

On the Equilibrium Spectrum of Gravity–Capillary Waves

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ABSTRACT

The equilibrium spectra of unidirectional gravity–capillary waves are derived from the wave-action balance equation. The calculations include nonlinear triad interactions, direct energy input from the wind, and viscous dissipation. Known equilibrium spectra for short gravity waves, which interact with the capillary waves, are taken as input. The results differ significantly from the standard power law solutions and indicate that the spectra for gravity–capillary waves in an oceanic environment can be rather different from those obtained in laboratory experiments.

1. Introduction

Better understanding of the statistical geometry of the ocean surface on scales from a couple of meters down to a few millimeters is crucial to the development of modern oceanography. Short gravity waves, waves in the gravity–capillary range, and ripples have wavelengths comparable to those of the measuring electromagnetic waves and serve as “mediators” in remote sensing of wind fields, currents, and sea levels. The statistical geometry of the free surface is intimately related to the nonlinear dynamics of the wavy flow field.

The dynamics of a homogeneous random wave field, in the gravity–capillary regime, is conveniently described by the balance of its action spectral density $N(\mathbf{k}, t)$, governed by

$$\frac{\partial N}{\partial t} = S_{nl} + S_d + S_i. \quad (1.1)$$

Here S stands for source or sink due to *nonlinear* wave–wave interactions, *dissipation* by internal friction or small-scale breaking, and direct *input* from the wind, respectively. In this paper it will be assumed: (i) S_{nl} is dominated by triad interactions, (ii) S_d is dominated by viscous dissipation, (iii) S_i has a linear dependence on N and a quadratic dependence on the wind friction velocity u_* , and (iv) the equilibrium range spectrum of short gravity waves N_g is not affected by the presence of the shorter gravity–capillary waves.

The appropriate expressions for S_{nl} , S_d , S_i , and N_g , as given in Stiassnie et al. (1991), Phillips (1977, p. 51), Plant (1982), and Phillips (1985), respectively, are:

$$\begin{aligned} \text{(i)} \quad S_{nl} = & 16\pi^3 \iint \{ N(\mathbf{k}_1)N(\mathbf{k}_2) - N(\mathbf{k})[N(\mathbf{k}_1) + N(\mathbf{k}_2)] \} \\ & \times [V^{(-)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)]^2 \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \delta(\omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)) d\mathbf{k}_1 d\mathbf{k}_2 \\ & + 32\pi^3 \iint \{ N(\mathbf{k}_2)[N(\mathbf{k}) + N(\mathbf{k}_1)] - N(\mathbf{k})N(\mathbf{k}_1) \} \\ & \times [V^{(-)}(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k})]^2 \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) \delta(\omega(\mathbf{k}) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)) d\mathbf{k}_1 d\mathbf{k}_2. \end{aligned} \quad (1.2)$$

The wave frequency ω is related to the wavenumber $\mathbf{k}$ through the linear dispersion relation

$$\omega^2 = gk + sk^3, \quad (1.3)$$

where $k = |\mathbf{k}|$, g is the gravitational acceleration and s is the ratio between the surface tension and the density of the water.

$$\text{(ii)} \quad S_d = -4\nu k^2 N, \quad (1.4)$$

where ν is the kinematic viscosity of the water;

$$\text{(iii)} \quad S_i = 0.04k^2 u_*^2 \cos \chi N / \omega, \quad (1.5)$$

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where χ is the angle between the propagation direction of waves and the wind; and

$$(iv) \quad N_g \approx 0.012 u_* k^{-4} \cos^2 \chi, \quad -\frac{\pi}{2} < \chi < \frac{\pi}{2}. \quad (1.6)$$

An equilibrium spectrum is defined as the stationary solution of (1.1), (i.e., $\partial N / \partial t = 0$, where t is the time). Its existence is still an open question; this has not hindered researchers from producing a few mathematical as well as experimental presentations of its structure (see Zhang 1995 and references therein). The best founded results are those by Zakharov and Filonenko (1966, 1967, hereafter ZF). Their exact mathematical solutions of $S_{nl} = 0$ [for gravity waves the triad interaction expression (1.2) is replaced by the appropriate quartet interaction term], for isotropic "pure" gravity and isotropic pure capillary wave fields, [see also Stiassnie et al. (1991)] are

$$N \propto k^{-\alpha}, \quad (1.7)$$

where $\alpha = 23/6$, or 4, for gravity waves and $\alpha = 17/4$ for capillary waves. A rather practical definition for pure gravity waves is $\lambda_g > 6$ cm, whereas for pure capillary, one could take $\lambda_c < 2$ mm, although ZF derivations ignore these boundaries altogether. One cannot extend the ZF method of solution, by taking into account simultaneously the effects of gravity and surface tension, because the dispersion relation (1.3) is not a homogeneous function of k anymore. Almost all researchers straighten out with ZF and present results similar to (1.7), with some variations in α .

It is well known that gravity waves and capillary waves interact with each other. As an example, a group of capillary waves 1-mm long interacts and exchanges energy with a gravity wave 60 cm long. Actually the group velocity of the former is equal to the phase velocity of the latter. The present note is an attempt to shed some light on this "cross boundary" mechanism and assess its significance. Unfortunately, the complexity of the mathematical problem forces us to restrict the treatment and study unidirectional wave fields (details are given in section 2).

The mathematical solution is derived using two different methods in sections 3 and 4. The discrepancy between our results and laboratory experiments is discussed in section 5.

2. Unidirectional wave fields

For unidirectional waves the system of equations that results from the first couple of delta functions in (1.2):

$$k - k_1 - k_2 = 0, \quad \omega(k) - \omega(k_1) - \omega(k_2) = 0 \quad (2.1)$$

has a solution only when $k \geq k_0$, where

$$k_0 = (2g/s)^{1/2}; \quad (2.2)$$

we denote this solution by $k_1^{(+)}(k)$, $k_2^{(+)}(k)$.

However, the system that results from the second couple of Delta functions in (1.2),

$$k + k_1 - k_2 = 0, \quad \omega(k) + \omega(k_1) - \omega(k_2) = 0, \quad (2.3)$$

has a solution for any k , denoted by $k_1^{(-)}(k)$, $k_2^{(-)}(k)$. These solutions are shown in Fig. 1. The values of $k_1^{(+)}$, $k_2^{(+)}$ in (1.2) are interchangeable but not those of $k_1^{(-)}$, $k_2^{(-)}$. Integrating (1.2) yields

$$\begin{aligned} \hat{S}_{nl} = & c^{(+)} \{ N(k_1^{(+)}) N(k_2^{(+)}) - N(k) [N(k_1^{(+)}) \\ & + N(k_2^{(+)})] \} h(k - k_0) + c^{(-)} \{ N(k_2^{(-)}) [N(k) \\ & + N(k_1^{(-)})] - N(k) N(k_1^{(-)}) \}, \end{aligned} \quad (2.4)$$

where h is the Heaviside unit function and

$$c^{(+)} = \frac{\pi \omega_1^{(+)} \omega_2^{(+)} k k_1^{(+)} k_2^{(+)}}{\omega |C_{g1}^{(+)} - C_{g2}^{(+)}|} \quad (2.5a)$$

$$c^{(-)} = \frac{\pi \omega \omega_1^{(-)} k k_1^{(-)} k_2^{(-)}}{\omega_2^{(-)} |C_{g2}^{(-)} - C_{g1}^{(-)}|}, \quad (2.5b)$$

where $C_g = d\omega/dk$ is the group velocity. In the derivation of (2.5) we have used the kernels $V^{(-)}$ given in Hogan et al. (1988).

The restriction to unidirectional fields does not affect the expression for S_d (1.4), but it reduces (1.5) by substituting $\chi = 0$ to

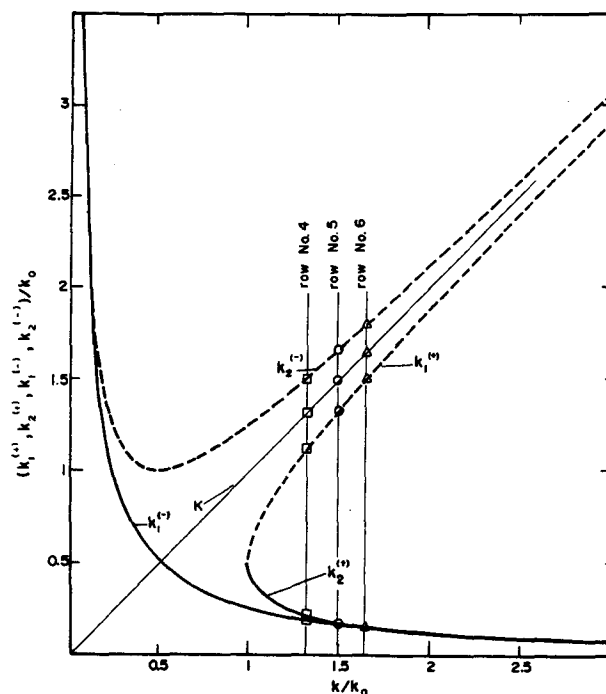


FIG. 1. Gravity capillary sum (+) and difference (-) wave triads. The row numbers are those in Table 1.

$$\hat{S}_i = 0.04k^2 u_*^2 \omega^{-1} N. \quad (2.6)$$

Integrating (1.6) gives

$$\hat{N}_g = \int_{-\pi/2}^{\pi/2} d\chi k N_g \approx 0.02 u_* k^{-3} = c_1 (k/k_0)^{-3},$$

$$k/k_0 \ll 1, \quad c_1 = 0.02 u_* k_0^{-3}. \quad (2.7)$$

The assumption of unidirectionality transforms the mathematical problem (1.1) from an integrodifferential equation to a functional equation by replacing the expression (1.2) for S_{nl} by $\hat{S}_{nl}$ given in (2.4).

3. Discretized solution of the functional equation

In order to solve the functional equation

$$\hat{S}_{nl} + S_d + \hat{S}_i = 0 \quad (3.1)$$

we use one (out of many) possible special sequences of wavenumbers k given in the first column of Table 1. The existence of such sequence was pointed out by van Gastel (1987).

Each row (from the third downward) of the table includes an additional four wavenumbers that form sum and difference triads together with the wavenumbers in the first column. Note that k in a certain row was chosen to be the same as $k_2^{(-)}$ in the preceding row and turns out to be the same as $k_1^{(+)}$ in the following one. Also note that $k_1^{(-)}$ and $k_2^{(+)}$ are both decreasing as one is moving down the table. The five wavenumbers in the fourth, fifth, and sixth rows are shown by $\square$, $\circ$, and $\triangle$ in Fig. 1.

For each of the k in the first column in Table 1 except the first two we apply (3.1), which gives $N(k_2^{(-)})$ in terms of the known $N(k)$, $N(k_1^{(-)})$, $N(k_1^{(+)})$, and $N(k_2^{(+)})$:

$$N(k_2^{(-)}) = \langle 4k^2 \nu^*(k) N(k) / c^{(-)}(k) + N(k) N(k_1^{(-)}) \\ + c^{(+)}(k) \{ N(k_1^{(+)}) N(k_2^{(+)}) \\ - N(k) [N(k_1^{(+)}) + N(k_2^{(+)})] \} / c^{(-)}(k) \rangle / \\ (N(k) + N(k_1^{(-)})), \quad (3.2)$$

where

$$\nu^* = \nu - 0.01 u_*^2 / \omega. \quad (3.3)$$

Note that $N(k)$ and $N(k_1^{(+)})$ are already known from applying (3.2) to the two preceding rows in the table, whereas $N(k_1^{(-)})$ and $N(k_2^{(+)})$ are assumed to be given by the gravity wave expression (2.7). The application of (3.1) to the first two rows in the table requires a somewhat different treatment. In this case, we solve two linear equations, which yield

$$N(0.85347k_0) \\ = \frac{0.28652^2 c^{(-)}(0.85347k_0) \nu^*(0.28652k_0)}{0.85347^2 c^{(-)}(0.28652k_0) \nu^*(0.85347k_0)} \\ \times N(0.28652k_0), \quad (3.4)$$

TABLE 1. A sequence of interrelated wavenumber triads.

k/k_0	$k_1^{(-)}/k_0$	$k_2^{(+)}/k_0$	$k_1^{(+)}/k_0$	$k_2^{(-)}/k_0$
0.286522	0.853478	—	—	1.140000
0.853478	0.286522	—	—	1.140000
1.140000	0.209529	0.286522	0.853478	1.349529
1.349529	0.174578	0.209529	1.140000	1.524107
1.524107	0.153170	0.174578	1.349529	1.677277
1.677277	0.138268	0.153170	1.524107	1.815544
1.815544	0.127102	0.138268	1.677277	1.942646
1.942646	0.118322	0.127102	1.815544	2.060969
2.060969	0.111178	0.118322	1.942646	2.172146
2.172146	0.105213	0.111178	2.060969	2.277359
2.277359	0.100133	0.105213	2.172146	2.377491
2.377491	0.095737	0.100133	2.277359	2.473228
2.473228	0.091883	0.095737	2.377491	2.565111
2.565111	0.088467	0.091883	2.473228	2.653578
2.653578	0.085413	0.088467	2.565111	2.738991
2.738991	0.082659	0.085413	2.653578	2.821649
2.821649	0.080159	0.082659	2.738991	2.901808
2.901808	0.077876	0.080159	2.821649	2.979684
2.979684	0.075780	0.077876	2.901808	3.055464

$$N(1.14k_0) = \langle 4\nu^*(0.28652k_0) N(0.28652k_0) \\ \times (0.28652k_0)^2 / c^{(-)}(0.28652k_0) \\ + N(0.28652k_0) N(0.85347k_0) \rangle / (N(0.28652k_0) \\ + N(0.85347k_0)). \quad (3.5)$$

Here again, we assume that $N(0.28652k_0)$ is given by the gravity wave equilibrium spectrum (2.7).

The discretized solutions for various values of u_* are shown in Fig. 2. According to Donelan and Peirson (1987), the expression for $\hat{S}_i$ given in (2.6) fails at small values of u_* , which probably explains the unacceptable results for $u_* = 0.15$ and 0.2 m s^{-1} .

For all other values of u_* , the curves for $N(k)$ in Fig. 2 have clear cutoffs. For values of k larger than these cutoffs (3.2) produces negative values for $N(k)$, which is defined as a positive physical quantity. Note that in plotting the curves of Fig. 2, we have used the gravity wave spectrum (2.7) for $k \leq 0.28652k_0$.

In the following section we convert the functional equation into a differential equation for large values of k and compare its solution with (3.2).

4. Asymptotic solution

Here we seek an asymptotic solution, valid for large (k/k_0) , to

$$\hat{S}_{nl} + S_d + \hat{S}_i = 0, \quad (4.1)$$

where $\hat{S}_{nl}$, S_d , and $\hat{S}_i$ are given by (2.4), (1.4), and (2.6).

To this end, we use the following asymptotic expansions. From (2.3) and (1.3)

$$k_1^{(-)} = \frac{2k_0^2}{9k} + \frac{25k_0^4}{729k^3}; \quad k_2^{(-)} = k + \frac{2k_0^2}{9k} + \frac{25k_0^4}{729k^3}. \quad (4.2a,b)$$

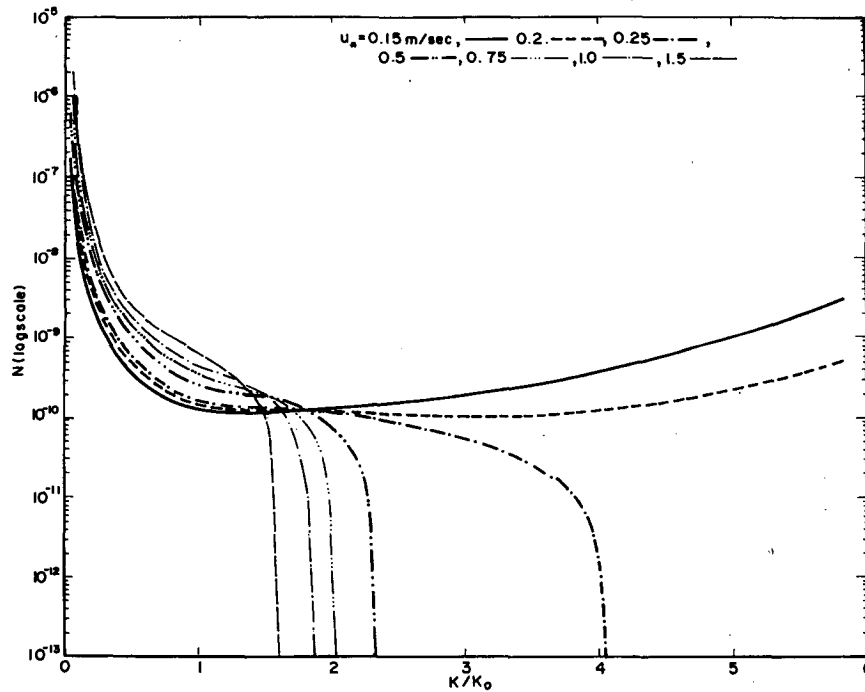


FIG. 2. Wave action spectral density $N(k)$, obtained from Eq. (3.2) for different values of the shear velocity u_* .

From (2.1) and (1.3)

$$k_2^{(+)} = \frac{2k_0^2}{9k} + \frac{61k_0^4}{729k^3}; \quad k_1^{(+)} = k - \frac{2k_0^2}{9k} - \frac{61k_0^4}{729k^3}. \quad (4.2c,d)$$

From (4.2d)

$$N(k_1^{(+)}) = N(k) - \left(\frac{2k_0^2}{9k} + \frac{61k_0^4}{729k^3} \right) N'(k)$$

From (2.7) and (4.2a)

$$N(k_1^{(-)}) = \hat{N}_g(k_1^{(-)})$$

$$= c_1 \left(\frac{9k}{2k_0} \right)^3 \left(1 - \frac{75k_0^2}{162k^2} \right). \quad (4.3a)$$

$$+ \frac{1}{2} \left(\frac{2k_0^2}{9k} + \frac{61k_0^4}{729k^3} \right)^2 N''(k). \quad (4.3d)$$

From (2.5a,b) and (4.2)

From (2.7) and (4.2c)

$$N(k_2^{(+)}) = \hat{N}_g(k_2^{(+)}) = c_1 \left(\frac{9k}{2k_0} \right)^3 \left(1 - \frac{183k_0^2}{162k^2} \right). \quad (4.3b)$$

$$c^{(+)} = \frac{8\pi k_0^4}{81} \left(1 + \frac{41k_0^2}{81k^2} \right);$$

From (4.2b)

$$N(k_2^{(-)}) = N(k) + \left(\frac{2k_0^2}{9k} + \frac{25k_0^4}{729k^3} \right) N'(k) + \frac{1}{2} \left(\frac{2k_0^2}{9k} + \frac{25k_0^4}{729k^3} \right)^2 N''(k). \quad (4.3c)$$

$$c^{(-)} = \frac{8\pi k_0^4}{81} \left(1 + \frac{23k_0^2}{81k^2} \right). \quad (4.4a,b)$$

Note that (4.3c,d) are Taylor expansions of $N(k_2^{(-)})$ and $N(k_1^{(+)})$ around $N(k)$, respectively.

Substituting (4.2), (4.3), and (4.4) into (4.1) gives to leading order

$$\left\{ -2\pi k_0^3 c_1 \left(1 - \frac{20k_0^2}{81k^2} \right) k^2 N' + \frac{2}{9} \pi k_0^5 c_1 k N'' - \frac{8}{81} \pi k_0^4 N^2 \right\} h(k - k_0) + \left\{ 2\pi k_0^3 c_1 \left(1 - \frac{2k_0^2}{81k^2} \right) k^2 N' + \frac{2}{9} \pi k_0^5 c_1 k N'' + \frac{8}{81} \pi k_0^4 N^2 \right\} - 4\nu k^2 N + 0.04 u_*^2 k^{1/2} N / s^{1/2}. \quad (4.5)$$

For $k > k_0$, there is a mutual cancellation of some of the terms in (4.5), which now gives

$$kN'' + N' + \frac{9}{\pi c_1 k_0^5} (-\nu k^2 + 0.01 u_*^2 k^{1/2} / s^{1/2}) N = 0. \quad (4.6)$$

Equation (4.6) has the following solutions in terms of confluent hypergeometric functions M and U (see Abramowitz and Stegun 1972).

$$N = \{a_1 M(\hat{a}, 1, \hat{\alpha} \hat{k}^{3/2}) + a_2 U(\hat{a}, 1, \hat{\alpha} \hat{k}^{3/2})\} \times \exp(-\hat{\alpha} \hat{k}^{3/2}/2), \quad (4.7)$$

where

$$\hat{k} = k/k_0; \quad \hat{\alpha} = \pi(\nu k_0 / 0.02 \pi u_*^2)^{1/2};$$

$$\hat{a} = \frac{1}{2} - 0.01 u_*^2 / (0.04 \pi g \nu u_*^2)^{1/2}. \quad (4.8)$$

In Figs. 3a–c we present a comparison between the numerical results from (3.2)–(3.5) and the asymptotic analytical expression (4.7). The overall agreement is very encouraging and increases our confidence in the mathematical approach. The caption of each figure includes the numerical values of the coefficients a_1, a_2 of (4.7) as well as the coordinates of the two adjoining points (k_a, N_a) and (k_b, N_b) on the numerical graphs through which the asymptotic graphs (4.7) were forced to pass.

The additional branch in Fig. 3c is physically meaningless; it is shown only in order to demonstrate the mathematical equivalence of the different methods of solution given in this and the preceding section.

Note that the equilibrium spectra (4.7) are a result of a balance between all three source terms (S_{nl} , S_d , and S_i), which turn out to be of equal importance. Previous authors made different and contradicting assumptions regarding the relative importance of the source terms. Donelan and Pierson (1987) set $S_{nl} = 0$, whereas Glazman (1995) assumed a purely inertial cascade and takes $S_d = S_i = 0$ (but includes higher order nonlinearities).

5. Discrepancy with laboratory experiments

Our results differ significantly from the standard power law “tails,” which were obtained in wave–wind flume experiments by Jahne and Riemer (1990), Hwang et al. (1993), and Zhang (1994). This discrepancy is probably related to the fact that the short gravity waves (in the wavelength range of 10 cm to 1 m) spectra in those experiments are very far from the fully developed state (2.7), which is the most dominant input in our calculations. We claim that our calculations indicate that the spectra for gravity–capillary waves in an oceanic environment are probably different than those obtained in laboratory experiments. To check the above argument we have solved the inviscid case

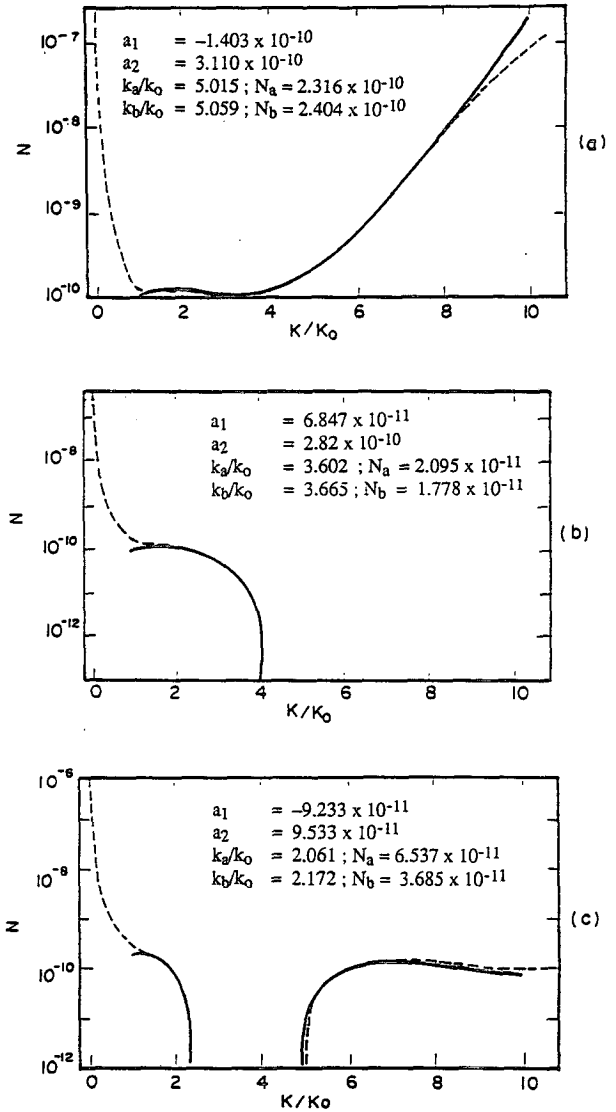


FIG. 3. Wave action spectral density. Comparison between the discretized solution of the functional equation (dashed) and the asymptotic solution of the differential equation (solid) for (a) $u_* = 0.2$, (b) $u_* = 0.25$, and (c) $u_* = 0.5 \text{ m s}^{-1}$.

$\hat{S}_{nl} = 0$, for large (k/k_0) , ignoring the effects of damping and wind input altogether.

For the inviscid problem, we require that each of the expressions in the curly parentheses in (4.5) should vanish, which yields for large (k/k_0)

$$c_1 k^2 N' + \frac{4k_0}{81} N^2 = 0 \quad (5.1)$$

to leading order. Note that (5.1) is satisfied by

$$N = c_1 / \left[1 + \frac{4}{81} \left(1 - \frac{k_0}{k} \right) \right], \quad k > k_0. \quad (5.2)$$

The above is true for $\hat{N}_g$ given by (2.7); however, for

other $\hat{N}_g$, say $\hat{N}_g = n(k)$, we have obtained instead of (5.1)

$$n\left(\frac{2k_0^2}{9k}\right)N' + \frac{9k}{2k_0^2}N^2 = 0, \quad (5.3)$$

which has the solution

$$N = N_0 / \left(1 + \frac{2}{9} k_0^2 N_0 \int_{2k_0^2/9k}^{2k_0^2/9} d\zeta / \zeta^3 n(\zeta) \right), \quad k \geq k_0, \quad (5.4)$$

where $N_0 = N(k_0)$.

The last result seems to be more applicable for comparisons with laboratory tests where (2.7) is usually not attained. Note that for $n \propto k^\alpha$, and $k < k_p$, where k_p is the pick of the spectrum, the asymptotic result for large k , that is, for $(k > 2k_0^2/9k_p)$ is

$$N \propto \left(\frac{k_0}{k} \right)^{2+\alpha}. \quad (5.5)$$

Equation (5.5) with $\alpha = 1.25$ gives for very short waves, the same power law as ZF obtained for pure capillary waves, albeit under different physical assumptions.

To summarize, the results we found differ significantly from those obtained in wind wave tank experiments. The results in Fig. 2 are characterized by a sharp cutoff and not by an algebraic decay law. This discrepancy leaves several open questions:

1) The short gravity waves (10 cm–1 m) in the experiments by Jahne and Riemer (1990), Hwang et al. (1993), and Zhang (1994) are far from the fully developed state that was assumed in the present computations. Is this the main reason? We claim that it probably is. Equation (5.5) indicates that the present model can produce spectra with a power law decay structure. Full details about the measured spectra are needed to prove this hypothesis.

2) How suitable is the expression S_i for the direct wind input, given in Eq. (1.5)? We do not know; however, the present model could easily be used to check other alternatives.

3) The negative values of N , indicated in Figs. 3b–c, may raise some doubt regarding the concept of equilibrium spectra. It may well be that we are looking for

something that does not exist. This is a rather difficult issue and beyond the scope of this short note.

4) The present model is for unidirectional wave fields only, which may raise some doubt regarding its physicality. Is this the main reason for the discrepancy? We do not know; to answer this question one needs to develop a rather cumbersome code and solve the full two-dimensional problem.

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