A MOMENTUM INTEGRAL FOR SURFACE WAVES IN DEEP WATER

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It has been demonstrated by the studies of Stokes (1847) and Levi-Civita (1925) that surface waves of finite height which are irrotational and periodic can exist in a liquid under the influence of gravity. Furthermore, it was pointed out by Stokes that such waves produce a transport of mass in the direction of propagation and hence possess a certain horizontal momentum relative to the undisturbed water at great depths. The present purpose of the writer is to show that there exists a simple relationship between the momentum and the kinetic energy in such wave motion. An equation of similar form was first derived by Levi-Civita (1924) using a different approach, but, so far as the writer is aware, no oceanographic applications of it have been made.[1] It is possible that the relationship may be of significance in the study of wave growth due to wind action.

The theory for waves of finite amplitude, developed by the writers mentioned above, indicates that the waves are symmetrical about the crests and troughs, not only at the surface but also at greater depths, although there is a rapid diminution of the amplitude downward. This circumstance implies that there is no variation of phase of the waves with depth. Also the theory indicates that the speed of propagation is constant, and that the waves travel without alteration of their form at the surface or below. The fluid motions are assumed to take place without any viscosity being present, and relative to a non-rotating coordinate system, so that the treatment does not include the effects of Coriolis forces. Since there are no motions or variations in the motions in the direction parallel to the crests of the waves, it suffices to consider a vertical section across the crests as shown schematically in Fig. 35. The problem may thus be considered as one in two-dimensional motion.

We shall suppose that the actual waves are propagated from left to right in the figure, but in order to treat the problem as one in steady-state motion we shall suppose that a constant translation from right

[1] A somewhat similar relationship was derived by Rayleigh (1914) for the case of *long* waves by approximate methods.

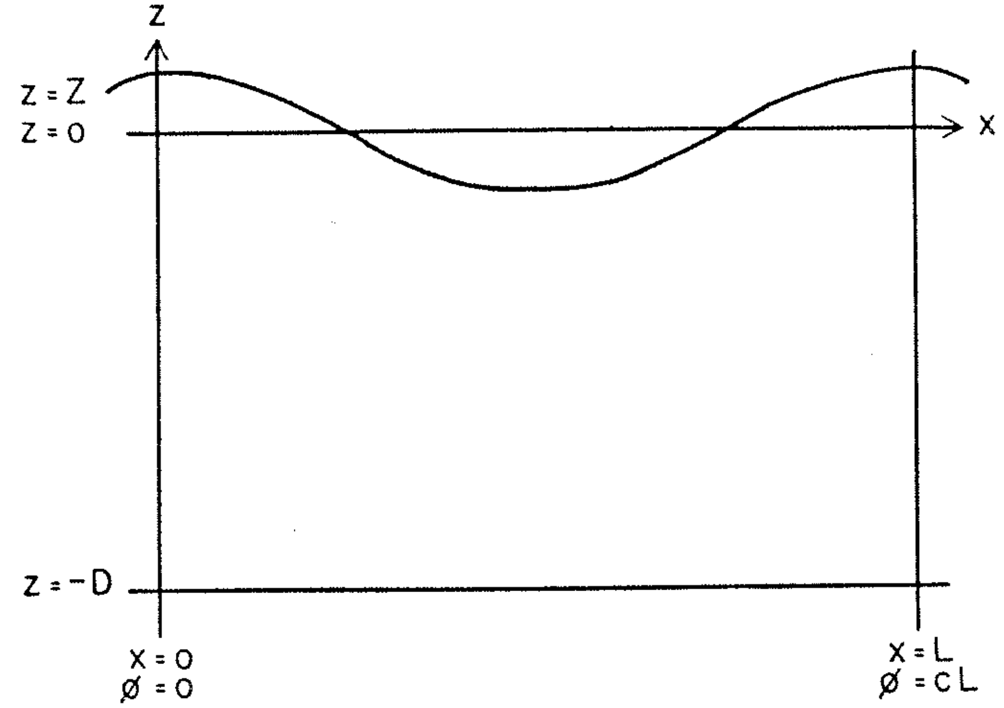

Figure 35. Schematic cross section of a wave normal to the wave crests.

to left, equal to the wave speed c, has been added to the actual motions. This artifice does not alter the fundamental properties of the dynamic system and produces greater simplicity in the picture of the motions. The vertical coordinate z is counted positive upward and has its origin in the undisturbed free surface of the water. The horizontal coordinate x is counted positive to the right and has its origin at a crest. The free surface Z is a streamline in the steady-state motion. Other streamlines similar to Z, but with progressively smaller amplitudes, lie below Z, but these are not shown in the diagram. The horizontal line $z = -D$ is assumed to be located at a sufficiently great depth where the wave disturbance is no longer of sensible intensity.

Since the motion is irrotational, it follows that we may introduce a velocity potential ϕ such that

$$U = -\frac{\partial\phi}{\partial x} \quad ; \quad w = -\frac{\partial\phi}{\partial z}. \tag{1}$$

Here U is the particle velocity in the x-direction, and w is the particle velocity in the z-direction, in the steady-state motion. If drawn in the diagram, lines along which ϕ is constant would constitute a set of curves orthogonal to the streamlines, and hence it follows that ϕ has a constant value at the verticals $x = 0$ and $x = L$ (where L is the wave length). The velocity potential is indeterminate to the extent of an arbitrary additive constant so that we may choose ϕ to be zero at $x = 0$. At the depth $z = -D$ the motion of the fluid is a simple horizontal translation at the rate c, so that here $U = -c$. Integration of the first equation in (1) thus gives the value for ϕ at $x = L$ to be cL.

By the use of Green's theorem it can be shown that the velocity potential obeys the following equation in a simply-connected space (see Lamb, 1932):

$$\iiint\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]dxdydz = -\iint\phi\frac{\partial\phi}{\partial n}ds. \quad (2)$$

In this equation ds is an element of surface area of the volume considered, and $\partial\phi/\partial n$ is the derivative along the inward normal to this area. In our application of this relationship $\partial\phi/\partial y = 0$, and the volume integral becomes an area integral, since we are concerned with a section of unit thickness. The right-hand member of (2) becomes a line integral. Choosing the region bounded by the verticals $x = 0$ and $x = L$, the free surface Z and the line $z = -D$, the only nonvanishing contribution to the right-hand member of (2) results from the integration along the vertical at $x = L$, because $\phi = 0$ at $x = 0$ and $\partial\phi/\partial n = 0$ along the free surface and the line (streamline) $z = -D$. Since $\partial\phi/\partial n = +U_L$ at $x = L$, we get

$$\int_0^L\int_{-D}^Z (U^2 + w^2)\,dzdx = -cL\int_{-D}^{Z_L} U_L\,dz \quad (3)$$

after substitution from (1) in the left-hand member (Z_L is the value of Z at $x = L$). The integral on the right of (3) is simply the volume transport between the two streamlines considered, and hence we may replace it by a similar integral at an arbitrary vertical so that

$$\int_{-D}^{Z_L} U_L\,dz = \int_{-D}^{Z} U\,dz = \text{constant}. \quad (4)$$

Introducing the horizontal velocity component u, which is present in the actual wave motion, and which is related to U by the equation

$$U = u - c, \quad (5)$$

we may rewrite (3), after substituting from (4), as follows:

$$\int_0^L \int_{-D}^{Z} (u^2 + w^2 - 2cu + c^2)\, dzdx = -\, cL \int_{-D}^{Z} (u - c)\, dz\,. \qquad (6)$$

Since the area over which the integration extends is equal to LD because of the choice of the origin for z, it is possible to simplify (6) so that we have

$$\int_0^L \int_{-D}^{Z} (u^2 + w^2 - 2cu)\, dzdx + c^2LD = -\, cL \int_{-D}^{Z} udz + c^2L\,(Z + D)\,,$$

and finally,

$$\int_0^L \int_{-D}^{Z} (u^2 + w^2)\, dzdx - 2c \int_0^L \int_{-D}^{Z} udzdx = -\, cL \int_{-D}^{Z} udz + c^2LZ = K\,, \qquad (7)$$

where K is a quantity which is independent of x. From the last equality in (7) we obtain by integration over one wave length

$$K = -\, c \int_0^L \int_{-D}^{Z} udzdx\,. \qquad (8)$$

If K is eliminated from (7) by means of (8) and the resulting equation multiplied by $\rho/2$, where ρ is the (uniform) density of the fluid, we obtain the relationship,

$$\int_0^L \int_{-\infty}^{Z} \rho \frac{u^2 + w^2}{2}\, dzdx = \frac{c}{2} \int_0^L \int_{-\infty}^{Z} \rho udzdx\,. \qquad (9)$$

In view of the fact that the disturbance in the actual wave disappears at great depths, it is permissible to extend the integration downward to $-\,\infty$.

This last equation states that *the kinetic energy per wave length and per unit distance along the crests of the waves is equal to one half the wave speed multiplied by the momentum of the same water mass in the direction of wave propagation.* Once the distribution of the velocity potential in Fig. 35 has been specified, equation (2) can be applied in the manner described, not only to the region between the surface and depth D, but to any region bounded above and below by two streamlines and by the two verticals at $x = 0$ and $x = L$. We might thus choose two streamlines, Z_1 and Z_2, which give a region whose area in the figure is numerically equal to L, and thus a volume in the section of unit thickness equal to L. For such a *material layer* we may then write

$$\frac{1}{L}\int_0^L\int_{Z_1}^{Z_2} \rho \frac{u^2 + w^2}{2}\, dzdx = \frac{c}{2L}\int_0^L\int_{Z_1}^{Z_2} \rho u dzdx. \qquad (10)$$

For the material layer considered, equation (10) states that on the average the kinetic energy per unit volume is equal to the momentum per unit volume multiplied by one half of the speed of propagation.[2]

It is of some interest to compare the value of the momentum given by equation (9) with the value obtained by Lamb (1932), who used the second order approximation to the wave solution presented by Stokes (1847). For this purpose it is necessary to have available an expression for the kinetic energy. Let us take for this quantity the approximate value given by the small-amplitude theory, namely, $\frac{1}{4} g\rho a^2 L$, where a is the amplitude and g is the acceleration of gravity. Eliminating the product gL from this expression by means of the relation that $2\pi c^2 = gL$, also given by the small-amplitude theory, and placing the result into (9), we obtain the expression $\pi\rho a^2 c$ for the momentum per wave length. This is in agreement with the result obtained by Lamb, except that in Lamb's result a is, strictly speaking, not the amplitude but rather an amplitude parameter which becomes very nearly equal to the amplitude for waves of small height. It should be remarked, however, that both the present method and the method used by Lamb for obtaining the momentum are approximate [although relation (9) is an exact one].

The extent to which the theoretical results obtained in this paper are directly applicable to surface waves which actually occur in the ocean is, of course, an open question. The waves which are found in nature are irregular in general, and the medium in which they are found departs considerably in its properties from an ideal fluid. Moreover, the motions take place in a rotating coordinate system so that it would appear that Coriolis forces are of importance in connection with the momentum associated with the waves, although such forces are probably of negligible consequence as far as the purely oscillatory components of motion are concerned. Temporarily laying

[2] This relationship is analogous to the principle in the electromagnetic theory of light, which states that in the case of plane waves the energy per unit volume is equal to the electromagnetic momentum per unit volume multiplied by the speed of propagation (see Page and Adams, 1931).

aside all such difficulties, it is a matter of at least some academic interest to see what use might be made of equation (9) in the study of the growth of waves due to wind action. The possible utility of the equation for this problem lies in the fact that, whereas it is a matter of great difficulty to estimate the energy imparted to the sea surface by a given wind, estimates of the momentum transfer are more easily made.

For this purpose let us substitute the approximate values mentioned above for the energy per wave length and for the wave speed c, given by the small-amplitude theory, into (9). The result may then be written in the form

$$g\rho a^2 = 2\sqrt{\frac{gL}{2\pi}}\,M\,, \tag{11}$$

where M is the average wave momentum per unit area of the sea surface. We shall assume that the wave system considered remains under the influence of a uniform wind which feeds energy and momentum into it during a given period of time. Since for a given wave length there is a limit to the amount of energy which can thus be fed into the waves, beyond which the waves break, it must be assumed that such breaking does take place and that longer waves with greater momentum and energy capacities are continually generated. For this reason it might be expected that the ratio of the amplitude a to the wave length L should be relatively large. The theoretical limiting value for this ratio is about 1/14, but it would be unreasonable to expect that this value would be reached due to the observed irregularities in the waves and to the presence of waves which have not attained the maximum height. As a more reasonable supposition, let us take an average value of 1/24, so that we have

$$\frac{a}{L} = \frac{1}{b} = \frac{1}{24}. \tag{12}$$

Eliminating the amplitude a from (11) by means of (12) and solving for L, we obtain

$$L = \sqrt[3]{\frac{2b^4M^2}{\pi g\rho^2}}. \tag{13}$$

Assuming, for purposes of orientation, that the total momentum transferred from a steady wind over a period of time t is used in creating waves on an originally undisturbed sea, we have

$$M = \tau t\,. \tag{14}$$

Here τ is the tangential wind stress and hence is the rate at which momentum is imparted to the sea. Rossby (1936) has given a relationship for obtaining the stress τ in terms of the wind velocity, namely

$$\tau = \rho^* \gamma_h^2 W_h^2 , \tag{15}$$

ρ^* being air density, W_h the wind at height h and γ_h a resistance coefficient appropriate for the level h. If h is approximately 15 meters, the value of γ_h is about 5×10^{-2} when c.g.s. units are used. According to this formula a wind of 30 knots should produce a stress of about 7 dynes per square centimeter.[3]

With such a stress equations (13) and (14) give the result that waves about 430 meters in length should be generated in 24 hours. The period of such waves would be about 16.6 seconds.[4] Since waves of this magnitude are seldom if ever observed in the generating areas, even with stronger winds than we have assumed, it seems that some of the premises made above are not proper.

The effect of the tangential wind stress on the motions of water in the oceans was treated by Ekman (1905), and the results of his studies are well known to oceanographers. In the present discussion the question arises whether the mass transport associated with wave motion is in some manner an integral part of the drift currents in the theory of Ekman, or whether it is superimposed on the drift currents. As a third possibility, it may perhaps be that, since the wave transport is not dependent on the presence of internal viscosity, but is, on the

[3] Formula (15) is applicable when the sea surface is hydrodynamically "rough." This condition is present with wind velocities above about 10 knots. The derivation of this equation indicates that τ is the total rate of momentum transfer to the sea surface regardless of the details of the mechanism by means of which this transfer is effected in the immediate vicinity of the water surface.

[4] According to the small-amplitude theory, there exists an equation relating the wave period T to the wave length L. The wave length may be eliminated from this equation by means of (13). We thus have $T = \sqrt{\frac{2\pi L}{g}}$, which becomes $T = \sqrt[3]{\frac{4\pi b^2 M}{g^2 \rho}}$. If τ and b are constant, it follows, with the aid of (14), that *the period T is proportional to the cube root of the time.* Under these circumstances we may also obtain by differentiation that

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{3t}, \text{ and } \frac{1}{L}\frac{dL}{dt} = \frac{2}{3t},$$

showing that the percentage rate of increase of the period or wave length is inversely proportional to the time. Similar equations may be written for the wave speed c.

other hand, subject to the effects of Coriolis forces, a composite solution to the problem can exist which takes into account these properties of the wave-transport components of the total motion. Whatever the answer to this question may be, it seems reasonable to suppose that not all of the momentum which is imparted to the sea by the wind stress goes into wave momentum, but that most of it is utilized in the generation and maintenance of drift currents. It is thus not surprising that the calculation of the magnitude of waves generated by a given wind on the assumption that all the momentum transferred becomes wave momentum should give too intense wave action. In order to use equation (13) for the purpose of securing an estimate of the waves which are actually generated, it would thus be necessary to have available a criterion to determine what fraction of the momentum goes into the wave motion.

Carrying these speculations a step further, it may be that during the growth of wave motion the fraction of the momentum which is used in creating waves changes as time progresses. Since the development of drift currents is dependent upon the presence of eddy viscosity in the water, and since this turbulent viscosity is relatively small during the early stages of the process, it is not unreasonable to suppose that at the beginning a relatively large part of the momentum received is utilized in a rapid development of wave motion. On the other hand, during the later stages of the process the continued presence of breaking waves probably brings about a large increase in the turbulent viscosity which in turn renders possible the existence of well developed drift currents whose maintenance requires a large part of the momentum received, so that but little remains for a further increase in the wave action. In the end some sort of steady state would thus be indicated, in which practically all of the momentum goes into the maintenance of drift currents and no further growth of the waves takes place, unless the wind conditions should change. That some such steady state does finally develop with a constant wind is supported by the fact that there is normally no progressive change in the wave regime in the large oceanic regions in the trade-wind belts.

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[*Note*—As this paper is going to press the writer has had opportunity to read a very interesting article by Sverdrup and Munk (1946) dealing with the subject of wave generation through wind action. These authors present curves depicting the increase of wave height and wave speed with time for an unlimited fetch under the influence of a constant wind (fig. 3 in their paper). These curves are derived on the basis of energy transfer calculations and have been used in the forecasting of wave formation for practical purposes.

If we accept these results obtained by Sverdrup and Munk as representing the actual process of wave generation, casual inspection of the curves shows that in the earlier stages the wave height increases as the two-thirds power of the time and the wave speed increases as the one-third power of time. This is in agreement with the equations contained in the present paper, provided that the parameter b is constant. Numerical computation shows, moreover, that with $b = 24$ (steepness about eight per cent), *approximately ten per cent of the momentum received by the water becomes wave momentum during these early stages.* During the later stages both curves tend to level off showing that probably a smaller and smaller percentage of the momentum received is utilized in wave growth.]

SUMMARY

In this paper an integral relationship between the kinetic energy and horizontal momentum of surface waves is derived by simple methods. A relation of similar form was first derived by T. Levi-Civita who used a different approach. The equation obtained by the writer states that the kinetic energy per wave length and per unit distance along the crests of the waves is equal to one half the wave speed multiplied by the momentum of the same water mass in the direction of wave propagation.

An attempt is made to utilize this equation for the study of the growth of waves due to wind action. The possible utility of the equation in this problem lies in the fact that, whereas it is difficult to estimate the energy imparted to the sea surface by a given wind, estimates of the momentum transfer are more easily made. For the case of an initially undisturbed ocean of large dimensions, subjected to a constant and uniform wind, a formula is obtained which states that the wave length should increase as the two-thirds power of the time. Numerical computations show that the wave lengths thus obtained are too large. It is suggested that the discrepancy may be due to the fact that a certain fraction of the momentum absorbed by the water is utilized in the generation and maintenance of drift currents. Further study may show that it is possible to introduce corrections for such effects.

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