	0·63587	0·821
0·83	0·66878	
0·82	0·69740	

The group-velocity is given by

$$\frac{C}{\beta} = \frac{c}{\beta} + \kappa h \cdot \frac{d(c/\beta)}{d\kappa h} = \{\xi + \tfrac{1}{2}\kappa h d\xi / d(\kappa h)\}/\xi^{\frac{1}{2}}.$$

Since the first differences of κh are all negative, the group-velocity is less than the wave-velocity. The value $\xi = 0{\cdot}92$ gives κh negative, corresponding, presumably, to a wave with a quasi-nodal plane.

As ξ diminishes, κh increases, but, as a glance at (24) will show, at a diminishing rate; consequently, while this approximation holds, C also continually diminishes, and there is no minimum group-velocity for $c > c_0$. For the computations five-figure tables only have been used, so that the last figure in the values of κh is likely to be seriously in error. As a rough check on the table as a whole, it may be noted that the first differences of C/β run fairly regularly.

6. *Summary.*

The effect of oceanic waters on Rayleigh waves, previously studied by Bromwich on the assumption of incompressibility, has been discussed for the case of compressible liquid, and the influence on group-velocity tabulated for values likely to occur in actual records. The modifications introduced on this account are small, but there appears as a physical feature the possibility of the formation of quasi-nodal planes, which in some ways resemble the nodes in an open organ-pipe. It is noted incidentally that gravity affects appreciably the wave-velocity, but not the group-velocity.

Note by Dr. Harold Jeffreys.

It can be proved from Stoneley's equation (24) that a minimum group-velocity exists. Writing this equation in the form

$$T(\xi)=F(\xi) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$T(\xi)\equiv\frac{\rho_0}{\rho_2}\frac{\tan \epsilon\kappa h}{\epsilon} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we see that if $c<c_0$ we may preserve the real form by writing

$$\epsilon=\iota\eta \quad 0<\eta<1 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$T(\xi)=\frac{\rho_0}{\rho_2}\frac{\tanh \eta\kappa h}{\eta} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Then $T(\xi)$, when κh is infinite, has its lowest infinity indefinitely near to $c=c_0$; while $F(\xi)$ is infinite when $c=0$ and decreases continually as c increases to c_0. Thus the limit of the wave-velocity when κh is great is less than c_0. Also h enters only through $\tanh \eta\kappa h$, and when η is finitely different from zero this differs from unity only by a quantity of order $e^{-2\eta\kappa h}$. When κh is great, therefore, we shall have

$$c=c_m+de^{-2\eta\kappa h} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $c_m<c_0$. But sufficient conditions for a minimum group-velocity to exist * are that c shall have finite limits when κ tends to zero or infinity, the former being the greater, and that when κ tends to infinity the difference between c and its limit decreases as rapidly as $1/\kappa^2$. All these conditions are satisfied in this problem, and therefore a minimum group-velocity exists. It seems probable, however, that the corresponding period is too short to be of much seismological interest.

The Elastic Yielding of the Earth. By R. Stoneley, M.A.

The object of this note is to summarise some calculations carried out several years ago and abandoned in the hope of finding a closer approximation; time has not yet permitted the completion of this work.

To make a comparison between the results of seismological and tidal investigations, the elastic constants may be found from earthquake data if some law of density is assumed; the deformation produced in a supposedly spherical earth by a disturbing potential represented by a second-degree zonal harmonic is next calculated, and the quantities h and k occurring in earth-tide theory are found. Further, the part of $(C-A)/A$ arising from elastic strain may be computed and compared with the value derived from a knowledge of the free period of latitude variation and the precessional constant.

* *M.N.R.A.S., Geophys. Suppl*, 1, 1925 Dec., 285.