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Partial reflexion of water waves by non-uniform adverse currents

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The propagation of linearized waves on a non-uniform slowly varying potential current is studied by converting the equations of flow into a Schrödinger ordinary differential equation in the complex plane. This equation, which is solved by the WKB method, indicates the existence of current barriers which allow partial energy transmission while reflecting the complementary part. The classical result of total transmission (Longuet-Higgins & Stewart 1961) as well as that of complete reflexion (Peregrine 1976) are recovered as limiting cases by the present, more general approach.

1. Introduction

We consider here the problem of water-wave propagation on a non-uniform slowly varying current. To simplify matters we limit the study to two-dimensional flows (in a vertical plane) in deep water, the current being represented by a potential flow beneath the free surface which is uniform at infinity. The subject has been reviewed comprehensively by Peregrine (1976, § II–D) and only a few relevant points of principle are recalled here.

The classical approach of Longuet-Higgins & Stewart (1961) and Phillips (1966), who used the equations of wavenumber and wave-action conservation, yields the well-known relationships between wavenumber, wave amplitude and current velocity. This approach is valid for all current velocities which are in the same direction as that of the wave propagation and for adverse currents with velocity magnitudes smaller than that of the critical velocity, which is defined as one-quarter of the incoming wave velocity in still water. The wave energy for a current which is uniform at infinity is completely transmitted, the incoming and outgoing waves being thus of the same wavenumber and amplitude. This property may be restated in terms of the work done by the radiation stresses, by observing that it is proportional to the velocity gradient and reversible.

The extension of the classical solution to the case of an adverse current of magnitude larger than the critical value has been carried out by Smith (1976) and Peregrine (1976). In such a case the entire wave energy is reflected by the current barrier, the relationship between the wavenumber and amplitude of the incoming and outgoing waves being given in Peregrine (1976). Both authors, Peregrine (1976) by a stream-function analysis and Fourier transformation and Smith (1976) by multiple-scale expansion, which led to a nonlinear Schrödinger equation, solved the problem locally (in the vicinity of the critical point) and matched their solution to an outer classical solution.

Between these two types of solution there is a gap, namely the regime prevailing when the current velocity is close to the critical value, either above or below. One would expect intuitively that there should exist a continuous transition from total transmission to total reflexion when the current velocity approaches the critical value. The aim of the present study is to determine precisely the dependence of the degree of reflexion upon the current velocity distribution. To achieve this goal we convert the mathematical problem into a Schrödinger equation which is valid in *the entire flow domain* and take advantage of the rich literature on asymptotic solutions by WKB approximations of such equations (our main reference is Fröman & Fröman 1965). The previous results, of complete transmission (classical approach) and total reflexion, are obtained as limiting forms of our solution for a weak subcritical current or a strong supercritical current barrier, respectively. In essence, the WKB procedure we have adopted succeeds in connecting the solution on the left and right sides of the critical points by contour integration in the complex plane even if the critical points of the current velocity are outside the flow domain.

2. Mathematical statement of the problem

We consider a two-dimensional potential flow of water of infinite depth. Variables are made dimensionless with respect to length and time scales $\lambda/2\pi$ and $(\lambda/2\pi g)^{\frac{1}{2}}$, respectively, where λ is the wavelength of a harmonic wave propagating in still water with the same frequency as the incoming wave. The velocity potential $\phi(x,y,t)$ (with x the horizontal co-ordinate, y the vertical co-ordinate, positive upwards, and t the time) is defined in the flow domain $-\infty < x < +\infty$, $-\infty < y \leqslant \eta$, where $y = \eta(x,t)$ is the free-surface equation. ϕ satisfies the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0 \quad (y \leqslant \eta) \tag{1}$$

in the flow domain and the boundary condition

$$\phi_{tt} + 2\phi_x \phi_{tx} + 2\phi_y \phi_{ty} + \phi_x^2 \phi_{xx} + 2\phi_x \phi_y \phi_{xy} + \phi_y^2 \phi_{yy} + \phi_y = 0 \quad (y = \eta). \tag{2}$$

η is related to ϕ by the Bernoulli equation

$$\eta = -[\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + \text{constant}] \quad (y = \eta), \tag{3}$$

whereas (2) results from eliminating η from (3) and the kinematical free-surface condition (see Phillips 1966, p. 23).

We consider a potential ϕ which is the sum of two functions: the potential Φ of a steady current and an unsteady term originating from a simple harmonic wave incident from $x \to -\infty$. The general problem is now simplified by making the following assumptions.

(i) The incoming wave is of small amplitude, i.e. $\alpha = 2\pi a/\lambda = o(1)$, where a is the amplitude of the incoming wave. This leads to the following linearized approximations:

$$\phi(x,y,t) = \Phi(x,y) + \alpha[\phi^{(w)}(x,y)\, e^{-jt} + \phi^{(w)*}(x,y)\, e^{jt}] + O(\alpha^2), \tag{4}$$

$$\eta(x,t) = N(x) + \alpha[\eta^{(w)}(x)\, e^{-jt} + \eta^{(w)*}\, e^{jt}] + O(\alpha^2), \tag{5}$$

where j is the complex unit† and an asterisk denotes a complex conjugate. Substitution of (4) and (5) into (1)–(3) and asymptotic and Taylor expansions yield the exact equations of steady flow at $O(\alpha^0)$ and the linearized wave equations at $O(\alpha)$. These are the only orders considered here and the corresponding equations are derived next.

(ii) The steady current is characterized by a length scale L such that the velocity components $U = \Phi_x$ and $V = \Phi_y$ are functions of the slow variables

$$\tilde{x} = x/\gamma, \quad \tilde{y} = y/\gamma, \tag{6}$$

where $\gamma = L/(\lambda/2\pi) \gg 1$. This representation expresses the assumption that the steady current is slowly varying with respect to the wave train. To simplify the problem of the steady motion further, without much loss of generality for wave propagation, we should assume that $U = O(1)$ and $N = O(1)$, i.e. that the velocity and free-surface elevation are of the order of the wave speed and length, respectively. Hence $U_x = \gamma^{-1}U_{\tilde{x}}$, $N_{\tilde{x}} = \gamma^{-1}N_x, \ldots$ are small quantities $O(\gamma^{-1})$ and substitution in the exact equation satisfied by the steady flow,

$$U^2U_x + 2UVU_y + V^2V_y + V = 0 \quad (y = N), \tag{7}$$

yields on the free surface

$$V = -\gamma^{-1}U^2U_x + O(\gamma^{-3}). \tag{8}$$

By the same token, we obtain from (3)

$$N = -\tfrac{1}{2}U^2 + O(\gamma^{-2}) \quad (y = N), \tag{9}$$

where the constant in (3) is taken to be zero so that $N = 0$ for $U = 0$ (still water). This ordering is equivalent to that which would be obtained by assuming that the steady current is a small Froude number flow. Indeed, defining Fr in terms of the horizontal velocity gives $Fr = U\gamma^{-\frac{1}{2}} = O(\gamma^{-\frac{1}{2}})$.

(iii) With U and N given, the wave potential $\phi^{(w)}$ satisfies the Laplace equation

$$\phi^{(w)}_{xx} + \phi^{(w)}_{yy} = 0 \quad (y \leqslant N) \tag{10}$$

while $\eta^{(w)}$ is given by (3) as

$$\eta^{(w)} = j\phi^{(w)} - U\phi^{(w)}_x + O(\gamma^{-1}) \quad (y = N). \tag{11}$$

$\phi^{(w)}$ also satisfies the following linearized boundary condition:

$$a_0\,\phi^{(w)} + a_1\,\phi^{(w)}_x + a_2\,\phi^{(w)}_y + a_{11}\,\phi^{(w)}_{xx} + a_{12}\,\phi^{(w)}_{xy} = 0 \quad (y = N). \tag{12}$$

The coefficients in (12) depend on the steady current velocity field and have the following expressions in $\tilde{x}$ and $\tilde{y}$ at $O(\gamma^{-1})$:

$$\left.\begin{aligned} a_0 &= -1 - j\gamma^{-1}U_{\tilde{x}}, \quad a_1 = -2jU + 3\gamma^{-1}UU_x, \\ a_2 &= 1 + 2j\gamma^{-1}U^2U_{\tilde{x}}; \quad a_{11} = U^2, \quad a_{12} = -2\gamma^{-1}U^3U_{\tilde{x}}. \end{aligned}\right\} \tag{12a}$$

In conclusion, the mathematical problem is defined by (10) and (12), it being assumed that the steady current velocity field is given and that $\phi^{(w)}$ satisfies appropriate conditions at infinity.

† It is necessary to distinguish the complex unit $j = \surd(-1)$ here from the complex unit i in the $(\tilde{x}, \tilde{y})$ plane introduced in § 3.

3. Formulation of the problem in terms of ordinary differential equations

There are two difficulties associated with the free-surface condition (12): the coefficients are variable and the upper boundary $y = N$ is not horizontal. However, we seek an asymptotic solution of $\phi^{(w)}$ for large γ and such a solution can be obtained explicitly. We shall make a few preparatory transformations to achieve this aim.

(i) Equation (10) and the boundary condition (12) are rewritten in terms of new variables X and Y which map conformally the plane $\tilde{z} = \tilde{x}+i\tilde{y} = \gamma^{-1}z$ onto the plane $\tilde{Z} = \tilde{X}+i\tilde{Y} = \gamma^{-1}Z$ such that the free surface of the current (9) is mapped on $\tilde{Y} = 0$:

$$\tilde{Z} = \tilde{z}-\frac{\gamma^{-1}}{2\pi}\int_{-\infty}^{\infty}\frac{U^2(\tau,0)}{\tau-\tilde{z}}\,d\tau+O(\gamma^{-2}). \tag{13}$$

(ii) Assuming the steady current potential to be regular (or at worst with a singularity at depth $\tilde{Y} = O(1)$ as implied by the length scale of the current) and solving a Dirichlet problem in the lower half $\tilde{Z}$ plane, a second-order linear differential equation for the new dependent variable $F(\tilde{Z})$ is obtained:

$$F_{\tilde{Z}\tilde{Z}}+\gamma BF_{\tilde{Z}}+\gamma^2CF = 0, \tag{14}$$

where

$$\left.\begin{aligned} B &= i(1+2Q)(1+\gamma^{-1}R)/Q^2+2\gamma^{-1}Q'/Q,\\ C &= -(1+2\gamma^{-1}R)/Q^2+i\gamma^{-1}Q'/Q^2 \end{aligned}\right\} \tag{14a}$$

and Q and R are analytical functions of $\tilde{Z}$ given by

$$Q(\tilde{Z}) = \tfrac{1}{2}[W(\tilde{Z})+W^*(\tilde{Z}^*)],\quad W(\tilde{Z}) = U-iV, \tag{15a}$$

$$R(\tilde{Z}) = \gamma\,\mathrm{Re}\{\tilde{z}_{\tilde{Z}}-1\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Q'Q}{\tau-\tilde{Z}}\,d\tau+iQQ', \tag{15b}$$

and $Q' = dQ/d\tilde{Z}$.

In order to recover the wave potential $\phi^{(w)}$ from F, one has to replace the complex unit i by j in F and then identify $\phi^{(w)}$ with the complex conjugate of F.

The reader interested in the mathematical details of the aforementioned steps is referred to the appendix. Our last step is to transform (14) into the standard WKB equation. With

$$F(\tilde{Z}) = G(\tilde{Z})\exp[-(\gamma/2)\textstyle\int B\,d\tilde{Z}] = \dfrac{G}{2Q}\exp\left[-i\gamma\displaystyle\int\frac{(1+2Q)(1+\gamma^{-1}R)}{2Q^2}\,d\tilde{Z}\right] \tag{16}$$

equation (14) becomes the one-dimensional Schrödinger equation

$$G_{\tilde{Z}\tilde{Z}}+q^2(\tilde{Z})\,G = 0, \tag{17}$$

where

$$q^2(\tilde{Z}) = \gamma^2(1+4Q)(1+2\gamma^{-1}R)/4Q^4. \tag{17a}$$

Summarizing this section, the problem of propagation of waves of small amplitude on a slowly varying, low Froude number current, formulated initially with the aid of (10) and (12), is now expressed with the aid of (17) in the transformed lower half-plane $(\tilde{X}, \tilde{Y})$. The coefficient q^2/γ^2 in (17), which is real for $\tilde{Y} = 0$, contains terms up to $O(\gamma^{-1})$ and to the order of accuracy of the mapping (13),

$$Q(\tilde{X},0) = U(\tilde{X},0) = U(x,\gamma^{-1}N)+O(\gamma^{-2})$$

is the horizontal velocity of the steady current on the free surface. (17) defines the wave problem globally, along the entire X axis, and an asymptotic solution for large γ in a strip $-\epsilon < \tilde{Y} < 0$, $\epsilon = O(\gamma^{-1})$ is sought. The advantage of the present formula-

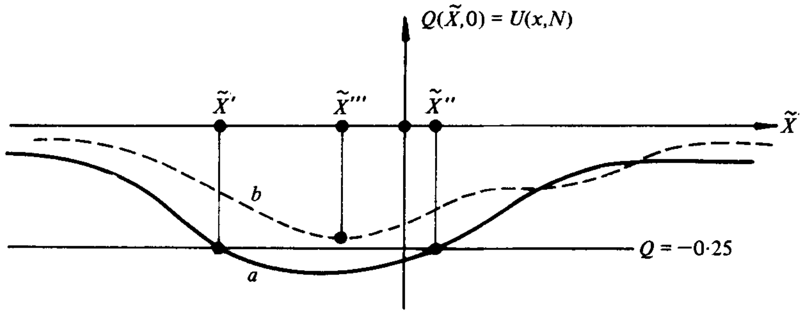

FIGURE 1. Current velocity distributions for: ——, supercritical barrier; ---, subcritical barrier.

tion stems from the possibility of employing the ready-made results of the well-known WKB analysis. An equation similar to (17) has been obtained by Smith (1976). Smith's representation is broader in the sense that it includes wave non-linearities as well as certain unsteady current terms. The validity of his Schrödinger equation is limited, however, to the neighbourhood of a strong supercritical current barrier, only.

4. Current barriers

In this section we will examine adverse currents, i.e. $U(x,N) < 0$ everywhere, without imposing the classical limitation $U > -0{\cdot}25$.

The neighbourhood of $U(x,N) = -0{\cdot}25$ $[U(\tilde{X},0) = -0{\cdot}25]$ corresponds to the 'potential barrier' of the Schrödinger equation, and we adopt a similar nomenclature here. We consider the two velocity distributions depicted in figure 1. In the case $U < -0{\cdot}25$ (curve a of figure 1) we refer to a supercritical barrier (overdense barrier for Schrödinger equation) whereas case b, in which U approaches the critical value from above ($U = -0{\cdot}25+\epsilon$, $\epsilon > 0$ and $\epsilon = o(1)$), is called the subcritical barrier (underdense barrier, respectively). The detailed mathematical treatment for both cases may be found in Fröman & Fröman (1965, chap. 9), and only their final results are given here.

(i) *Supercritical barrier*

In the case of the supercritical barrier (curve a, figure 1) the general solution of (17) is given by

$$G = c_1 g_1(\tilde{Z}) + c_2 g_2(\tilde{Z}), \tag{18}$$

where g_1, g_2 are the WKB solutions

$$g_1 = q(\tilde{Z})^{-\frac{1}{2}} \exp\{iS(\tilde{Z})\}, \quad g_2 = q(\tilde{Z})^{-\frac{1}{2}} \exp\{-iS(\tilde{Z})], \tag{19a, b}$$

$$S(\tilde{Z}) = \int^{\tilde{Z}} q(\tau)\, d\tau. \tag{20}$$

The coefficients c_1, c_2 in (18) are functions of $\tilde{Z}$ and the lower limit of the integral (20) is at the left-side turning point $(\tilde{X}', 0)$, (figure 1). Far on the left, from the barrier, g_1 corresponds to a wave with a wavenumber k_1 and g_2 to a wave with wavenumber K_1,

$$k_1 = 4/[1+(1+4U(x,N))^{\frac{1}{2}}]^2, \quad K_1 = 4/[1-(1+4U(x,N))^{\frac{1}{2}}]^2. \tag{21a, b}$$

Far to the right the opposite is true (this is a result of the definition of $q^{\frac{1}{2}}$ along the real axis, see Fröman & Fröman (1965), chap. 6). Both wave trains (k_1 and K_1) propagate to the right, and so does the energy flux (or wave action flux) of the k_1 wave. On the other hand, the energy flux of the K_1 wave, for adverse currents, points to the left, i.e. in a direction opposite to the wave celerity, see Peregrine (1976).†

Let us assume that the incoming wave is of type k_1 such that $c_1(-\infty, 0) = 1$. The radiation condition does not allow energy inflow from $x \to +\infty$, so that $c_1(+\infty, 0) = 0$. From these two conditions we obtain, as in Fröman & Fröman [1965, chap. 9, equations (9.13), (9.15), (9.17)‡] far from the turning points

$$G(\tilde{X}) = \frac{1}{|q|^{\frac{1}{2}}} \exp\left\{i\int_{\tilde{X}'}^{\tilde{X}} |q|\, d\tilde{X}\right\} - \frac{iR_c}{|q|^{\frac{1}{2}}} \exp\left\{-i\int_{\tilde{X}'}^{\tilde{X}} |q|\, dX\right\} \quad (\tilde{X} < \tilde{X}'), \qquad (22a)$$

$$G(\tilde{X}) = \frac{T_c}{|q|^{\frac{1}{2}}} \exp\left\{i\int_{\tilde{X}''}^{\tilde{X}} |q|\, d\tilde{X}\right\} \quad (\tilde{X} > \tilde{X}''). \qquad (22b)$$

If the two turning points $(\tilde{X}', 0)$ and $(\tilde{X}'', 0)$ are separated, $\tilde{X}'' - \tilde{X}' = O(1)$, we also have between the turning points

$$G(\tilde{X}) = \frac{R_c e^{-\frac{1}{4}i\pi}}{|q|^{\frac{1}{2}}} \exp\left\{-\int_{\tilde{X}'}^{\tilde{X}} |q|\, d\tilde{X}\right\} \quad (\tilde{X}' < \tilde{X} < \tilde{X}''). \qquad (22c)$$

The quantities T_c and R_c are given by

$$R_c = 1/[1+\exp(-2K_c)]^{\frac{1}{2}}; \quad T_c = \exp(-K_c)[1+\exp(-2K_c)]^{\frac{1}{2}}, \qquad (23a, b)$$

$$K_c = \int_{\tilde{X}'}^{\tilde{X}''} |q(\tilde{X})|\, d\tilde{X} = \int_{x'}^{x''} \frac{|1+4U(x,N)|^{\frac{1}{2}}}{2U^2(x,N)}\, dx + O(\gamma^{-1}). \qquad (23c)$$

When $\tilde{X}'' - \tilde{X}' = O(1)$, $K_c \gg 1$, and consequently $T_c \simeq 0$, $R_c \simeq 1$ which corresponds to the case of a strong supercritical barrier, discussed by Smith (1976).

From (22*a*, *b*), (16), (11) and (5) we have for the free-surface

$$\eta = -U^2/2 + C\frac{1-Uk_1}{|1+4U|^{\frac{1}{4}}} \cos\left[\int_{x'}^{x} k_1\, dx - t\right] + CR_c \frac{1-UK_1}{|1+4U|^{\frac{1}{4}}} \cos\left[\int_{x'}^{x} K_1\, dx - t + \pi/2\right] \quad (x < x', \quad y = N), \qquad (24a)$$

$$\eta = -U^2/2 + CT_c \frac{1-Uk_1}{|1+4U|^{\frac{1}{4}}} \cos\left[\int_{x''}^{x} k_1\, dx - t\right] \quad (x > x'', \quad y = N). \qquad (24b)$$

The constant C is determined by the amplitude of the incoming wave (α) throughout the relationship

$$C = \alpha|1+4U_\infty|^{\frac{1}{4}}/[1-U_\infty k_1(U_\infty)], \qquad (25)$$

where U_∞ is the current velocity at infinity (see A 2).

† The two additional wavenumbers mentioned by Peregrine are those connected with equation (A 11), i.e. with incoming waves from $x = \infty$ which we disregard for physical reasoning. Indeed, the radiation condition allows no energy flux from $|x| \to \infty$, except that of the incoming wave k_1.

‡ It is pointed out that an error has crept in equation (9.17), namely the expression appearing in the second line should be multiplied by $-i$.

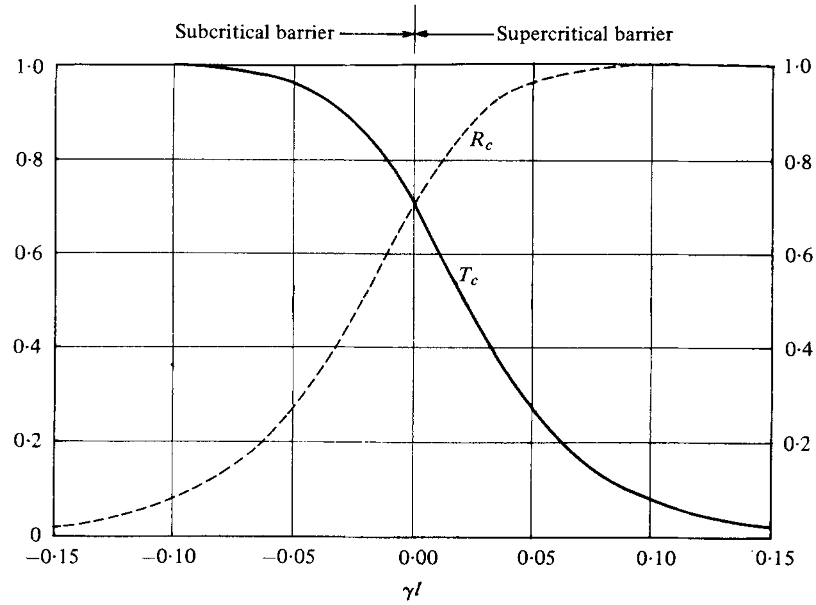

FIGURE 2. Reflexion (R_c) and transmission (T_c) coefficients for a parabolic barrier.

(ii) *Subcritical barrier*

In the case of a subcritical barrier, the solution far from the barrier is the same as before, i.e. (24*a*) is valid for $x < x'''$ and (24*b*) for $x > x'''$ with the lower limit of the integrals at an arbitrary point on the real axis. Equations (23*a*, *b*) are also valid and only (23*c*) is replaced by

$$K_c = -\gamma \int_{-\tilde{Y}'''}^{\tilde{Y}'''} \frac{[1+4Q(\tilde{X}'''+i\tilde{Y})]^{\frac{1}{2}}}{2Q^2(\tilde{X}'''+i\tilde{Y})}\, d\tilde{Y} \leqslant 0. \tag{26}$$

The points $(\tilde{X}'''-i\tilde{Y}''', \tilde{X}'''+i\tilde{Y}''')$ are the roots of the equation $Q(\tilde{Z}) = -0{\cdot}25$, being assumed that, in the neighbourhood of the saddle $(\tilde{X}''', 0)$, Q has the expansion $Q(\tilde{Z}) = Q(\tilde{X}''') + Q_{\tilde{Z}\tilde{Z}}(\tilde{X}''').(\tilde{Z}-\tilde{X}''')^2/2+\ldots)$ with $Q_{\tilde{Z}\tilde{Z}}(\tilde{X}''') > 0$.

Because of the negative sign of K_c and because $\gamma \gg 1$, the coefficient $R_c = O(\exp(K_c))$ is generally very small and $T_c \simeq 1$, so that the classical solution (Phillips 1966, p. 56) is obtained. If $\tilde{Y}''' \ll 1$, however, e.g. $\tilde{Y}''' = O(\gamma^{-1})$, and the current approaches the critical value, the result is essentially different: when the two points for which $Q = -0{\cdot}25$ collapse on the real axis $R_c = T_c = 1/\sqrt{2}$ and the same result is obtained for a supercritical barrier at the limit $\tilde{X}''-\tilde{X}' \to 0$.

(iii) *Illustration*

To illustrate the results let us consider an adverse current with a parabolic velocity distribution near the critical point

$$Q = -(0{\cdot}25+l)+(\tilde{Z}-\tilde{X}_{\max})^2, \tag{27}$$

where for $\tilde{Z} = \tilde{X}_{\max}$ the strength of the current is the largest and the constant $l = o(1)$

might be positive or negative, for supercritical or subcritical barriers, respectively. According to (23c) and also (26) we have

$$K_c = \pi\gamma l/(0{\cdot}25+l)^{\frac{3}{2}} \approx 8\pi\gamma l. \tag{28}$$

The relationship between T_c, R_c and γl based on (28) and (23a,b) is represented graphically in figure 2. The transition from the classical solution ($R_c \to 0$, $T_c \to 1$), Longuet-Higgins & Stewart (1961) or Phillips (1966), towards the strong supercritical case with $\tilde{X}'' - \tilde{X}' = O(1)$ ($T_c \to 0$, $R_c \to 1$), Smith (1976) or Peregrine (1976), is clearly illustrated.

(iv) *Energy balance*

The wave action flux through a vertical plane is given, as shown by Peregrine (1976), p. 40, by

$$e = \frac{\alpha^2}{2\sigma}(U+c_g), \tag{29}$$

where α, σ and c_g are the amplitude, frequency and deep water group velocity as seen by an observer moving with the current, respectively. Using (24a, b), the well-known Doppler condition, (21a, b) and (29) the fluxes e_l and e_r, far to the left of the barrier and far to the right, are as follows:

$$e_l = \tfrac{1}{4}C^2(1-R_c^2); \quad e_r = \tfrac{1}{4}C^2T_c^2 \tag{30a, b}$$

with C given by (25).

Using (23a, b) it is seen that e_l and e_r obey the requirement of wave action conservation

$$e_l - e_r = \tfrac{1}{4}C^2[1-(R_c^2+T_c^2)] = 0. \tag{31}$$

It is also seen that T_c and R_c can be regarded as coefficients of energy transmission and reflexion, respectively.

Hence, the two curves of figure 2, interdependent through (31), give a picture of the amount of energy reflected by a current barrier. It is worthwhile to emphasize again that energy is reflected by the K_1 waves.

5. Summary and discussion

The present study shows by analytical means that the energy of waves which encounter a barrier of an adverse current may be partially transmitted while the remaining amount is reflected by waves of type K_1. If, however, the slope of the incoming wave exceeds a certain value, the slope growth near the barrier may cause the wave breaking and part of the energy is dissipated (as was shown by Peregrine (1976) for the case of a strong supercritical barrier).

To grasp the physical implications of the results we shall recall the three typical cases:

(i) *Supercritical barrier with total reflexion.* It might be instructive to compare the current barrier with a vertical wall reflecting a monochromatic wave in still water. In both cases, the wave energy is totally reflected but there is a profound difference between the two mechanisms. The wall reflects a wave of same wavenumber and amplitude as the incoming k_1 wave, resulting in a standing wave of twice the amplitude. If the flow is extended to infinity beyond the wall, a discontinuity in the wavenumber is introduced to represent the effect of the wall. A current barrier reflects energy by a

K_1 wave which is the only one which ensures a continuous transition from the k_1 incoming wave to a wave carrying energy to the left. In fact, the amplitude and wavenumber of the K_1 wave far from the barrier could be found easily, in the frame of the present linear analysis, from the dispersion equation and energy flux balance, respectively.

(ii) *Subcritical barrier with weak reflexion.* At the other extreme, a weak subcritical barrier causes negligible reflexion and wave propagation may be analysed with the aid of the classical radiation-stress concept of Longuet-Higgins & Stewart (1961). Still, if the current distribution regarded on the free-surface as the real part of an analytical function, has a saddle point behaviour, there is an exponentially weak reflexion of the wave energy. This type of reflexion is similar to that considered by Meyer (1975) for the simple case of the solution of the one-dimensional wave equation with a variable phase velocity.

(iii) *Near critical barrier with partial reflexion.* This is precisely the intermediate case analysed here for the first time. Unlike (i) and (ii) a quantitative evaluation of the reflexion or transmission coefficient is not possible simply by applying wave action conservation far from the barrier, but requires analysis along the entire free-surface.

The present work is part of a thesis by M. Stiassnie submitted to the Department of Applied Mathematics, Technion, in partial fulfilment of the requirements for a Doctor of Science degree.

Appendix

The steps (i) and (ii) of § 3 which lead from (10), (12) to (14) are given here in detail.

(i) The flow domain of the steady current is mapped conformally onto a half-plane. A 'slow' new vertical variable $\tilde{Y}$ is defined by

$$\tilde{Y} = \tilde{y} + \tfrac{1}{2}\gamma^{-1}U^2(\tilde{x}, \gamma^{-1}N) + O(\gamma^{-2}) \qquad \text{(A 1)}$$

such that for $\tilde{y} = \gamma^{-1}N$, $\tilde{Y} = 0$ [see (8)]. But $U(\tilde{x}, \gamma^{-1}N) = U(\tilde{x}, 0) + O(\gamma^{-1})$ so that at the same order, (A 1) can be rewritten as

$$\tilde{Y} = \tilde{y} + \gamma^{-1}U^2(\tilde{x}, 0)/2.$$

The mapping $\tilde{Z} = \tilde{X} + i\tilde{Y}$ is therefore given by (13).

At infinity we have

$$U_\infty = \lim_{|\tilde{x}|\to\infty} U(\tilde{x}, \gamma^{-1}N) \quad \text{and} \quad N_\infty = -\tfrac{1}{2}U_\infty^2. \qquad \text{(A 2)}$$

Keeping the same symbols in the transformed planes

$$U(\tilde{X}, \tilde{Y}) \equiv U(\tilde{x}, \tilde{y}); \quad V(\tilde{X}, \tilde{Y}) \equiv V(\tilde{x}, \tilde{y}); \quad \phi^{(w)}(X, Y) \equiv \phi^{w}(x, y), \qquad \text{(A 3)}$$

we observe that $U_{\tilde{x}} = U_{\tilde{X}} + O(\gamma^{-1})$. Hence, the coefficients a_0, a_1, ... in (12*a*) can be rewritten at once at same order with the aid of the new variables by just replacing $U_{\tilde{x}}$ by $U_{\tilde{X}}$. Furthermore, regarding the analytical functions $Q(Z)$ and $R(Z)$ (see (15*a*, *b*)), we have $Q(\tilde{X}, 0) = U(\tilde{X}, 0)$ and

$$R(\tilde{X}, 0) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{UU_\tau}{\tau - \tilde{X}}\, d\tau,$$

where PV stands for the principal value.

By chain differentiation and by using (13), (A 2), (A 3) and (15*a*, *b*) the free-surface condition (12) is now rewritten with the aid of the new variables as follows,

$$(A_0^R+jA_0^J)\,\phi^{(w)}+(A_1^R+jA_1^J)\,\phi_X^{(w)}+A_2^R\,\phi_Y^{(w)}+A_{11}^R\,\phi_{XX}^{(w)}=0 \quad (Y=0). \qquad \text{(A 4)}$$

The coefficients $A_0^R, A_0^J, \ldots, A_{11}^R$ are real on $Y=0$ and they are given at $O(\gamma^{-1})$ by

$$\left.\begin{aligned} &A_0^R=-1;\quad A_0^J=-\gamma^{-1}Q';\quad A_1^R=2\gamma^{-1}QQ';\quad A_1^J=-2Q(1-\gamma^{-1}R);\\ &A_2^R=1-\gamma^{-1}R;\quad A_{11}^R=Q^2(1-2\gamma^{-1}R).\end{aligned}\right\} \qquad \text{(A 4a)}$$

(ii) We express now the potential $\phi^{(w)}$ with the aid of analytical functions of Z by defining

$$\phi^{(w)}=\Phi^R+j\Phi^J;\quad \Phi^R=Re_i\{F^R(Z)\};\quad \Phi^J=Re_i\{F^J(Z)\}, \qquad \text{(A 5)}$$

where F^R and F^J are holomorphic in the lower Z half-plane and Re_i stands for real part with respect to complex variables in i. The free-surface condition (A 4) generates two equations for F^R and F^J as follows:

$$Re_i\{q_1\}=0;\quad Re_i\{q_2\}=0 \quad (Y=0), \qquad \text{(A 6a, b)}$$

$$q_1=A_0^R\,F^R-A_0^J\,F^J+(A_1^R+iA_2^R)\,F_Z^R-A_1^J\,F_Z^J+A_{11}^R\,F_{ZZ}^R, \qquad \text{(A 7)}$$

$$q_2=A_0^R\,F^J+A_0^J\,F^R+(A_1^R+iA_2^R)\,F_Z^J+A_1^J\,F_Z^R+A_{11}^R\,F_{ZZ}^J. \qquad \text{(A 7b)}$$

If the coefficients $A_0^R, A_0^J, \ldots, A_{11}^R$ are holomorphic, q_1 and q_2 are also holomorphic for $Y\leqslant 0$ and (A 6*a*, *b*) give at once

$$q_1(Z)\equiv 0;\quad q_2(Z)\equiv 0 \quad (Y\leqslant 0). \qquad \text{(A 8a, b)}$$

Generally, the potential of the steady current is singular and so are the coefficients (A 4*a*). But the singularity is at a depth $\tilde{Y}=O(1)$, as implied by our basic assumption regarding the length-scale of the current. It can be shown (Stiassnie 1977), that such singularities would introduce terms $O(e^{-\gamma})$ in the right-hand side of (A 8*a*, *b*) which are negligible at the order considered here. We adopt (A 8*a*, *b*), therefore, as identities valid in any case and eliminate F^R and F^J from (A 8*a*, *b*) by defining the new functions

$$F=\tfrac{1}{2}(F^R-iF^J);\quad \mathscr{F}=\tfrac{1}{2}(F^R+iF^J). \qquad \text{(A 9)}$$

Equations (A 8*a*, *b*) now become, in terms of the 'slow' variable $\tilde{Z}$,

$$F_{\tilde{Z}\tilde{Z}}+\gamma BF_{\tilde{Z}}+\gamma^2 CF=0, \qquad \text{(A 10)}$$

$$\mathscr{F}_{\tilde{Z}\tilde{Z}}+\gamma\mathscr{B}\mathscr{F}_{\tilde{Z}}+\gamma^2\mathscr{C}\mathscr{F}=0, \qquad \text{(A 11)}$$

with B and C given in (14*a*), and

$$\mathscr{B}=i(1-2Q)\,(1+\gamma^{-1}R)/Q^2+2\gamma^{-1}Q'/Q, \qquad \text{(A 11a)}$$

$$\mathscr{C}=-(1+2\gamma^{-1}R)/Q^2-i\gamma^{-1}Q'/Q^2.$$

At this point it is emphasized that the separation of (A 8*a*, *b*) into the two equations (A 10) and (A 11) would have not been possible if terms $O(e^{-\gamma})$ were present in the right-hand side of (A 8*a*, *b*) (a detailed discussion is given in Stiassnie 1977).

Furthermore, the equations (A 10) and (A 11) pertain to two distinct cases: equation (A 10) for F together with the condition $\mathscr{F}\equiv 0$ is related to waves incoming from the left ($X\to-\infty$); whereas (A 11) for $\mathscr{F}$ with the condition $F\equiv 0$ represents waves incoming from the right ($X \to \infty$). It is enough, therefore, to solve one of the two equations (A 10), (A 11) for the case of a one-sided incoming wave from infinity and this is precisely the aim of the present study. We take, therefore, $\mathscr{F} \equiv 0$, and concentrate on solving (A 10) which is identical to (14).

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