THE TRANSMISSION OF RAYLEIGH WAVES ACROSS AN OCEAN FLOOR WITH TWO SURFACE LAYERS

PART I: THEORETICAL

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ABSTRACT

The theoretical part of this paper is a discussion of the propagation of waves of Rayleigh type in an elastic medium with a horizontal double surface layer, above which is a uniform layer of liquid This model is based on seismic determinations of the velocities of explosion waves in the layers below the ocean bottom, and the equation giving the wave velocity as a function of wave length is derived as a determinantal equation of the eleventh order.

The numerical solution of this equation and the application to the propagation of Rayleigh waves across the ocean floor will be given in Part II.

Introduction

The effect of a compressible ocean on the propagation of Rayleigh waves was first investigated some thirty years ago, under the simplifying assumption that the ocean floor was a uniform elastic solid extending to a great depth.[1] The chief results found were that the wave propagation is dispersive, and that for sufficiently short waves there may exist in the water "quasi-nodal planes," at which the motion will be everywhere horizontal; the effect on the wave velocity of waves of period 15 seconds is small, and as the period shortens the group velocity becomes decidedly less than the wave velocity.

In an addendum to that paper Jeffreys pointed out that for very short waves a minimum group velocity exists; this, in fact, I searched for at that time in earthquake records, without success, but it appears now to correspond to the T phase. The wave-velocity equation has been found independently by later writers.

Longuet-Higgins[2] has investigated in great detail the motion in the water, with particular reference to second-order terms. Roughly speaking, for long waves such as the surface waves of earthquakes the ocean introduces a relatively small correction, and for waves of short wave length the special interest is in the motion of the water.

With increasing information about the constitution of the ocean floor, notably that given by the seismological researches of M. Ewing, T. F. Gaskell, M. N. Hill, and others, it is now quite inadequate to treat the ocean floor as uniform in composition; indeed, some particular cases of layering have been worked out.[3] A sufficient number of areas of the oceans have been sampled to give a good general idea of the suboceanic structure, which is in marked contrast to that of the continents. For

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[1] R. Stoneley, "The Effect of the Ocean on Rayleigh Waves," *Mon. Not. Roy. Astron. Soc., Geophys. Suppl.*, 1: 349–356 (1926).

[2] M. S. Longuet-Higgins, "A Theory of the Origin of Microseisms," *Philos. Trans. Roy. Soc.*, A, 243: 1–35 (1950).

[3] See, e.g., W. S. Jardetzky and F. Press, "Crustal Structure and Surface-Wave Dispersion (Part III)," *Bull. Seism. Soc. Am.*, 43: 137–144 (1953).

example, seismic refraction measurements by Ewing, Sutton, and Officer[4] in the North America Basin of the Atlantic Ocean indicate that beneath 1 to 2 km. of sediments there lies a thickness of some 4 km. of rock which, from the velocity of compressional waves, seems to be basic in composition; at the base of these rocks is a transition to rocks that are presumably ultrabasic, and this transition may be correlated with the Mohorovičić discontinuity below the continents. Accordingly, it is desirable to work out the wave-velocity equation for a double surface layer surmounted by a layer of water.

The problem without the water layer has recently been discussed in detail with reference to the continents.[5] The wave velocity is found from a determinantal equation of order 10. The analogous problem, with the addition of an oceanic layer, can be treated similarly, and will lead in the first instance to a determinant of order 11.

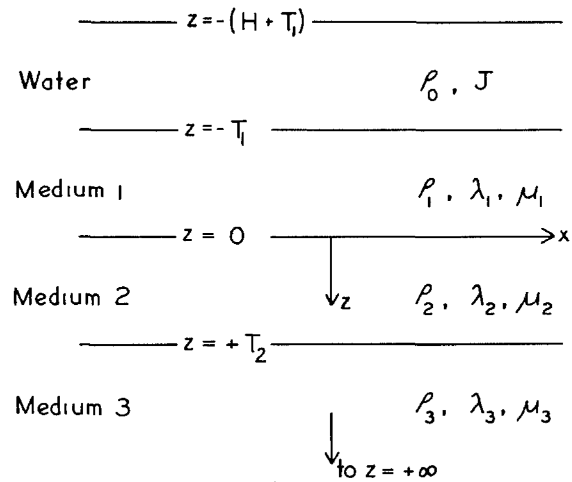

Fig. 1.

The Wave-Velocity Equation

i) *Displacements.* The notation is indicated in figure 1, and corresponds to that of RW 1954 with the addition of a layer of sea water of depth H, of density ρ_0 and of bulk modulus J, extending from the free surface $z = -(H + T_1)$ to the upper surface $z = -T_1$ of the sea bed. This region of water is denoted by the suffix 0; the velocity c_0 of sound in water is $(J/\rho_0)^{\frac{1}{2}}$, and, like ρ_0, is a function of the depth; but since the increase with pressure amounts in 3 km. to less than 1 per cent, c_0 will be treated as constant. The variation of gravity (g) with z will be neglected.

It has been shown in my paper of 1926 already cited that if the small disturbances of velocity from the state of undisturbed equilibrium are written

$$u_0 = \partial\phi/\partial x; \quad w_0 = \partial\phi/\partial z, \tag{1}$$

[4] M. Ewing, G. H. Sutton, and C. B. Officer, Jr., "Seismic Refraction Measurements in the Atlantic Ocean, Part VI: Typical Deep Stations, North America Basin," *Bull. Seism. Soc. Am.*, 44: 21–35 (1954).

[5] R. Stoneley, "Rayleigh Waves in a Medium with Two Surface Layers" (First Paper), *Mon. Not. Roy. Astron. Soc., Geophys. Suppl.*, 6: 610–615 (1954), and (Second Paper) *ibid.*, 7: 71–75 (1955). The First Paper will hereafter be referred to as RW 1954.

and the velocity potential ϕ_0 and the pressure variation p_0 are written as

$$\phi_0 = \Phi \exp i\kappa(x - ct) \tag{2}$$

$$p_0 = \Pi \exp i\kappa(x - ct)\ , \tag{3}$$

then

$$\kappa c \Phi = -i\Pi/\rho_0 \tag{4}$$

and

$$\Phi = \exp(-mz) \cdot (M \cos \kappa_0 z + N \sin \kappa_0 z) \tag{5}$$

where

$$m = g/2\, c_0^2; \quad \kappa_0^2 = \kappa^2 \left(\frac{c^2}{c_0^2} - 1\right) - \frac{g^2}{4c_0^4} \tag{6}$$

The formula (5) differs from that given in the 1926 paper in having the sign of m changed; the difference arises because in that paper z was measured in an upward direction. For waves of the periods prevailing in earthquakes the gravity term $g^2/4\ c_0^4$ may be dropped; likewise m may be put equal to zero.

Following the notation of RW 1954, the displacements[6] in medium 1, with Lamé constants λ_1, μ_1, and density ρ_1, are taken to be

$$U_1 = [-(\kappa/r_1)(A_1 \sinh r_1 z + B_1 \cosh r_1 z) - (s_1/\kappa)(C_1 \sinh s_1 z + D_1 \cosh s_1 z)] \sin \kappa(x - ct)\ ; \tag{7}$$

$$W_1 = [A_1 \cosh r_1 z + B_1 \sinh r_1 z + C_1 \cosh s_1 z + D_1 \sinh s_1 z] \cos \kappa(x - ct) \tag{8}$$

where

$$r_1^2 = \kappa^2(1 - c^2/\kappa_1^2); \quad s_1^2 = \kappa^2(1 - c^2/\beta_1^2); \quad \alpha_1^2 = (\lambda_1 + 2\mu_1)/\rho_1;$$

$$\beta_1^2 = \mu_1/\rho_1, \quad \text{and} \quad A_1, B_1, C_1, D_1 \text{ are constants.} \tag{9}$$

The displacements in medium 2 have the same form, except that the suffix 2 replaces the suffix 1.

In medium 3, in order that the displacements shall tend to zero as z tends to infinity we put

$$U_3 = [(\kappa/r_3)E \exp(-r_3 z) + (s_3/\kappa)F \exp(-s_3 z)] \sin \kappa(x - ct)\ ; \tag{10}$$

$$W_3 = [E \exp(-r_3 z) + F \exp(-s_3 z)] \cos \kappa(x - ct)\ , \tag{11}$$

where E, F are constants and r_3, s_3 are defined analogously to r_1, s_1.

In the ocean the small displacements will be taken as

$$U_0 = -(P/\kappa_0 c) \sin \kappa_0(z + L) \sin \kappa(x - ct) \tag{12}$$

$$W_0 = (P/\kappa c) \cos \kappa_0(z + L) \cos \kappa(x - ct) \tag{13}$$

[6] Note that in hydrodynamics the symbols u, w, ordinarily indicate velocities; accordingly, capital letters are used throughout this paper for displacements.

and the corresponding velocities as

$$u_0 = \partial\phi_0/\partial x = P(\kappa/\kappa_0)\sin\kappa_0(z+L)\cos\kappa(x-ct) \tag{14}$$

$$w_0 = \partial\phi_0/\partial z = P\cos\kappa_0(z+L)\sin\kappa(x-ct) \tag{15}$$

derived from a velocity potential

$$\phi_0 = (P/\kappa_0)\sin\kappa_0(z+L)\sin\kappa(x-ct)\,, \tag{16}$$

in which P is a constant and L is written for $H + T_1$. In view of the neglect of gravity $\kappa_0^2 = \kappa^2(c^2c_0^{-2} - 1)$.

Boundary Conditions

At the free surface the pressure variation p_0 must vanish. This has already been secured through the form assumed for ϕ_0, which must vanish at the free surface since the rate of dilatation $(\partial^2/\partial x^2 + \partial^2/\partial z^2)\phi_0$ vanishes for $z = -L$.

At the bottom of the ocean, $z = -T_1$, the tangential stress vanishes, while the vertical displacement and the normal stress are continuous. Thus at $z = -T_1$,

$$\partial U_1/\partial z + \partial W_1/\partial x = 0\,; \tag{17}$$

$$W_0 = W_1 \tag{18}$$

$$-\frac{\partial p_0}{\partial t} = J\nabla^2\phi_0 = \frac{\partial}{\partial t}\left\{\lambda_1\frac{\partial U_1}{\partial x} + (\lambda_1 + 2\mu_1)\frac{\partial W_1}{\partial z}\right\} \tag{19}$$

Writing for brevity $\cosh r_1T_1 = Cr_1$; $\sinh r_1T_1 = Sr_1$; $\cosh s_1T_1 = Cs_1$; $\sinh s_1T_1 = Ss_1$, and correspondingly for suffix 2, and setting $b_1 = (\kappa^2 + s_1^2)/\kappa^2 = 2 - c^2/\beta_1^2$, and similarly for b_2, b_3, these three boundary conditions give

$c^{-1}\cos\kappa_0 H$	Cr_1	$-Sr_1$	Cs_1	$-Ss_1$
$(J/\mu_1)(\kappa/\kappa_0)(c/c_0^2)\sin\kappa_0 H$	$b_1(\kappa/r_1)Sr_1$	$-b_1(\kappa/r_1)Cr_1$	$2(s_1/\kappa)Cr_1$	$-2(s_1/\kappa)Cs_1$
0	$2Cr_1$	$-2Sr_1$	b_1Cs_1	$-b_1Ss_1$
0	0	κ/r_1	0	s_1/κ
0	1	0	1	0
0	$2\mu_1$	0	μ_1b_1	0
0	0	$\mu_1\kappa b_1/r_1$	0	$2\mu_1s_1/\kappa$
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$$2(A_1Cr_1 - B_1Sr_1) + b_1(C_1Cs_1 - D_1Ss_1) = 0\,; \qquad \text{I}$$

$$(P/\kappa c)\cos\kappa_0 H + A_1Cr_1 - B_1Sr_1 + C_1Cs_1 - D_1Ss_1 = 0\,; \qquad \text{IIa}$$

$$\begin{aligned} -(J/\kappa_0\mu_1)(c/c_0^2)P\sin\kappa_0 H + (b_1\kappa/r_1)(-A_1Sr_1 + B_1Cr_1) \\ + 2(s_1/\kappa)(-C_1Ss_1 + D_1Cs_1) = 0\,. \end{aligned} \qquad \text{IIb}$$

Of these last three equations, I is the same as equation I of RW 1954. Equation IIa is an additional equation, arising from the continuity of the vertical displacement at the ocean floor, and IIb is a modification of the former equation II. The remaining eight equations, representing the continuity of the displacement and stress at $z = 0$ and $z = T_2$, are the same as equations III to X of RW 1954, and will here be quoted for convenience, lettered as before. They are:

$$(\kappa/r_1)B_1 + (s_1/\kappa)D_1 = (\kappa/r_2)B_2 + (s_2/\kappa)D_2 \qquad \text{III}$$

$$A_1 + C_1 = A_2 + C_2 \qquad \text{IV}$$

$$\mu_1(2A_1 + b_1C_1) = \mu_2(2A_2 + b_2C_2) \qquad \text{V}$$

$$\mu_1 b_1(\kappa/r_1)B_1 + 2\mu_1(s_1/\kappa)D_1 = \mu_2 b_2(\kappa/r_2)B_2 + 2\mu_2(s_2/\kappa)D_2 \qquad \text{VI}$$

$$(\kappa/r_2)(A_2Sr_2 + B_2Cr_2) + (s_2/\kappa)(C_2Ss_2 + D_2Cs_2) = -(\kappa/r_3)E' - (s_3/\kappa)F' \qquad \text{VII}$$

$$A_2Cr_2 + B_2Sr_2 + C_2Cs_2 + D_2Ss_2 = E' + F' \qquad \text{VIII}$$

$$\mu_2(2A_2Cr_2 + 2B_2Sr_2 + b_2C_2Cs_2 + b_2D_2Ss_2) = \mu_3(2E' + b_3F') \qquad \text{IX}$$

$$\begin{aligned} \mu_2(\kappa/r_2)b_2(A_2Sr_2 + B_2Cr_2) + 2\mu_2(s_2/\kappa)(C_2Ss_2 + D_2Cs_2) = \\ -\mu_3\{(\kappa/r_3)b_3E' + 2(s_3/\kappa)F'\} \end{aligned} \qquad \text{X}$$

$$\left.\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\kappa/r_2 & 0 & -s_2/\kappa & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 \\
-2\mu_2 & 0 & -\mu_2 b_2 & 0 & 0 & 0 \\
0 & -\mu_2\kappa b_2/r_2 & 0 & -2\mu_2 s_2/\kappa & 0 & 0 \\
(\kappa/r_2)Sr_2 & (\kappa/r_2)Cr_2 & (s_2/\kappa)Ss_2 & (s_2/\kappa)Cs_2 & \kappa/r_3 & s_3/\kappa \\
Cr_2 & Sr_2 & Cs_2 & Ss_2 & -1 & -1 \\
2\mu_2 Cr_2 & 2\mu_2 Sr_2 & \mu_2 b_2 Cs_2 & \mu_2 b_2 Ss_2 & -2\mu_3 & -\mu_3 b_3 \\
(\mu_2 b_2\kappa/r_2)Sr_2 & (\mu_2 b_2\kappa/r_2)Cr_2 & (2\mu_2 s_2/\kappa)Ss_2 & (2\mu_2 s_2/\kappa)Cs_2 & \mu_3 b_3\kappa/s_3 & 2\mu_3 s_3/\kappa
\end{array}\right| = 0 \qquad (20)$$

In equations VII, VIII, IX, and X, E' and F' have been written for $E \exp(-r_3T_2)$ and $F \exp(-s_3T_2)$ respectively.[7]

The elimination of the eleven constants P, A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, E', F' among these eleven equations gives equation (20), which expresses in determinantal form the relation between κ and the wave velocity c. If c is known for any given value of κ, then any ten of the foregoing set of equations will give the ratios of the constants, so that the amplitude of the motion may be determined for all depths. This equation reduces for $H = 0$ to the corresponding equation (19) of RW 1954.

By way of verification, we may consider as in RW 1954 the particular case of large values of κ, i.e., very short wave lengths. In these circumstances the functions $\sinh r_1T_1$ and $\cosh r_1T_1$ tend to equality, and similarly with the other hyperbolic functions, and the simplified determinant can easily be shown to break down into the product of three determinants. Two of these correspond to the propagation of Rayleigh waves of short wave length along the interfaces $z = 0$ and $z = T_2$, and the third determinant should give formally the equation of Rayleigh waves in an ocean resting on uniform material corresponding to layer 1, but of infinite depth.

For the limiting case, it suffices to add column 2 of (20) to column 3, and column 4 to column 5. Then the top three elements of columns 3 and 5 are all zeros, as likewise are the first three elements of columns 6, 7, 8, 9, 10, 11. Thus the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \dfrac{J\kappa c^2}{\mu_1\kappa_0 c_0^2}\tan\kappa_0 H & b_1\kappa/r_1 & 2s_1/\kappa \\ 0 & 2 & b_1 \end{vmatrix} \tag{21}$$

formed from the surviving elements of the first three rows (after removal of exponentials by division through the second and fourth columns) will be a factor of (20) and when equated to zero should be equivalent to equation (24) of my 1926 paper.

On equating (21) to zero and multiplying out, the equation reduces to

$$b_1^2 - 4\left(1 - \frac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}}\left(1 - \frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}} + \frac{\rho_0 c^4(1 - c^2/\alpha_1^2)^{\frac{1}{2}}}{\rho_1\beta_1^4(c^2/c_0^2 - 1)^{\frac{1}{2}}}\tan\kappa_0 H = 0\ ,$$

which, with suitable change of notation, is the wave-velocity equation referred to. The remaining two factors will be the four-row determinants previously considered.

Part II of this paper, which is being prepared in collaboration with Dr. U. Hochstrasser, will deal with the application of the theory to the passage of earthquake surface waves across the ocean.

[7] In RW 1954, T_2 is misprinted as T_3 in these definitions.