

# MOMENTUM AND ENERGY INTEGRALS FOR GRAVITY WAVES OF FINITE HEIGHT

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1. *Introduction.* The analytical study of gravity waves of finite height which are periodic and move without change in shape in an ideal irrotational fluid is a subject which has occupied the attention of physicists and mathematicians ever since Stokes (9) published his classic paper entitled "On the Theory of Oscillatory Waves" in 1847. However, it was not until 1925 that an existence proof for such waves in the case of a basin of infinite depth was published by Levi-Civita (4). The corresponding proof for the case of limited depth was presented by Struik (10) in 1926.

Much of the effort that has been expended on this wave problem was directed toward the computation of the exact wave form which results from the physical circumstances involved. In this paper we shall not pursue such a course, but rather we shall seek certain simple integral properties of the wave motion, in a sense, directly from the differential equations governing the dynamic system considered, using only a few general properties of the solutions of these equations. In this process we shall assume, of course, the existence of a detailed solution, relying for this upon the proofs given by the investigators alluded to earlier. In the last section the solitary wave will be studied from the same point of view.

2. *General characteristics of the wave motion.* The waves whose study is contemplated here are plane waves, with the crests straight and with every vertical section normal to the crests presenting the same appearance, namely that given in Fig. 45 by the full lines. The basin is assumed to be of a depth  $D$ , with a horizontal plane bottom and a horizontal plane free surface when there are no waves present. A constant and uniform vertical force of gravity is assumed to act downward on all the fluid particles. The basin is assumed to be of very large dimensions in the direction normal to the crests, but may be limited by plane parallel vertical walls normal to the crests. All the motions are assumed to take place relative to a nonrotating coordinate system. By-and-large these are the same conditions as were

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assumed by Levi-Civita and Struik, the case of infinite depth being approached by making  $D$  very large.

We shall make use of a cartesian coordinate system, with the origin below a crest at the mean level of the fluid, the vertical coordinate  $z$  being counted positive upward and the horizontal coordinate  $x$  counted positive to the right in Fig. 45.

When waves are present the free surface is deformed more or less as shown, so that at a given instant its shape is given by  $z = z_0(x)$  where  $z_0(x)$  is a periodic function which is symmetric about the crests and the troughs. We shall assume that the waves are propagated to the right in the figure. Since this propagation is not accompanied by

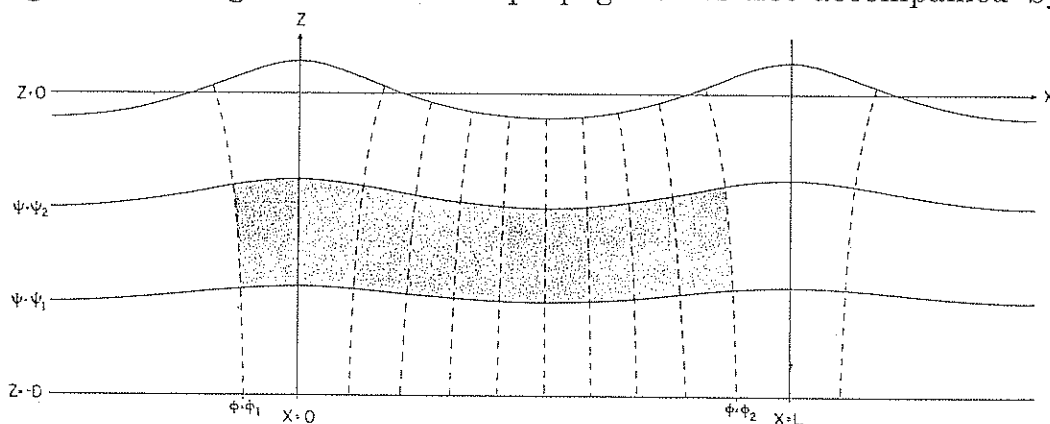


Figure 45. Schematic vertical cross section representing periodic gravity waves in a basin of depth  $D$ . The full lines are streamlines while the dashed lines are equipotential lines, both for the motion relative to the moving wave. The origin for the vertical coordinate is taken at the mean level of the free surface. The shaded area represents a characteristic region over which various integrals are taken as explained in the text.

any change of structure at the surface or below, it follows that the only effect is to cause a translation of the picture with the (constant) velocity of propagation  $c$ .

Suppose now that we add a constant horizontal translation,  $-c$ , to all the particles of fluid involved. The effect of this fictitious added translation to the left makes the wave structure stationary, and the problem is reduced to one of steady motion in a plane. The constant translation does not introduce any change in the internal dynamics of the system and can be allowed for by purely kinematic considerations later.

From the considerations presented above it is apparent that the wave problem can be reduced to a special case of a two-dimensional steady-state jet problem under gravity, having a "free" streamline along which the pressure is constant (or zero) at the top, and a rigid horizontal boundary below. Assuming that, by suitable means, a

proper solution to this jet problem is specified, there still remains a certain arbitrariness in regard to the corresponding progressive wave (except for the case of infinite depth). This difficulty arises from the fact that any constant horizontal translation added to the solution of the jet problem will result in a possible moving wave. This arbitrariness has been mentioned by Stokes (9) and also by Lamb (2). One convention which can be followed is to define the wave speed so that the resulting progressive waves do not produce a net transport of fluid. We shall, however, adopt an alternative which will be discussed presently.

3. *A momentum integral for periodic gravity waves.* In an article published previously, the writer (8) presented a simple derivation of a momentum integral for gravity waves in deep water. An integral of the same nature was first derived by Levi-Civita (3) using a different method. A generalization of the writer's approach will now be given for the case of a basin of arbitrary depth.

Considering the steady-state two-dimensional motion represented in Fig. 45, we have at our disposal the continuity equation which may be written in the form

$$(1) \quad \frac{\partial U}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

since the fluid is incompressible. In (1)  $U$  is the  $x$ -component of the particle velocity and  $w$  is the  $z$ -component. Since the motion is irrotational, we also have the relationship

$$(2) \quad \frac{\partial w}{\partial x} - \frac{\partial U}{\partial z} = 0.$$

The steady-state velocity component  $U$  is related to the wave speed  $c$ , which will be discussed later, by the equation,

$$(3) \quad U = u - c,$$

where  $u$  is the  $x$ -component of the particle velocity in the actual progressive wave. The vertical velocity component  $w$  is, of course, the same both in the steady motion and in the progressive wave.

By virtue of equations (1) and (2) we may introduce a velocity potential  $\Phi$  and a stream function  $\Psi$  for the steady motion, such that

$$(4) \quad U = -\frac{\partial \Phi}{\partial x} = -\frac{\partial \Psi}{\partial z}; \quad w = -\frac{\partial \Phi}{\partial z} = +\frac{\partial \Psi}{\partial x}.$$

Lines along which  $\Psi$  is constant constitute streamlines for the steady motion, and are shown by solid lines in Fig. 45. The free surface and

the bottom at  $z = -D$  are two members of this family of curves. The lines along which  $\Phi$  is constant form a family of orthogonal trajectories to the steady-state streamlines and are shown by dashed lines in Fig. 45. Both  $\Phi$  and  $\Psi$  are undefined by (4) to the extent of an additive constant in each case.

The velocity components in the progressive wave being  $u$  and  $w$ , it follows from (1), (2) and (3) that we may also introduce a velocity potential  $\phi$  and a stream function  $\psi$  for the progressive wave, such that

$$(5) \quad u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial z}; \quad w = -\frac{\partial \phi}{\partial z} = +\frac{\partial \psi}{\partial x}.$$

From (3), (4) and (5) it follows that

$$(6) \quad \Phi = \phi + cx + k_\phi; \quad \Psi = \psi + cz + k_\psi,$$

$k_\phi$  and  $k_\psi$  being constants. Owing to the fact that the actual wave motion is strictly periodic in  $x$ , we have that, at a given instant,

$$(7) \quad [U, u, w]_{x+L, z} = [U, u, w]_{x, z},$$

where  $L$  is the wave length. Defining the quantities  $\Delta\Phi$  and  $\Delta\phi$  by the equations

$$(8) \quad \Phi_{x+L, z} = \Phi_{x, z} + \Delta\Phi; \quad \phi_{x+L, z} = \phi_{x, z} + \Delta\phi,$$

it is seen from (4), (5) and (7) that  $\Delta\Phi$  and  $\Delta\phi$  are *independent of  $x$  and  $z$* . As a further consequence of (4), (5) and (7) we may write that

$$(9) \quad \Psi_{x+L, z} = \Psi_{x, z}; \quad \psi_{x+L, z} = \psi_{x, z}.$$

No constants appear in (9), since the stream function is a measure of transport which must be the same for successive wave lengths by virtue of (7). It follows from (8) and (9) that the pattern of streamlines and equipotential lines for regions separated by an integral number of wave lengths is congruent. From the first equation in (6) and from (8) we obtain the relation that

$$(10) \quad \Delta\phi = \Delta\Phi - cL.$$

Let us now consider a region in Fig. 45 bounded below by the steady-state streamline  $\Psi = \Psi_1$  and above by the streamline  $\Psi = \Psi_2$ , where the choice of  $\Psi_1$  and  $\Psi_2$  is arbitrary. Let us further consider a closed portion of this region bounded at the left by the arbitrary equipotential line  $\Phi_1$  and at the right by the equipotential line  $\Phi_2$  separated from  $\Phi_1$  by one wave length so that  $\Phi_2 - \Phi_1 = \Delta\Phi$ . The kinetic energy  $E$  of the steady-state motion represented by this closed region, per unit distance along the crests of the waves, is given by

$$(11) \quad \frac{2}{\rho} E = \int \int (U^2 + w^2) dx dz = \int \int \frac{\partial(\Phi, \Psi)}{\partial(x, z)} dx dz = \int \int d\Phi d\Psi = \Delta\Phi \Delta\Psi,$$

where use has been made of (4) and  $\rho$  is the (uniform) density.<sup>1a</sup> With the conventions that have been introduced, both  $\Delta\Phi$  and  $\Delta\Psi$  are positive. In similar fashion it is possible to obtain an expression for the horizontal momentum  $M$  of the steady motion represented by the closed region. We thus have that

$$(12) \quad \frac{1}{\rho} M = \int \int U dx dz = - \int \int \frac{\partial\Psi}{\partial z} dx dz = - \int \int dx d\Psi = -L\Delta\Psi.$$

In order to pass from the steady-state motion to the actual motion, we may either transform the integrals  $E$  and  $M$  by means of (3), or we may write directly that<sup>2</sup>

$$(13) \quad E = e - cm + \frac{1}{2} c^2 \mu,$$

and

$$(14) \quad M = m - c\mu.$$

Here we define  $e$ ,  $m$  and  $\mu$  as follows:

$$(15) \quad e = \frac{\rho}{2} \int \int (u^2 + w^2) dx dz \quad ; \quad m = \rho \int \int u dx dz \quad ; \\ \mu = \rho \int \int dx dz,$$

so that  $e$  is the kinetic energy and  $m$  the horizontal momentum of the mass  $\mu$  (per unit thickness) in the actual motion. Upon eliminating  $\mu$  between (13) and (14) and then substituting from (11) and (12) for  $E$  and  $M$ , and finally by using (10), we obtain

<sup>1a</sup> In general, limits of integration have been omitted where they are clear from the text.

<sup>2</sup> Equation (13) is obtained from the mechanical principle that the kinetic energy of a system of particles is the sum of the kinetic energy due to the motions with respect to the common center of mass and the kinetic energy due to the motion of the center of mass. Thus, if in our problem we denote the first quantity by  $T$  and the velocity of the center of mass in the actual motion by  $\gamma$ , we have that

$$e = T + \frac{1}{2} \gamma^2 \mu \quad ; \quad E = T + \frac{1}{2} (\gamma - c)^2 \mu,$$

from which (13) follows when  $T$  is eliminated.

$$(16) \quad 2e - cm = 2E + cM = \rho(\Delta\Phi - cL)\Delta\Psi = \rho\Delta\phi\Delta\Psi.$$

In the case of an infinitely deep basin, the physically significant solution to the wave problem is one in which all the actual motions disappear at great depths. This means that the constant  $\Delta\phi$  must be zero in this case. *We shall make the convention that  $\Delta\phi = 0$  also for the case of a basin of finite depth.* This convention is equivalent to the relationship that

$$(17) \quad c = \frac{\Delta\Phi}{L},$$

which we assume to be true for all cases whether the depth be infinite or not. We thus have the two relationships

$$(18) \quad 2E + cM = 0,$$

and

$$(19) \quad \boxed{2e - cm = 0}.$$

The last equation states that the kinetic energy per wave length of the deformed layer considered, when multiplied by two, is equal to the horizontal momentum of the layer per wave length multiplied by the speed of propagation.

There is no limitation in the development which would preclude the possibility of applying (19) to the entire depth of a finite basin or to the total kinetic energy and total momentum for the case of infinite depth.<sup>3</sup>

Let us apply (19) to the entire depth of fluid between the free surface  $z_0$  and  $-D$ . Introducing average values  $\tau$  and  $\gamma$  defined by

$$(20) \quad \tau = \frac{1}{LD} \int \int \frac{1}{2} (u^2 + w^2) dx dz,$$

and

$$(21) \quad \gamma = \frac{1}{LD} \int \int u dx dz,$$

so that  $\tau$  is the mean kinetic energy divided by the density and  $\gamma$  is the mean transport velocity, we may write (19) in the form

<sup>3</sup> Although the region considered above is bounded laterally by equipotential lines for the steady motion, this is not essential. Any region, bounded above and below by lines along which  $\Psi$  is constant and whose lateral boundaries are congruent in such a way that they could be brought into coincidence by horizontal translation, may serve equally well, provided that the lateral boundaries are separated by an integral number of wave lengths. This follows from the fact that the more general region may be considered as being composed of a large number of elementary regions of the simple type discussed in the text.

$$(22) \quad \frac{\gamma}{c} = \frac{\tau}{\frac{1}{2}c^2}.$$

Equation (22) was first derived by Levi-Civita (3) by other methods. From our analysis it is clear that a relationship of the form (22) is true not only for the entire layer but also for *any* layer which has the shape indicated earlier.

It is of interest to examine what further kinematic implications follow from the choice of  $c$  expressed by (17). If the first equation in (4) is integrated along the bottom streamline at  $z = -D$ , we have, with the aid of (17),

$$(23) \quad \int_{x_1}^{x_1+L} U_D dx = -\Delta\Phi = -cL.$$

Using the transformation

$$(24) \quad dx = U_D dt,$$

where  $t$  is time, we may write

$$(25) \quad \int_{t_1}^0 U_D^2 dt = -\bar{U}_D^2 t_1 = -cL,$$

from (23), where  $t_1$  is the time necessary for a particle to move from  $x = x_1 + L$  to  $x = x_1$  in the steady motion, and  $\bar{U}_D^2$  is the time average of the square of the velocity of the particle during this period  $t_1$ . We may also write

$$(26) \quad \int_0^{t_1} U_D dt = -L = \bar{U}_D t_1,$$

where  $\bar{U}_D$  is the time average of  $U_D$  for a particle during the period  $t_1$ . From (25) and (26) we may now form the relation

$$(27) \quad c \bar{U}_D = -\bar{U}_D^2.$$

By averaging (3) over the period  $t_1$  we may write (27) in the form

$$(28) \quad c \bar{u}_D = -\bar{u}_D^2,$$

showing that  $\bar{u}_D$ , the average particle velocity at the bottom in the actual wave, is non-negative so that the particles progress in the direction of propagation when waves are present.

From (23), when combined with (3), it follows that

$$(29) \quad \int_{x_1}^{x_1+L} u_D dx = 0.$$

Since in irrotational motion all closed circuits must have zero circulation, it follows from (29) that the line integral from  $x = 0$  to  $x = L$  of the velocity in the progressive wave along *any* curve joining these verticals (which are located at crests) is zero, or that

$$(30) \quad \int_0^L (u dx + w dz) = 0.$$

4. *An energy integral for periodic gravity waves.* For the purposes of the subsequent discussion it is necessary to consider the dynamical aspects of the motions studied. The dynamical properties of the motions are, in general, specified by the two hydrodynamical equations of motion for the  $x$  and  $z$ -directions and the equation of continuity (1). For a nonrotating coordinate system the two former equations, in the order mentioned, are:

$$(31) \quad \rho \frac{dU}{dt} = - \frac{\partial p}{\partial x}; \quad \rho \frac{dw}{dt} = - g\rho - \frac{\partial p}{\partial z}.$$

Here  $p$  is pressure and  $g$  is the acceleration of gravity. If we introduce a potential  $\kappa$  defined by

$$(32) \quad \kappa = p + g\rho z,$$

equations (31) become

$$(33) \quad \rho \frac{dU}{dt} = - \frac{\partial \kappa}{\partial x}; \quad \rho \frac{dw}{dt} = - \frac{\partial \kappa}{\partial z}.$$

Equations (33) possess an integral which we may write as

$$(34) \quad \frac{\rho}{2} (U^2 + w^2) + \kappa = \frac{\rho}{2} c_1^2,$$

which is a statement of Bernoulli's equation for the steady-state motion in our problem. In general the constant  $c_1^2$  may vary from streamline to streamline, but for the case of irrotational motion it is the same for the whole plane.

Let us rewrite (34), first in the form

$$(35) \quad \frac{\rho}{2} (U^2 + w^2) + \frac{\partial \kappa z}{\partial z} - z \frac{\partial \kappa}{\partial z} = \frac{\rho}{2} c_1^2,$$



and then, with the aid of the second equation in (33), in the form

$$(36) \quad \frac{\rho}{2} (U^2 + w^2) + \frac{\partial \kappa z}{\partial z} + \rho z \frac{dw}{dt} = \frac{\rho}{2} c_1^2,$$

or

$$(37) \quad \frac{\rho}{2} (U^2 - w^2) + \frac{\partial \kappa z}{\partial z} + \rho \frac{dzw}{dt} = \frac{\rho}{2} c_1^2.$$

We shall now integrate (37) over the area mentioned in the previous section, bounded by two streamlines and two equipotential lines separated by one wave length, so that

$$(38) \quad E_x - E_z + \iint \frac{\partial \kappa z}{\partial z} dx dz + \rho \iint \frac{dzw}{dt} dx dz = \frac{1}{2} c_1^2 \mu,$$

where  $E_x$  and  $E_z$  are the kinetic energies of the horizontal and vertical components of motion respectively, in the steady-state motion over the area taken. From (34) and (7) we know that

$$(39) \quad [\kappa z, wz]_{x+L, z} = [\kappa z, wz]_{x, z}.$$

It can easily be shown that under these circumstances the last integral on the left-hand side of (38) vanishes.<sup>4</sup> With the aid of Stokes theorem and (39) we may write that

$$(40) \quad \iint \frac{\partial \kappa z}{\partial z} dx dz = - \oint \kappa z dx = \int_L \kappa_2 z_2 dx - \int_L \kappa_1 z_1 dx,$$

where the contributions of the lateral boundaries vanish because of the periodicity of  $\kappa z$  over a wave length. We may now eliminate  $\kappa$  by means of (32) and substitute the result in (38) so that

<sup>4</sup> Let us consider any property  $H$  characteristic of the steady-state picture. We may write that

$$\frac{dH}{dt} = U \frac{\partial H}{\partial x} + w \frac{\partial H}{\partial z} = \frac{\partial}{\partial x} (UH) + \frac{\partial}{\partial z} (wH),$$

the last equality being obtained with the aid of (1). If we now perform an integration over the area involved, we have by Stokes theorem that

$$\iint \left[ \frac{\partial}{\partial x} (UH) + \frac{\partial}{\partial z} (wH) \right] dx dz = \oint (UH dz - wH dx).$$

Since the top and bottom of the area are streamlines along which  $Udz - wdx = 0$ , no contribution results from the integration along these portions of the circuit. If, further, the quantity  $H$  is periodic in  $x$ , as is the case with  $U$  and  $w$ , the contributions from the remaining two portions of the circuit will cancel so that the value of the entire integral is zero. These conditions are fulfilled when  $H = wz$ .

$$(41) \quad E_x - E_z + \int_L (p_2 z_2 - p_1 z_1) dx + g\rho \int_L (z_2^2 - z_1^2) dx = \frac{1}{2} c_1^2 \mu.$$

By virtue of the fact that the vertical velocity  $w$  is the same in both the actual motion and the steady-state motion, we may write that

$$(42) \quad E_z = e_z; \quad E_x = e_x - cm + \frac{1}{2} c^2 \mu.$$

Here  $e_x$  and  $e_z$  are the kinetic energies of the horizontal and vertical components of motion in the actual wave over the area considered. The second relation in (42) is obtained by subtracting  $e_z$  from both sides of equation (13). Eliminating  $E_x$  and  $E_z$  from (41), one obtains the result

$$(43) \quad e_x - e_z - cm + \int_L (p_2 z_2 - p_1 z_1) dx + g\rho \int_L (z_2^2 - z_1^2) dx = \frac{1}{2} (c_1^2 - c^2) \mu.$$

This equation is true for any material layer of the shape considered, whether the depth is infinite or not. Moreover, it is independent of any convention in regard to the wave speed  $c$ . In the case of infinite depth, no sensible derangement of the pressure occurs at great distances below the surface, and hence the pressure must be that which is produced hydrostatically when no waves are present. Although it is not necessary to do so, we shall take the pressure at the free surface to be zero for convenience. Accordingly,  $\kappa$  in equation (32) must be zero at great depths and  $c_1^2$  in equation (34) must be equal to  $c^2$ , since here  $w = 0$  and  $U^2 = c^2$ . Inasmuch as  $c_1^2$  and  $c^2$  are constants, it follows that for the case of infinite depth the right-hand side of (43) vanishes for all the material layers and hence also for the case when the entire depth is considered.

The most interesting case arises when (43) is applied to the entire fluid from the free surface  $z_0$  to the bottom. The pressure at the top is now zero while  $z_1 = -D$ , so that

$$(44) \quad e_x - e_z - cm + g\rho \int_L z_0^2 dx + D \int_L \kappa_D dx = \frac{1}{2} (c_1^2 - c^2) \mu.$$

The term involving  $z_0^2$  is twice the potential energy per wave length for the wave motion. It can be shown that the integral of  $\kappa$  with respect to  $x$  along any steady-state streamline vanishes if the pressure at the free surface is zero.<sup>5</sup> We thus have

<sup>5</sup> Suppose that we integrate the second of the equations (33) over an area bounded by the free surface on top, by a steady-state streamline below and by two equipotential lines separated by one wave length. According to the discussion in the foot-

$$(45) \quad e_x - e_z - cm + 2v = \Delta ,$$

in which  $v$  represents the potential energy per wave length and  $\Delta = \frac{1}{2} (c_1^2 - c^2) \mu$ . Eliminating  $cm$  by means of (19) we have

$$(46) \quad \boxed{e_x + 3e_z + \Delta = 2v} .$$

In this relationship  $\Delta = 0$  for the case of infinite depth and the equation itself becomes *an exact integral for periodic gravity waves relating the quantities  $e_x$ ,  $e_z$  and  $v$* . It thus follows that in this case *the sum of the kinetic energy of the horizontal motion and three times the kinetic energy of the vertical motion is equal to twice the potential energy of the waves*.

5. *The partition of energy in periodic gravity waves.* Considering first the case of infinite depth for which  $\Delta = 0$ , equation (46) may be written in the form

$$(47) \quad e_x - e_z = 2(e - v) = 2\epsilon ,$$

where  $\epsilon$  is defined as the difference between the kinetic and potential energy of the waves. So far as the writer has been able to find there appears to be no simple means for obtaining the magnitude or algebraic sign of  $\epsilon$  from general considerations without making use of the detailed solution to the wave problem. According to Rayleigh (7), who carried out a computation of  $\epsilon$  from the detailed solution, the first term in the power series representing this quantity is positive. This first term is furthermore of the fourth order. Rayleigh's work has been extended by Platzman (5) so as to include the term of the eighth order. Platzman's results corroborate those of Rayleigh, and it seems that, to the eighth order of approximation, the kinetic energy of the waves exceeds the potential energy by as much as about twelve per

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note on page 183 this integral will vanish since  $w$  is periodic in  $x$ . We then have by Stokes theorem that

$$-\int \int \frac{\partial \kappa}{\partial z} dx dz = \oint \kappa dx = 0 .$$

Because  $\kappa$  is also periodic, no net contribution results from the integration along the lateral boundaries. In the expression (32) for  $\kappa$  the pressure along the free surface is zero and the mean level of the free surface is also zero so that no contribution is obtained here. It then follows that the integral of  $\kappa$  along the lower streamline must vanish.

This result is merely a statement of the fact that on the average the fluid within a material layer between two streamlines of the steady motion is in hydrostatic equilibrium. It is also possible to arrive at this conclusion by applying the momentum theorem of classical hydrodynamics to the motions of the fluid.

cent when the waves approach the extreme form at the surface. For waves of small amplitude this difference becomes exceedingly small.

Equation (47) gives the result that the kinetic energy of horizontal motion exceeds the kinetic energy of the vertical motion by an amount equal to  $2\epsilon$ .

In the case when the surface amplitude is infinitely small with respect to the wave length and also when the depth is small compared to  $L$ , we know from the small-amplitude theory that the horizontal motion is sensibly the same at all depths and that the kinetic energy of the vertical motions is negligible. This case corresponds to the so-called "long waves." If we integrate (34) along the bottom streamline, we obtain

$$(48) \quad \int_{x_1}^{x_1+L} U_D^2 dx = c_1^2 L,$$

since the integral of  $\kappa$  vanishes and  $w$  is zero. With the aid of (3) and (29) we can also write (48) as follows:

$$(49) \quad \int_{x_1}^{x_1+L} u_D^2 dx = (c_1^2 - c^2) L,$$

which, incidentally, shows that  $\Delta$  is a positive quantity. If we now form the kinetic energy integral for long waves over one wave length, say, between crests, we have

$$(50) \quad e \doteq \frac{\rho}{2} \int_{x_1}^{x_1+L} \int_{-D}^{z_0} u^2 dz dx \doteq \frac{\rho}{2} \int_{x_1}^{x_1+L} u_D^2 (z_0 + D) dx \doteq \frac{1}{2} (c_1^2 - c^2) \rho = \Delta,$$

approximately, since  $z_0$  is a small quantity. It then follows from (46) that  $e \doteq v$ . This is in agreement with the results of the small-amplitude theory of long waves.

More generally, for the case of small amplitude but arbitrary depth, we may accept the principle of equipartition of energy given by the classic theory, namely that  $e \doteq v$ . When this relation is combined with (46) so as to eliminate  $v$ , we obtain that  $\Delta \doteq e_x - e_z$ .

A discussion of the most general case—that of arbitrary amplitude and depth—would presumably require an analysis of the detailed solution such as was made by Rayleigh and by Platzman for the case of infinite depth.

6. *Applications to the solitary wave.* Some of the procedures used in this paper may be applied also to the phenomenon of the solitary wave studied principally by Boussinesq (1) and Rayleigh (6). This (plane) wave consists of a single elevation above the undisturbed level of the free surface and is symmetrical about the crest so that it presents an appearance similar to that shown in Fig. 46. In the case of the extreme amplitude the crest forms a sharp wedge of  $120^\circ$ . The writer is not aware of any direct proof of the existence of this wave similar to the demonstrations of Levi-Civita and Struik for periodic waves. We shall assume, however, that the solitary wave represents a rigorous solution of the dynamic equations of motion, and seek certain integral properties of the wave.

We shall again make use of the artifice of steady motion and consider a region bounded by two steady-state streamlines above and

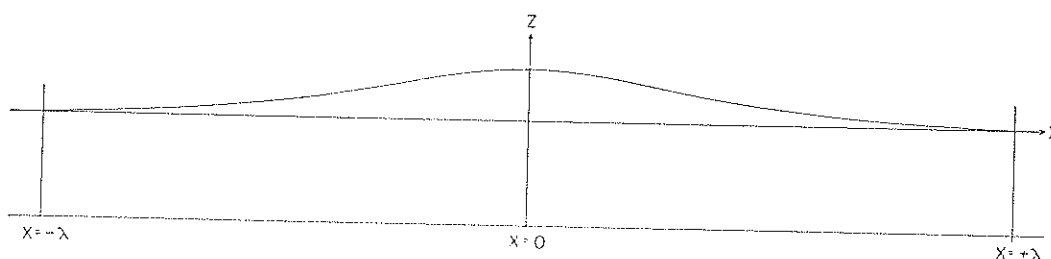


Figure 46 Schematic vertical cross section representing a solitary wave in a basin of depth  $D$ . The origin for the vertical coordinate is taken at the undisturbed level of the free surface.

below. As the lateral boundaries we shall take two verticals at  $x = +\lambda$  and  $x = -\lambda$ , where  $\lambda$  is sufficiently large to place the lateral boundaries in fluid which is not sensibly disturbed. Under these circumstances the vertical boundaries will also be equipotential lines. By means of reasoning analogous to that given in Section 3, we see that the kinetic energy  $E$  and momentum  $M$  of the steady-state motion are again given by equations (11) and (12), provided we replace  $L$  by  $2\lambda$ . With this slight change equation (16) also holds, but we see that now  $\Delta\phi$  cannot be equated to zero, since in the progressive wave all the particles influenced by the wave have a component of motion forward. It therefore follows that equations (18) and (19) are not valid for the solitary wave.

When the entire depth is considered the quantity  $\Delta\Psi$  appearing in (12) has, in the present case, a simple interpretation, being equal to  $cD$ , since the steady motion is a uniform translation for large positive or negative values of  $x$ . From (12) and (3) we may then write that

$$\begin{aligned}
 (51) \quad -2\lambda cD &= \int_{-D}^{z_0} \int_{-\lambda}^{+\lambda} U dx dz = \int_{-D}^{z_0} \int_{-\lambda}^{+\lambda} (u - c) dx dz = \\
 &= \frac{1}{\rho} m - 2\lambda cD - c \int_0^{z_0} \int_{-\lambda}^{+\lambda} dx dz.
 \end{aligned}$$

If we denote the superficial mass—the mass of the fluid above the undisturbed free surface per unit distance along the crest—by  $\mu_0$ , it follows that

$$(52) \quad \boxed{m = c\mu_0},$$

where the integrations may be taken over the entire range of  $x$ . *The momentum of the solitary wave is therefore equal to the momentum of the superficial mass moving with the wave speed.*

In view of the fact that the steady motion is a translation at the rate  $c$  at great distances from the crest, the constant  $c_1$  in equation (37) must now be equal to  $c$ , and the other considerations leading to the formulation of this equation are valid for the solitary wave. Furthermore, an equation analogous to (43) is valid, so that when the entire depth is considered (44) may be written as

$$(53) \quad e_x - e_z - c^2\mu_0 + 2v + D \int_{-\lambda}^{+\lambda} \kappa_D dx = 0,$$

where use has been made of (52). As explained in the footnote on page 184 we have that

$$(54) \quad 0 = \oint \kappa dx = \int_{-\lambda}^{+\lambda} \kappa_D dx - g\rho \int_{-\lambda}^{+\lambda} z_0 dx,$$

so that

$$(55) \quad \int_{-\lambda}^{+\lambda} \kappa_D dx = g\mu_0.$$

Equation (53) thus may be written as

$$(56) \quad e_x - e_z + 2v + (gD - c^2)\mu_0 = 0.$$

The first equation in (31) may be written in the form

$$(57) \quad \rho \left( \frac{\partial U^2}{\partial x} + \frac{\partial U w}{\partial z} \right) = - \frac{\partial p}{\partial x},$$

because the motion is steady and free of divergence. We shall form the integral of (57) over the area bounded by the bottom, the free surface, the vertical at  $x = -\lambda$  and an arbitrary vertical at some value of  $x$  within the wave. The area integrals are at once transformable into line integrals by means of Stokes theorem so that we have

$$(58) \quad \rho \oint (U^2 dz - U w dx) = - \oint p dz.$$

Since no contribution to the integral on the left results from the integration along the top and bottom streamlines, and since the pressure is taken to be zero along the free surface, evaluation of (58) gives that

$$(59) \quad \left[ \int_{-D}^{z_0} (\rho U^2 + p) dz \right]_x = \left[ \int_{-D}^{z_0} (\rho U^2 + p) dz \right]_{-\lambda}.$$

The quantity on the right of (59) is immediately capable of simplification, because at  $x = -\lambda$

$$(60) \quad U^2 = c^2; \quad p = -g\rho z; \quad z_0 = 0.$$

We thus have that at any arbitrary vertical

$$(61) \quad \int_{-D}^{z_0} (\rho U^2 + p) dz = \rho c^2 D + \frac{1}{2} g \rho D^2 = \text{constant}.$$

The pressure  $p$  may be eliminated from (61) by means of Bernoulli's equation (34), so that, with some reduction,

$$(62) \quad \rho \int_{-D}^{z_0} (U^2 - w^2) dz + \rho c^2 z_0 - g \rho z_0^2 = \rho c^2 D.$$

If this equation be integrated once more with respect to  $x$  from  $-\lambda$  to  $+\lambda$ , it follows that

$$(63) \quad E_x - E_z + \frac{1}{2} c^2 \mu_0 - v = \rho c^2 D \lambda.$$

We may now pass to the actual motion by using equations (42). Thus

$$(64) \quad e_x - e_z - cm + \frac{1}{2} c^2 \mu + \frac{1}{2} c^2 \mu_0 - v = \rho c^2 D \lambda.$$

Keeping in mind the expression for the momentum  $m$  given in (52) and also the fact that  $\mu = \mu_0 + 2\rho D \lambda$ , we finally have that

$$(65) \quad \boxed{e_x - e_z = v}.$$

In the solitary wave the difference between the kinetic energy of the horizontal and vertical motions is equal to the potential energy.

By adding  $2e_z$  to both members of (65) we have that

$$(66) \quad e = v + 2e_z,$$

so that the kinetic energy is greater than the potential energy.

Equations (56) and (65) may be combined so as to give an expression for the wave speed by eliminating the quantity  $e_x - e_z$  between them. The result is

$$(67) \quad c^2 = gD + 3 \frac{v}{\mu_0},$$

showing that the speed of a solitary wave is greater than the speed of the corresponding long waves.

For the highest possible solitary wave it follows from Bernoulli's equation (34) when applied to the free surface that  $c^2 = 2gz_T$ , where  $z_T$  is the height of the crest above the mean level. By inserting this value of  $c^2$  in (67), it follows that

$$(68) \quad z_T = \frac{D}{2} + \frac{3}{4} \int z_o^2 dx / \int z_o dx.$$

The extreme height is thus greater than half the depth of the basin. The ratio of the integrals in (68) may be interpreted as a mean height  $\bar{z}_o$ . According to an estimate quoted by Lamb (2),  $z_T = 0.78D$ . The corresponding value of  $\bar{z}_o$  turns out to be equal to nearly one-half of  $z_T$ .

Equations (52) and (65) of this section were derived in such a manner as to be applicable to the entire wave. There is, however, no restriction in the development which would prevent their application to a closed region bounded by *any* two verticals, by the free surface and by the bottom. In the limit we may replace them by integral relationships along a single arbitrary vertical. They then become, respectively,

$$(69) \quad \int_{-D}^{z_o} u dz = cz_o; \quad \int_{-D}^{z_o} (u^2 - w^2) dz = gz_o^2.$$

If equations (69) are applied to the vertical at the crest,  $w$  vanishes and  $z_o$  is the "amplitude" of the wave which we shall denote by  $a$ . We may then write

$$(70) \quad \bar{u}^2 = c^2 \frac{a^2}{(D+a)^2}; \quad \bar{u}^2 = g \frac{a^2}{D+a}.$$



From the long-wave theory we know that for small surface deformations of large horizontal extent there is no sensible variation of  $u$  with depth. For larger amplitudes  $u$  varies with depth, so we may write, with the aid of the Schwarz inequality, that  $\overline{u^2} \geq \bar{u}^2$ . We then obtain from (70) the result that<sup>6</sup>

$$(71) \quad c^2 \leq g(D + a),$$

where the equality holds for very small values of  $a$ .

If we retain only the inequality in (71) and eliminate  $c^2$  by means of the relation that  $c^2 = 2gz_T$  for a wave of extreme height, we get an upper bound for  $z_T$ , namely  $D$ . Since we have already found a lower bound we may write that  $D/2 < z_T < D$ .

### SUMMARY

In this paper certain integral properties of gravity waves in a homogeneous and incompressible fluid are deduced from the differential equations governing such phenomena. The treatment is restricted to plane waves in an irrotational medium which move without change of form. No assumption is made concerning the amplitude so that the results are applicable to waves of finite height. Both periodic and solitary waves are considered. The more important results are the following:

1. For periodic waves the product of the momentum and the wave speed is equal to twice the kinetic energy. This statement is valid not only for the total depth of fluid but also for the fluid contained between any two streamlines of the corresponding steady-state motion.
2. For periodic waves in a basin of infinite depth the sum of the kinetic energy of the horizontal motions plus three times the kinetic energy of the vertical motions is equal to twice the potential energy. Since it has been shown by Rayleigh and by Platzman that the total kinetic energy is greater than the potential energy, this statement implies that the kinetic energy of the horizontal motions is greater than that of the vertical motions.
3. For the solitary wave the momentum is equal to the superficial mass multiplied by the wave speed. This is valid not only for the entire wave but also for any portion of it bounded laterally by two verticals.
4. For the solitary wave the kinetic energy of the horizontal motions is greater than the kinetic energy of the vertical motions by an amount

<sup>6</sup> According to Weinstein (11), the next term in the series representing  $c^2$ , in terms of  $a$ , is negative.

equal to the potential energy. From this it follows that the total kinetic energy is larger than the potential energy. These results are valid not only for the entire wave but also for any portion of it bounded laterally by two verticals.

5. For the solitary wave the square of the speed of propagation is equal to the depth of the basin multiplied by the acceleration of gravity plus three times the ratio of the potential energy to the superficial mass.

6. For the solitary wave it can be shown by means of integral relationships that the square of the speed of propagation is less than the height of the crest above the bottom multiplied by the acceleration of gravity, although equality of these two quantities is approached for small amplitudes.

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