Coupling of Internal Wave Motion with Entrainment at the Density Interface of a Two-Layer Lake

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ABSTRACT

A solution is presented to the problem of a two-layer rectangular basin subject to a suddenly applied, uniform wind stress; Coriolis effects are ignored. The solution is obtained for the case in which the time scales of internal wave motion, wave decay and entrainment are widely separated; in this range the entrainment across the interface is a perturbation on the mean motions.

The solution includes an oscillatory initial response, followed by wave decay and a steady-state interface setup with baroclinic circulation, all superimposed on a slow deepening of the top layer by entrainment. The entrainment in turn affects the frequency of interfacial waves as the mixed layer deepens—the deepening alters top and bottom layer thickness and the density jump between them. For the range of mixed-layer Richardson number considered, entrainment is energized by wind stirring at the water surface, which does work at a rate proportional to u_*^3.

1. Introduction

This paper presents an analytical solution to the initial value problem of a two-layer rectangular basin subject to a suddenly applied uniform wind stress. The solution considers entrainment at the base of the mixed layer (the top layer in this case) as well as the interfacial waves and circulation caused by the wind stress, but is valid only for the case in which entrainment acts as a small perturbation on the baroclinic motions.

The physical parameters which determine the solution include the wind stress $\tau_0 = \rho_0 u_*^2$, where u_* is shear velocity and ρ_0 is the density of water; top and bottom layer depths h_1 and h_2, respectively; basin depth H and length L; and the density jump $\Delta\rho$ between top and bottom layers. Spigel and Imberger (1979, hereafter referred to as SI) have developed time scales for many of the processes relevant to wind mixing in lakes. They found that the overall dynamics of a wind event could be predicted by considering the ratio of a mixed-layer Richardson number, $\mathrm{Ri} = g'h_1/u_*^2$ (where $g' = \Delta\rho g/\rho_0$) to the basin aspect ratios L/h_1 and h_1/H. In particular, for $\mathrm{Ri} > (L^2/4h_1^2)(H/h_2)$, stratification is so strong that mixed-layer deepening can be effectively ignored in determining the response to wind. Application of the wind stress causes the interface between upper and lower layers to oscillate about an equilibrium setup position with period $T_i = 2L/(g'h_1h_2/H)^{1/2}$. The oscillations decay in a time we shall call T_d, leaving a steady state setup of the interface and a wind-driven circulation in the top layer. For this case, the time for a turbulent front to reach the interface from the water surface is longer than the time $T_i/4$ for the interface to first reach the setup position. Heaps and Ramsbottom (1966) have obtained a complete solution for a variable wind stress, but neglect mixed-layer deepening, for the initial value problem considered here.

For $\mathrm{Ri} < (L^2/4h_1^2)(H/h_2)$ the effects of wind-induced entrainment at the base of the mixed layer cannot be ignored. To account for these effects we will use a one-dimensional (in the vertical) integral entrainment model which is a synthesis of those presented by Niiler (1975) and Zeman and Tennekes (1977). The model is described in a review article by Sherman *et al.* (1978), and is derived by integrating the one-dimensional turbulent kinetic energy (TKE) budget over the mixed layer. The model incorporates the effects of heat exchange with the atmosphere, though we will restrict ourselves here to the mechanical effects of the wind. In the model, the mixed layer is assumed to move as a slab with velocity ΔU relative to the hypolimnion, bounded by shear zones at the mixed-layer base and at the water surface. The rate at which the wind does work in producing TKE in the surface waters is $\tau_0 u_s$, where u_s is the surface drift velocity. Based on Bye's (1965) and Wu's (1975) work showing the resemblance of velocity profiles in the very surface waters to that of an equilibrium flat plate turbulent boundary layer, we assume $u_s \approx u_*$, so the rate of TKE production in the surface waters is proportional to $\rho_0 u_*^3$. However, TKE is also produced by the interaction

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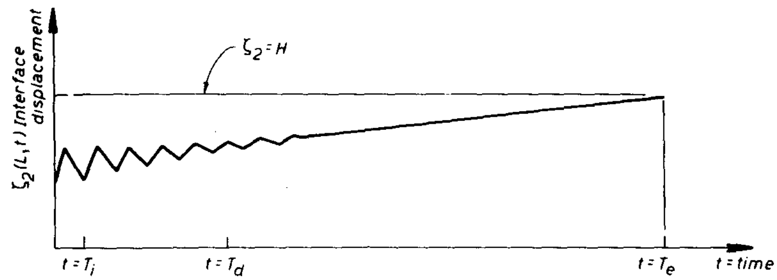

FIG. 1. Conceptual sketch of interface displacement ζ_2 at the downwind end of a rectangular basin of length L as a function of time, following the switching on of a uniform wind stress at time $t = 0$. The long time scale T_e is the time for the mixed layer to deepen to the bottom; superimposed on the slow deepening are internal waves of period T_i, which decay with an e-folding time T_d, with $T_i < T_d < T_e$.

of Reynolds stresses with velocity shear in the large-scale mean motions, both within and at the base of the mixed layer. The velocity scale for this production is ΔU rather than u_*, and the rate of TKE production for entrainment is proportional to $\rho_0 \Delta U^2 u_e$, where u_e is the one-dimensional entrainment velocity. Controversy exists over which of the above two mechanisms—stirring in the surface waters or shear production within and at the base of the mixed layer—dominate mixed-layer deepening. The model described by Sherman *et al.* (1978) incorporates both mechanisms and may be written as

$$\underset{1}{\tfrac{1}{2}(C_T\eta^2 u_*^2} + \underset{2}{(\Delta\rho/\rho_0)gh_1)}dh_1/dt = \underset{3}{(C_K\eta^3/2)u_*^3} + \underset{4}{(C_S/2)\Delta U^2 dh_1/dt} - \underset{5}{\Lambda_L}. \quad (1)$$

Here $\Delta\rho$ is the density jump at the base of the mixed layer and $dh_1/dt = u_e$, the one-dimensional deepening rate. C_K, η, C_T and C_S are O(1) coefficients measuring the efficiencies of the various processes involved in entrainment; their values as determined by experiment are fully discussed by Sherman *et al.* (1978) and Fischer *et al.* (1979). The left-hand side of (1) gives the rate at which energy must be supplied for entrainment to proceed, while the right-hand side gives the rate at which energy is made available. Term 1 accounts for the rate of change of TKE in the water column, and shows that some energy is required for entrainment by a turbulent front even in homogeneous fluid. Term 1 has relatively little importance in lakes, where it is usually dominated by the buoyancy term 2. Terms 3 and 4 give energy available from stirring and shear production, respectively. Λ_L accounts for losses of TKE from the mixed layer via generation of internal waves in the hypolimnion; parameterization of this term is discussed by Sherman *et al.* (1978) and the reader may also refer to Thorpe (1973). Λ_L is not thought to be of primary importance in describing mixed layers in lakes, and in the present study we neglect losses due to Λ_L by assuming that there is no stratification in the hypolimnion.

Using (1), SI investigated the circumstances under which either internal shear production or surface stirring dominates mixed-layer deepening in lakes. They found that, for a two-layer basin with mixed-layer Richardson number in the range

$$(L/2h_1)(H/h_2)^{1/2} < \mathrm{Ri} < (L^2/4h_1^2)(H/h_2), \quad (2)$$

that shear production has negligible effect on entrainment, which is energized by surface stirring of the wind. [A short argument to this end, utilizing the estimate from Heaps and Ramsbottom (1966) for the maximum value of ΔU as $u_*^2T_i/4h_1$, is presented in Section 2b.] For the range of Ri given by (2), entrainment acts as a small perturbation on the mean motions, on a longer time scale than either the wave period T_i or the decay time T_d. Fig. 1 depicts qualitatively the behavior to be expected for interface displacement at the end of the basin. The oscillations and setup described earlier are superimposed on a slow deepening process energized by the stirring effects of the wind. The entrainment time T_e for the interface to erode to the bottom of the basin is much longer than either T_i or T_d; this wide separation of time scales permits an analytical solution to be obtained, valid for the range of Ri given by (2).

As stratification becomes weaker, Ri $< (L/2h_1) \times (H/h_2)^{1/2}$, the effects of entrainment and interfacial instability dominate the response. Shear production within and at the base of the mixed layer is the most important energy source for mixed-layer deepening, which proceeds at a much faster rate than for (2). Displacements of the interface can be large—on the order of the basin depth—and the linearizing assumptions we shall employ here break down. The solution presented here is also restricted to cases for which the earth's rotation is not important, i.e., for $T_i < T_\Omega$, where $T_\Omega = \pi(\Omega \sin\phi)^{-1}$ is the inertial period, ϕ is latitude and Ω is the angular velocity of the earth's rotation.

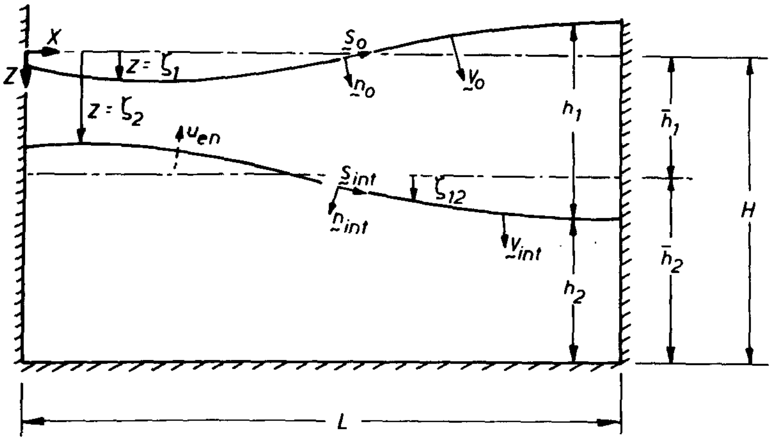

FIG. 2. Two-layer rectangular basin—definition sketch.

2. Equations and scales

Fig. 2 illustrates the two-layer rectangular basin and associated terminology. The z axis has origin at the undisturbed water surface and is positive downward, $z = \zeta_1$ and $z = \zeta_2$ being the coordinates of the free surface and interface, respectively. Subscripts 1 and 2 denote the top and bottom layer, respectively, while the overbar denotes an average over basin length; thus, $\bar{h}_1$, $\bar{h}_2$ are average top and bottom layer thicknesses. ζ_{12} is the displacement of the interface from the horizontal. Hence, $h_1 = \bar{h}_1 + \zeta_{12} - \zeta_1$, where $h_1(x,t)$ is top-layer thickness. Water particle velocities in top and bottom layers are $\mathbf{u}_1 = (u_1,w_1)$, $\mathbf{u}_2 = (u_2,w_2)$, where u_1, w_1 and u_2, w_2 are x, z components. Velocities in the x direction are decomposed as $u_1 = U_1 + u_1'$, $u_2 = U_2 + u_2'$, where U_1, U_2 are velocities averaged over layer thicknesses and give net longitudinal transport at any section, and u_1', u_2' are the residual circulation velocities in top and bottom layers.

Terms $\mathbf{v}_{\text{int}}$ and $\mathbf{v}_0$ denote the velocities of the interface and free surface, respectively, while $\mathbf{n}_{\text{int}}$ and $\mathbf{n}_0$ are unit vectors normal to the interface and free surface. We assume that the free surface is a material surface, always made up of the same water particles, so that

$$\mathbf{v}_0 \cdot \mathbf{n}_0 = \mathbf{u}_1 \cdot \mathbf{n}_0 \quad \text{at} \quad z = \zeta_1, \tag{3}$$

i.e., velocities of the free surface itself and of water particles on the free surface are identical.

Conditions at the interface are slightly more complicated. The velocity of the interface in the $\mathbf{n}_{\text{int}}$ direction is (Stoker, 1957, p. 11)

$$\begin{aligned} \mathbf{v}_{\text{int}} \cdot \mathbf{n}_{\text{int}} &= \zeta_{2t}(\zeta_{2x}^2 + 1)^{-1/2} \\ &= (\zeta_{12t} + \bar{h}_{1t})(\zeta_{2x}^2 + 1)^{-1/2}, \end{aligned} \tag{4a}$$

where subscripts x, t denote differentiation. The normal velocity of the interface is thus due to motion ζ_{12t} about a horizontal level $z = \bar{h}_1$ as well as to an overall thickening $d\bar{h}_1/dt$ of the top layer by entrainment of fluid from the bottom layer. Because of entrainment, the interface cannot be a material surface—entrainment can only occur if there is a relative velocity between water particles at the interface and the interface itself. If the fluid is incompressible and volume flux is conserved across the interface, then

$$(\mathbf{u}^+ - \mathbf{v}_{\text{int}}) \cdot (-\mathbf{n}_{\text{int}}) = (\mathbf{u}^- - \mathbf{v}_{\text{int}}) \cdot (-\mathbf{n}_{\text{int}}) = u_{\text{en}}, \tag{4b}$$

where superscript +, − denote fluid properties just above and below the interface. $(\mathbf{u}^- - \mathbf{v}_{\text{int}}) \cdot (-\mathbf{n}_{\text{int}})$ is then the velocity of water particles in the bottom layer relative to the interface in the $-\mathbf{n}_{\text{int}}$ direction, out of the bottom layer and into the top layer. This defines the entrainment velocity u_{en} normal to the interface, which must equal the relative normal velocity of water particles just above the interface. These considerations are taken into account when formulating boundary conditions at the interface.

a. *Equations of motion*

In order to subtract out the effects of hydrostatic pressure, pressures in the top and bottom layers are decomposed as

$$p_1 = \rho_1 g z + p_1', \tag{5a}$$

$$p_2 = \rho_2 g z - (\rho_2 - \rho_1) g \bar{h}_1 + p_1' + p_2'. \tag{5b}$$

Then the momentum equations for the top and bottom layers are, to the Boussinesq approximation,

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_1}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p_1'}{\partial x} + \frac{1}{\rho_0}\left(\frac{\partial \tau_{1xx}}{\partial x} + \frac{\partial \tau_{1zx}}{\partial z} \right), \tag{6a}$$

$$\frac{\partial w_1}{\partial t} + u_1 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial w_1}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p_1'}{\partial z} + \frac{1}{\rho_0}\left(\frac{\partial \tau_{1xz}}{\partial x} + \frac{\partial \tau_{1zz}}{\partial z}\right), \tag{6b}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z} = -\frac{1}{\rho_0}\left(\frac{\partial p_1'}{\partial x} + \frac{\partial p_2'}{\partial x}\right) + \frac{1}{\rho_0}\left(\frac{\partial \tau_{2xx}}{\partial x} + \frac{\partial \tau_{2zx}}{\partial z}\right), \tag{6c}$$

$$\frac{\partial w_2}{\partial t} + w_2 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial w_2}{\partial x} = -\frac{1}{\rho_0}\left(\frac{\partial p_1'}{\partial z} + \frac{\partial p_2'}{\partial z}\right) + \frac{1}{\rho_0}\left(\frac{\partial \tau_{2xz}}{\partial x} + \frac{\partial \tau_{2zz}}{\partial z}\right). \tag{6d}$$

Here τ_{zx} represents stress in the x direction on a plane whose normal is in the $+z$ direction. The equations of continuity for the top and bottom layers are

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \tag{7a}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial z} = 0. \tag{7b}$$

We can use (3) and (4) as boundary conditions to integrate the equations of motion over the layer depths. Expanding (3) and (4b) yields expressions for vertical velocities at the free surface and interface:

$$w_1 = \zeta_{1t} + u_1\zeta_{1x} \quad \text{at} \quad z = \zeta_1, \tag{8a}$$

$$\left.\begin{aligned} w^+ &= \zeta_{2t} + u^+\zeta_{2x} - u_{\text{en}}(1 + \zeta_{2x}{}^2)^{1/2} \quad (8\text{b}) \\ w^- &= \zeta_{2t} + u^-\zeta_{2x} - u_{\text{en}}(1 + \zeta_{2x}{}^2)^{1/2} \quad (8\text{c}) \end{aligned}\right\} \text{ at } z = \zeta_2,$$

where +, − denote properties just above and below the interface. We also specify zero velocity normal to the bottom and side walls, i.e.,

$$w_2 = 0 \quad \text{at} \quad z = H, \tag{9a}$$

$$u_1 = u_2 = U_1 = U_2 = 0 \quad \text{at} \quad x = 0, L. \tag{9b}$$

We define

$$u_e = u_{\text{en}}(1 + \zeta_{2x}{}^2)^{1/2}, \tag{10}$$

where u_e may be interpreted as volume flux across the interface per unit of horizontally projected area of interface. To see this, consider path length ds along the interface $z = \zeta_2$; then

$$ds^2 = dx^2 + dz^2 = dx^2[1 + (dz/dx)^2] = dx^2(1 + \zeta_{2x}{}^2),$$

so $u_e dx = u_{\text{en}} ds$.

The simultaneous occurrence of entrainment and a velocity discontinuity at the interface influences the proper formulation of shear stress at the interface. If $\boldsymbol{\tau}^+$, $\boldsymbol{\tau}^-$ are stress tensors just above and below the interface, then (Slattery, 1972, p. 40)

$$-\mathbf{n}_{\text{int}}(\boldsymbol{\tau}^+ - \boldsymbol{\tau}^-) = (\mathbf{u}^+ - \mathbf{u}^-)\rho_0 u_{\text{en}}, \tag{11a}$$

which may be expanded as

$$-\zeta_{2x}(\tau_{xx}^+ - \tau_{xx}^-) + (\tau_{zx}^+ - \tau_{zx}^-) = -(u^+ - u^-)\rho_0 u_e, \tag{11b}$$

$$-\zeta_{2x}(\tau_{xz}^+ - \tau_{xz}^-) - (\tau_{zz}^+ - \tau_{zz}^-) = -(w^+ - w^-)\rho_0 u_e. \tag{11c}$$

The above equations show that a stress discontinuity must occur if there are both entrainment and a velocity jump across the interface.

We assume that at the free surface the wind stress τ_0 acts parallel to the free surface and is continuous across the free surface. Pressure is taken as uniformly zero along the free surface; pressure is assumed continuous across the interface.

With the above boundary conditions the equations of continuity may be integrated over the thickness of each layer to give

$$h_{1t} + (U_1 h_1)_x = u_e, \tag{12a}$$

$$h_{2t} + (U_2 h_2)_x = -u_e. \tag{12b}$$

Averaging these equations over the length of the basin gives

$$d\bar{h}_1/dt = \bar{u}_e, \quad d\bar{h}_2/dt = -\bar{u}_e. \tag{13}$$

A useful form of the continuity equations may be obtained by making use of (13) and recalling that $h_1 = \bar{h}_1 + \zeta_{12} - \zeta_1$ and $h_2 = \bar{h}_2 - \zeta_{12}$:

$$(\zeta_{12} - \zeta_1)_t + [U_1(\zeta_{12} - \zeta_1)]_x + \bar{h}_1 U_{1x} = u_e - \bar{u}_e, \tag{14a}$$

$$\zeta_{12t} + (U_2\zeta_{12})_x - \bar{h}_2 U_{2x} = u_e - \bar{u}_e. \tag{14b}$$

In this form it is easier to isolate the behavior of the baroclinic mode, for which interface displacements ζ_{12} are much larger than those of the free surface ζ_1.

Similarly, the x momentum equations may be integrated over each layer thickness to give

$$\frac{\partial U_1}{\partial t} + U_1 \frac{\partial U_1}{\partial x} = \frac{1}{h_1}\int_{\zeta_1}^{\zeta_2} -\frac{1}{\rho_0}\frac{\partial p_1'}{\partial x}dz + \frac{\tau_{zx}^+ - \tau_{zx}^0}{\rho_0 h_1} + \frac{M_1}{h_1}, \tag{15a}$$

$$\frac{\partial U_2}{\partial t} + U_2 \frac{\partial U_2}{\partial x} = \frac{1}{h_2}\int_{\zeta_2}^{H} -\frac{1}{\rho_0}\left(\frac{\partial p_1'}{\partial x} + \frac{\partial p_2'}{\partial x}\right)dz + \frac{\tau_{zx}^H - \tau_{zx}^-}{\rho_0 h_2} + \frac{M_2}{h_2}, \tag{15b}$$

in which

$$M_1 = \frac{1}{\rho_0}\int_{\zeta_1}^{\zeta_2} \frac{\partial \tau_{1xx}}{\partial x} dz - \frac{\partial}{\partial x}\int_{\zeta_1}^{\zeta_2} (u_1')^2 dz - (u_1'^+)u_e, \tag{15c}$$

$$M_2 = \frac{1}{\rho_0}\int_{\zeta_2}^{H} \frac{\partial \tau_{2xx}}{\partial x} dz - \frac{\partial}{\partial x}\int_{\zeta_2}^{H} (u_2')^2 dz - (u_2'^-)u_e. \tag{15d}$$

Here τ_{zx}^0, τ_{zx}^H denote shear stresses at the free surface and the bottom; and τ_{zx}^+, τ_{zx}^- are stresses just above and below the interface. Notice that the explicit effects of entrainment have all but vanished from the momentum equations, being confined to the product of u_e with the residual circulation velocities u_1', u_2'; as we shall see, this represents a higher order effect for both the motion and entrainment.

b. Scaling

In order to scale the equations we must assume that the vertical length scales are approximated by $h_1 \sim h_2 \sim H$, the total depth, which is a constant. We assume that the short time scale is that for internal waves, which we approximate as $T_0 = L/(g_0'H)^{1/2}$, $g_0' = \Delta\rho_0 g/\rho_0$ giving the initial density jump. We thus effectively exclude consideration of the barotropic response, which occurs on a much faster time scale, $L/(gH)^{1/2}$, almost instantaneously so far as the baroclinic response is concerned. We introduce an overall Richardson number, $\mathrm{Ri}_0 = g_0'H/u_*^2$, which is of the same order as the mixed layer Richardson number, $\mathrm{Ri} = g'h_1/u_*^2$. From (2), the solution we seek is thus valid for the range

$$O(L/H) < \mathrm{Ri}_0 < O(L/H)^2). \tag{16}$$

The following scales are based on the work of SI:

Length	$x \sim L, \quad z \sim H, \quad h_1 \sim H, \quad h_2 \sim H$
Time	$t \sim T_0 \sim L/(g_0'H)^{1/2} \sim L/(u_*\mathrm{Ri}_0)^{1/2}$
Velocity	$u_1 \sim u_2 \sim U_1 \sim U_2 \sim \Delta U\ (=U_1 - U_2) \sim u_*^2 T_0/H = u_* L/(H\mathrm{Ri}_0^{1/2})$
	$w_1 \sim w_2 \sim u_1 H/L, \quad u_1' \sim u_2' \sim u_*.$ (17)
Displacement	$\zeta_1 \sim Lu_*^2/gH \sim (\Delta\rho_0/\rho_0)(L/\mathrm{Ri}_0)$
	$\zeta_{12} \sim Lu_*^2/g_0'H \sim L/\mathrm{Ri}_0$
Pressure	$p_1'/\rho_0 \sim g\zeta_1 \sim g_0'L/\mathrm{Ri}_0$
	$p_2'/\rho_0 \sim g_0'\zeta_{12} \sim g_0'L/\mathrm{Ri}_0$
Stresses	$\tau_0 \sim \tau_{zx}^0 \sim \tau_{zx}^H \sim \rho_0 u_*^2$
	$\tau_{xx} \sim \tau_{zz} \sim (H/L)\rho_0 u_*^2$
	$\tau^+ \sim \tau^- \sim \rho_0 u_e \Delta U \sim \rho_0(u_*/\mathrm{Ri}_0)[Lu_*/(H\mathrm{Ri}_0^{1/2})]$.

In the scaling for interfacial stresses we have assumed that entrainment scales as $u_e \sim u_*/\mathrm{Ri}$. This is based on the entrainment model (1), which may be solved for u_e:

$$u_e/u_* = C_K\eta^3[\mathrm{Ri}(1 + C_T\eta^2/\mathrm{Ri} - C_S\Delta U^2/g'h_1)]^{-1}. \tag{18}$$

Using the scales introduced above we can estimate the magnitude of the shear production term in (18) as $\Delta U^2/(g'h_1) \sim [L/(H\mathrm{Ri}_0)]^2 \ll 1$ for Ri_0 given by (16). To lowest order, then (18) simplifies to

$$u_e/u_* = C_K\eta^3/\mathrm{Ri} \tag{19}$$

for the range of interest. This is the deepening law proposed by Kraus and Turner (1967). The scale for entrainment is thus $u_e \sim u_*/\mathrm{Ri}_0$.

3. The lowest order solution

Substituting the scales into the equations for z momentum (an asterisk denotes dimensionless variable) gives

$$\left(\frac{H}{L}\right)^2 \frac{\partial w_1^*}{\partial t^*} + \frac{H}{L\mathrm{Ri}_0}\left(u_1^* \frac{\partial w_1^*}{\partial x^*} + w_1^* \frac{\partial w_1^*}{\partial z^*}\right)$$

$$= -\frac{\partial p_1'^*}{\partial z^*} + \left(\frac{H}{L}\right)^2\left(\frac{\partial \tau_{xz}^*}{\partial x^*} + \frac{\partial \tau_{zz}^*}{\partial z^*}\right), \tag{20a}$$

$$\left(\frac{H}{L}\right)^2 \frac{\partial w_2^*}{\partial t^*} + \frac{H}{L\mathrm{Ri}_0}\left(u_2^* \frac{\partial w_2^*}{\partial x^*} + w_2^* \frac{\partial w_2^*}{\partial z^*}\right) = -\left(\frac{\partial p_1'^*}{\partial z^*} + \frac{\partial p_2'^*}{\partial z^*}\right) + \left(\frac{H}{L}\right)^2\left(\frac{\partial \tau_{xz}^*}{\partial x^*} + \frac{\partial \tau_{zz}^*}{\partial z^*}\right). \quad (20b)$$

Since we are interested only in the lowest order solution and will carry our expansion no farther than $O(H/L)$, we will neglect at the outset terms smaller than $O(H/L)$. Keeping in mind that $\mathrm{Ri}_0 > L/H$, Eqs. (20a,b) give $\partial p_1'^*/\partial z^* = \partial p_2'^*/\partial z^* = 0$, i.e., pressures are a function of x only. From the definitions of p_1', p_2', and the dynamic free-surface condition that pressures be constant (zero) along the free surface, we have $p_1' = -\rho g \zeta_1$ for all z. Similarly, requiring continuity of pressure across the interface gives $p_2' = -\Delta\rho g \zeta_{12}$ for all z. Thus, pressures are hydrostatic for the scaling we have adopted, and pressure gradients arise only because of departures of the free surface and interface from the horizontal.

The integrated equations (15a,b) for x momentum then become, in dimensionless terms and neglecting terms smaller than $O(H/L)$:

$$\frac{\partial U_1^*}{\partial t^*} + \frac{L}{H\mathrm{Ri}_0} U_1^* \frac{\partial U_1^*}{\partial x^*} = \frac{\partial \zeta_1^*}{\partial x^*} + \frac{1}{h_1^*}\left(\frac{L}{H\mathrm{Ri}_0^{3/2}} \tau_{zx}^{+*} - \tau_{zx}^{0*} - \frac{H}{L}\frac{\partial}{\partial x^*}\int_{\zeta_1/H}^{\zeta_2/H} u_1'^{*2} dz^*\right), \quad (21a)$$

$$\frac{\partial U_2^*}{\partial t^*} + \frac{L}{H\mathrm{Ri}_0} U_2^* \frac{\partial U_2^*}{\partial x^*} = \frac{\partial \zeta_1^*}{\partial x^*} + \frac{\partial \zeta_{12}^*}{\partial x^*} + \frac{1}{h_2^*}\left(\tau_{zx}^{H*} - \frac{L}{H\mathrm{Ri}_0^{3/2}} \tau_{zx}^{-*} - \frac{H}{L}\frac{\partial}{\partial x^*}\int_{\zeta_2/H}^{1} u_2'^{*2} dz^*\right). \quad (21b)$$

The integrated equations for conservation of mass, in dimensionless form to $O(H/L)$, are

$$\frac{\partial\left(\zeta_{12}^* - \frac{\Delta\rho_0}{\rho_0}\zeta_1^*\right)}{\partial t^*} + \frac{L}{H\mathrm{Ri}_0}\frac{\partial U_1^*\left(\zeta_{12}^* - \frac{\Delta\rho_0}{\rho_0}\zeta_1^*\right)}{\partial x} + \bar{h}_1^* \frac{\partial U_1^*}{\partial x^*} = \frac{u_e^* - \bar{u}_e^*}{\mathrm{Ri}_0^{1/2}}, \quad (22a)$$

$$\frac{\partial \zeta_{12}^*}{\partial t^*} + \frac{L}{H\mathrm{Ri}_0}\frac{\partial U_2^* \zeta_{12}^*}{\partial x^*} - \bar{h}_2^* \frac{\partial U_2^*}{\partial x^*} = \frac{u_e^* - \bar{u}_e^*}{\mathrm{Ri}_0^{1/2}}. \quad (22b)$$

We also have the compatibility relations $h_1 = \bar{h}_1 + \zeta_{12} - \zeta_1$ and $h_2 = \bar{h}_2 - \zeta_{12}$, which are nondimensionalized as

$$h_1^* = \bar{h}_1^* + \frac{L}{H\mathrm{Ri}_0}\left(\zeta_{12}^* - \frac{\Delta\rho}{\rho_0}\zeta_1^*\right), \quad (23a)$$

$$h_2^* = \bar{h}_2^* - \frac{L}{H\mathrm{Ri}_0}\zeta_{12}^*. \quad (23b)$$

Keeping in mind that $\mathrm{Ri}_0 > L/H$ we see that entrainment enters the equations directly only as a higher order term. We therefore need only include the lowest order effects in the entrainment model. From (23a) it can be seen that $h_1^* = \bar{h}_1^*$ to lowest order; we can thus invoke the one-dimensional conservation of mass condition for a two-layer system, $\Delta\rho h_1$ = constant. This implies that Ri = constant as the mixed layer deepens. Hence, to lowest order, the entrainment model (19) simplifies to

$$u_e/u_* = C_K\eta^3/\mathrm{Ri} = \text{constant}. \quad (24)$$

The problem is simplified tremendously, since now $u_e = \bar{u}_e$ to lowest order, and entrainment enters the equations only through the slow variation in $\bar{h}_1$ and $\bar{h}_2$ given by (13), i.e.,

$$d\bar{h}_1^*/dt^* = -d\bar{h}_2^*/dt^* = \epsilon\bar{u}_e^*, \quad (25a)$$

where

$$\epsilon = L/[H(\mathrm{Ri}_0^{3/2})]. \quad (25b)$$

This immediately suggests the introduction of two time scales, a slow one for entrainment

$$\xi = \epsilon t^* \quad (26a)$$

and a fast time scale for the wave motion

$$\mu = f(\xi)/\epsilon, \quad (26b)$$

where f is an unknown function of ξ, to be determined in the course of analysis (see Nayfeh, 1973, pp. 282–284). If $f(\xi) = \xi$, for example, then $\mu = t^*$. We have chosen the above form for μ so that derivatives with respect to the fast time will be O(1), as will be seen below. Derivatives with respect to dimensionless time t^* must now be expanded as

$$\frac{\partial}{\partial t^*} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t^*} + \frac{\partial}{\partial \mu}\frac{\partial \mu}{\partial \xi}\frac{\partial \xi}{\partial t^*} = \epsilon\frac{\partial}{\partial \xi} + f'\frac{\partial}{\partial \mu},$$

where $\partial\xi/\partial t^* = \epsilon$ and a prime denotes differentiation with respect to ξ. We thus have the following set of equations, correct to $O(H/L)$ for $O(L/H) < \mathrm{Ri}_0 < O((L/H)^2)$, $O((H/L)^{1/2}) < \epsilon < O((H/L)^2)$:

Continuity
$$f'\frac{\partial}{\partial\mu}\left(\zeta_{12}^* - \frac{\Delta\rho_0}{\rho_0}\zeta_1^*\right) + \epsilon\frac{\partial}{\partial\xi}\left(\zeta_{12}^* - \frac{\Delta\rho_0}{\rho_0}\zeta_1^*\right) + \epsilon\mathrm{Ri}_0^{1/2}\frac{\partial}{\partial x^*}\left[U_1^*\left(\zeta_{12}^* - \frac{\Delta\rho_0}{\rho_0}\zeta_1^*\right)\right] + \bar{h}_1^*\frac{\partial U_1^*}{\partial x^*} = 0 \quad (27a)$$

$$f'\frac{\partial\zeta_{12}^*}{\partial\mu} + \epsilon\frac{\partial\zeta_{12}^*}{\partial\xi} + \epsilon\mathrm{Ri}_0^{1/2}\frac{\partial U_1^*\zeta_{12}^*}{\partial x^*} - \bar{h}_2^*\frac{\partial U_2^*}{\partial x^*} = 0. \quad (27b)$$

Momentum
$$f'\frac{\partial U_1^*}{\partial\mu} + \epsilon\frac{\partial U_1^*}{\partial\xi} + \epsilon\mathrm{Ri}_0^{1/2}U_1^*\frac{\partial U_1^*}{\partial x^*} = \frac{\partial\zeta_1^*}{\partial x^*} + \frac{\epsilon\tau_{zx}^{+*} - \tau_{zx}^{0*}}{h_1^*} - \frac{H}{L}\frac{\partial}{\partial x^*}\int_{\zeta_1/H}^{\zeta_2/H}(u_1'^*)^2dz^*. \quad (28a)$$

$$f'\frac{\partial U_2^*}{\partial\mu} + \epsilon\frac{\partial U_2^*}{\partial\xi} + \epsilon\mathrm{Ri}_0^{1/2}U_2^*\frac{\partial U_2^*}{\partial x^*} = \frac{\partial\zeta_1^*}{\partial x^*} + \frac{\partial\zeta_{12}^*}{\partial x^*} + \frac{\tau_{zx}^{H*} - \epsilon\tau_{zx}^{-*}}{h_2^*} - \frac{H}{L}\frac{\partial}{\partial x}\int_{\zeta_2/H}^{1}(u_2'^*)^2dz^*. \quad (28b)$$

Entrainment $d\bar{h}_1^*/d\xi = -d\bar{h}_2^*/d\xi = u_e^*.$ (29)

together with the compatibility relations (23a,b).

If we consider ϵ an expansion parameter, we can expand all variables in a series, as

$$U_1^* = U_{1(0)} + \epsilon U_{1(1)} + \ldots. \quad (30)$$

For example, if we assume $\Delta\rho/\rho_0$ and H/L are small, of $O(\epsilon)$, we can write down the $O(\epsilon^0)$ problem at once, as

$$f'\frac{\partial U_{1(0)}}{\partial\mu} = \frac{\partial\zeta_{1(0)}}{\partial x^*} - \frac{\tau_{zx}^{0*}}{\bar{h}_{1(0)}}, \quad (31a)$$

$$f'\frac{\partial U_{1(0)}}{\partial\mu} = \frac{\partial\zeta_{12(0)}}{\partial x^*} + \frac{\partial\zeta_{1(0)}}{\partial x^*} + \frac{\tau_{zx}^{H*}}{\bar{h}_{2(0)}}, \quad (31b)$$

$$f'\frac{\partial\zeta_{12(0)}}{\partial\mu} + \bar{h}_{1(0)}\frac{\partial U_{1(0)}}{\partial x^*} = 0, \quad (31c)$$

$$f'\frac{\partial\zeta_{12(0)}}{\partial\mu} - \bar{h}_{2(0)}\frac{\partial U_{2(0)}}{\partial x^*} = 0. \quad (31d)$$

These are similar to the linearized equations solved by Heaps and Ramsbottom (1966), but now $\bar{h}_1, \bar{h}_2$ are functions of the slow time variable ξ, as is the unknown function $f(\xi)$. In order to solve the lowest order problem we need to determine f. To do this we must go to the next higher order, even though we need not solve the higher order problem completely. The $O(\epsilon^1)$ problem is

$$f'\frac{\partial U_{1(1)}}{\partial\mu} = \frac{\partial\zeta_{1(1)}}{\partial x^*} + \frac{\tau_{zx(0)}^{+}}{\bar{h}_{1(0)}} - \frac{\partial U_{1(0)}}{\partial\xi} - \mathrm{Ri}_0^{1/2}U_{1(0)}\frac{\partial U_{1(0)}}{\partial x^*} - \frac{\partial}{\partial x^*}\int_0^{h_{2(0)}}u_{1(0)}'^2dx, \quad (32a)$$

$$f'\frac{\partial U_{2(1)}}{\partial\mu} = \frac{\partial\zeta_{1(1)}}{\partial x^*} + \frac{\partial\zeta_{12(1)}}{\partial x^*} + \frac{1}{\bar{h}_{2(0)}}(\tau_{zx(1)}^{H} - \tau_{zx(0)}^{-} - \tau_{zx(0)}^{H}\mathrm{Ri}^{1/2}\zeta_{12(0)}) - \frac{\partial U_{2(0)}}{\partial\xi} - \mathrm{Ri}_0^{1/2}U_{2(0)}\frac{\partial U_{2(0)}}{\partial x^*} + \frac{\tau_{zx(0)}^{H}}{\bar{h}_{1(1)}} - \frac{\partial}{\partial x^*}\int_{h_{2(0)}}^{1}u_{2(0)}'^2dz, \quad (32b)$$

$$f'\frac{\partial\zeta_{12(1)}}{\partial\mu} + \bar{h}_{1(0)}\frac{\partial U_{1(1)}}{\partial x^*} = -\frac{\partial\zeta_{12(0)}}{\partial\xi} - \mathrm{Ri}_0^{1/2}\frac{\partial U_{1(0)}\zeta_{12(0)}}{\partial x^*}, \quad (32c)$$

$$f'\frac{\partial\zeta_{12(1)}}{\partial\mu} - \bar{h}_{2(0)}\frac{\partial U_{2(1)}}{\partial x^*} = -\frac{\partial\zeta_{12(0)}}{\partial\xi} - \mathrm{Ri}_0^{1/2}\frac{\partial U_{2(0)}\zeta_{12(0)}}{\partial x^*}. \quad (32d)$$

We notice that the left-hand sides of the $O(\epsilon^1)$ equations are identical in form to the left-hand sides of the $O(\epsilon^0)$ equations. The right-hand sides of the $O(\epsilon^1)$ equations, however, contain terms which do not appear in the form of the lower order equations. Nonlinear combinations of $O(\epsilon^0)$ terms occur.

Derivatives of $O(\epsilon^0)$ terms with respect to the slow time variable ξ appear as well, for the first time at $O(\epsilon^1)$. As will be seen below, it is these terms which determine the form of $f(\xi)$ for the lower order solution. For the moment we treat f' as a parameter in (31a)–(31d) and solve the $O(\epsilon^0)$ problem.

For the remainder of the paper we drop the lowest order subscript (0) and the superscript asterisk for dimensionless variables except where ambiguity is possible.

Subtracting (31d) from (31c) and integrating from $x = 0$ to 1, with $U_1 = U_2 = 0$ at $x = 0$, 1, gives conservation of mass for the baroclinic mode as

$$\bar{h}_1 U_1 = -\bar{h}_2 U_2. \tag{33}$$

The transports are equal and opposite in the top and bottom layers. Because of our choice of scales for ζ_1 and ζ_{12}, the free surface coordinate ζ_1 has vanished from the continuity equations [(31c) and (31d)] indicating that so far as continuity is concerned the free surface is flat. However, the scales for the pressure are the same for the free surface as for the interface so that dynamically the effects of free-surface displacements are as important as those of the interface. Thus ζ_1 appears in the momentum equations (31a,b). We can eliminate both ζ_1 and U_2 from the momentum equations by subtracting (31b) from (31a) and by making use of the relation for conservation of mass (33) to get

$$\frac{f'}{\bar{h}_2}\frac{\partial U_1}{\partial \mu} + \frac{\partial \zeta_{12}}{\partial x} = \frac{-\tau_{zx}^0}{\bar{h}_1} - \frac{\tau_{zx}^H}{\bar{h}_2}. \tag{34}$$

Recalling that the stress at the free surface in the $+x$ direction is equal to $\tau_0 = \rho u_*^2$, the wind stress, we have $-\tau_{zx}^0 = \tau_0/\rho u_*^2 = 1$. In addition, we can parameterize τ_{zx}^H, the bottom boundary stress, by equating the decay of internal wave energy to the work done by the bottom stress in the boundary layer at the bottom of the basin. In physical variables, the energy E in the wave field is proportional to $E \sim \rho_0(U_1^2 h_1 + U_2^2 h_2)L \sim \rho_0 U_2^2 h_2 HL/h_1$ [see Eq. (33)]. The work W done by the stresses in the boundary layer over one wave period is proportional to $W \sim \tau_{zx}^H U_2 L T_i$. Equating W to the loss in wave energy dE over one period yields an estimate for the fraction of wave energy dissipated per wave period as

$$dE/E \sim -\tau_{zx}^H T_i/\rho_0 U_1 H \sim 2\alpha_d, \tag{35}$$

where α_d is a decay modulus, assumed constant (see Keulegan, 1959). As Keulegan (1959) points out, and as confirmed by the solution derived here, the decay modulus α_d has a physical significance related to the e-folding decay time T_d for wave amplitude or particle velocity, given by $\alpha_d = T_i/T_d$. The decay modulus may be thought of as the ratio between the time scales of wave period and wave decay, so that for periodic response we must have $\alpha_d < 1$. Assuming $4T_0 \sim T_i$ (this will be exactly true for $h_1 = h_2 = H/2$), we let $\tau_{zx}^H = \alpha_d \rho_0 U_1 H/2T_0$. In dimensionless form,

$$\tau_{zx}^{H*} = \alpha_d U_1^*/2. \tag{36}$$

Relating wave decay to dissipation in the bottom boundary layer thus leads to a linear bottom stress law.

Substituting for τ_{zx}^0, τ_{zx}^H in (34) gives the momentum equation as

$$\frac{f'}{\bar{h}_2}\frac{\partial U_1}{\partial \mu} + \frac{\partial \zeta_{12}}{\partial x} = \frac{1}{\bar{h}_1} - \frac{\alpha_d}{2}\frac{U_1}{\bar{h}_2}. \tag{37}$$

Eqs. (37) and (31c) now form a set of two equations in the two unknowns U_1, ζ_{12}. Eliminating ζ_{12} by cross differentiation gives

$$\frac{\partial^2 U_1}{\partial \mu^2} + \frac{\alpha_d}{2f'}\frac{\partial U_1}{\partial \mu} - \frac{\bar{h}_1\bar{h}_2}{f'^2}\frac{\partial^2 U_1}{\partial x^2} = 0. \tag{38}$$

This is a linear wave equation with damping.

To obtain initial conditions for U_1 we require that $\mu = 0$ at $t = 0$, and assume that the motion starts from rest: $U_1 = \zeta_{12} = 0$ at $\mu = 0$. Using these conditions to evaluate (37) at $\mu = 0$ gives the second initial condition for U_1, $(\partial U_1/\partial \mu)|_{\mu=0} = \bar{h}_2/(\bar{h}_1 f')$. If we assume a solution of the form

$$U_1 = \sum_{n=1}^{\infty} A_n(\mu,\xi)\sin n\pi x,$$

then the boundary conditions are satisfied and the problem may be readily solved to give the solution for U_1:

$$U_1 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4\bar{h}_2}{n\pi f'\bar{h}_1} \times \exp\left(\frac{-\alpha_d}{4f'}\mu\right)\sin\gamma\mu \sin n\pi x, \tag{39}$$

where

$$\gamma(\xi) = \frac{(n^2\pi^2\bar{h}_1\bar{h}_2 - \alpha_d^2/16)^{1/2}}{f'}. \tag{40}$$

The solution for ζ_{12} may now be found by integrating the expression for continuity (31c) to get

$$\zeta_{12} = \frac{1}{\bar{h}_1}\left[\left(x - \frac{1}{2}\right) + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\gamma n^2\pi^2}\exp\left(\frac{-\alpha_d}{4f'}\right)\mu \times \cos n\pi x\left(\gamma\cos\gamma\mu + \frac{\alpha_d}{4f'}\sin\gamma\mu\right)\right]. \tag{41}$$

In a similar fashion ζ_1 may be determined from (31a) together with the conservation of volume condition $\int_0^1 \zeta_1 dx = 0$, and U_2 may be determined from (33).

We are still left with the problem of determining $f(\xi)$, and thus μ and γ. As we mentioned earlier, (32a)–(32d) for the $O(\epsilon^1)$ problem have $O(\epsilon^0)$ terms on the right-hand side, including some containing derivatives of $U_{1(0)}$, etc, with respect to ξ. Now all these $O(\epsilon^0)$ terms act as nonhomogeneous terms in the equations for the $O(\epsilon^1)$ terms; if they were not there, the $O(\epsilon^1)$ equations would be identical to those of $O(\epsilon^0)$. Following Nayfeh (1973, pp. 280–284), we now argue as follows. The homogeneous solutions to (32) [i.e., the solutions of the equations without the $O(\epsilon^0)$ terms on the right-hand side] will contain terms $\exp(-\alpha_d\mu/4f') \times \cos\gamma\mu$, $\exp(-\alpha_d\mu/4f')\sin\gamma\mu$ since the form of the homogeneous equations is exactly that of the $O(\epsilon^0)$ equations. The nonhomogeneous terms, however, also contain terms which are proportional to these homogeneous solutions. This gives rise to particular solutions with terms like $\mu^2\exp(-\alpha_d\mu/4f')\cos\gamma\mu$, etc. In order for the perturbation expansion to be valid, the ratio of $O(\epsilon^1)$ terms to $O(\epsilon^0)$ terms must remain small for large time, μ; we must therefore eliminate the μ^2 terms. We note that among the nonhomogeneous terms, those with derivatives with respect to ξ will give rise to the offending terms multiplied by the coefficient γ'. Since the homogeneous terms give rise to terms with coefficients f' and γ, but not γ', the only way to suppress these terms is to require that $\gamma' = 0$. Other conditions will be required as well to solve the $O(\epsilon^1)$ problem, but for the $O(\epsilon^0)$ problem it is sufficient to set $\gamma' = 0$.

If $\gamma' = 0$, then Nayfeh (1973, p. 278) points out that $\gamma = 1$ without loss of generality, and from the expression (40) for γ we have

$$f(\xi) = \int_0^{\xi}\left(n^2\pi^2\bar{h}_1\bar{h}_2 - \frac{\alpha_d^2}{16}\right)^{1/2} dY, \qquad (42)$$

where Y is a dummy variable. Recalling that $\xi = \epsilon t^*$, this satisfies the requirement that $\mu = f/\epsilon = 0$ when $t^* = 0$. Using the expressions for $\xi = \epsilon t^*$ and $\mu = f/\epsilon$, we can write (42) in terms of μ and t^* as

$$\mu(t^*) = \int_0^{t^*}\left(n^2\pi^2\bar{h}_1\bar{h}_2 - \frac{\alpha^2}{16}\right) dT^*, \qquad (43)$$

with T^* a dummy variable, and the dependence of $\bar{h}_1$, $\bar{h}_2$ on t^* is given by the entrainment law (29) together with (26a).

The solution is summarized in terms of physical variables below; the conversion is rather tedious but straightforward. To lowest order, in terms of physical variables,

$$U_1 = \frac{u_*^2}{H} T_0 \sum_{\substack{n=1\\ n\ \mathrm{odd}}}^{\infty} \frac{4\bar{h}_2}{n\pi\psi\bar{h}_1} \times \exp\left(-\frac{\alpha_d}{4}\frac{\phi}{\psi}\right)\sin\phi \sin\frac{n\pi x}{L}, \qquad (44a)$$

$$U_2 = -\frac{U_1\bar{h}_1}{\bar{h}_2}, \qquad (44b)$$

$$\zeta_{12} = \frac{L}{\mathrm{Ri}}\left[\left(\frac{x}{L} - \frac{1}{2}\right) + \sum_{\substack{n=1\\ n\ \mathrm{odd}}}^{\infty} \frac{4}{n^2\pi^2}\exp\left(-\frac{\alpha_d}{4}\frac{\phi}{\psi}\right) \times \cos\frac{n\pi x}{L}\left(\cos\phi - \frac{\alpha_d}{4\psi}\sin\phi\right)\right], \qquad (44c)$$

$$\zeta_1 = -\frac{\Delta\rho}{\rho_0}\frac{L}{\mathrm{Ri}}\left[\left(\frac{x}{L} - \frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{4\bar{h}_2}{n^2\pi^2 H}\exp\left(-\frac{\alpha_d}{4}\frac{\phi}{\psi}\right) \times \cos\frac{n\pi x}{L}\left(\cos\phi - \frac{\alpha_d}{4\psi}\sin\phi\right)\right], \qquad (44d)$$

where

$$\phi = \frac{1}{L}\int_0^t \left(n^2\pi^2\frac{g'\bar{h}_1\bar{h}_2}{H} - \frac{\alpha_d^2 g_0' H}{16}\right)^{1/2} dt = \frac{(g_0'H)^{1/2}}{L}\int_0^t \psi dt \qquad (44e)$$

$$\psi = (g_0'H)^{-1/2}\left(\frac{n^2\pi^2 g'\bar{h}_1\bar{h}_2}{H} - \frac{\alpha_d^2}{16}g_0'H\right)^{1/2}. \qquad (44f)$$

The solution represents standing waves which decay with time, leaving zero mean velocity and a steady state setup in both top and bottom layers. Both damping and entrainment affect the frequency of the oscillations, the frequency changing slowly as the mixed layer deepens and hence $\bar{h}_1$, $\bar{h}_2$ and g' change. The time variable does not appear simply as an argument of the harmonic sine and cosine functions, but must be evaluated via an integral which accounts for changes in $\bar{h}_1$, $\bar{h}_2$ and g' with entrainment. After the wave motion has decayed, the interface remains tilted with slope $\zeta_{12x} = \mathrm{Ri}^{-1}$. Since Ri is constant to this approximation, the angle of tilt does not change as the interface continues to deepen slowly according to $d\bar{h}_1/dt = C_K\eta^3 u_*\mathrm{Ri}^{-1}$. The solution reduces to that obtained by Heaps and Ramsbottom (1966) for the baroclinic mode for $\bar{h}_1$, $\bar{h}_2$ and g' constant.

The solutions for U_1, U_2 at mid-basin ($x = L/2$), and interface displacement $\zeta_2 = \zeta_{12} + \bar{h}_1$ at the leeward end of the basin ($x = L$) are illustrated in Fig. 3 for the following initial conditions: $h_1 = 12$ m, $h_2 = 38$ m, $g' = 7.8(10^{-3})$ m s^{-2}, $L = 6600$ m, $u_*^2 = 2.5(10^{-4})$ m^2 s^{-2}. These values correspond to the temperature profile of 15 September 1951 for Lake Windermere given by Heaps and Ramsbottom (1966); on that day the mixed layer extended to a depth of 12 m, overlying a sharp thermocline and a nearly uniform hypolimnion. From the above values we calculate $T_i = 13.7$ h, compared with an ob-

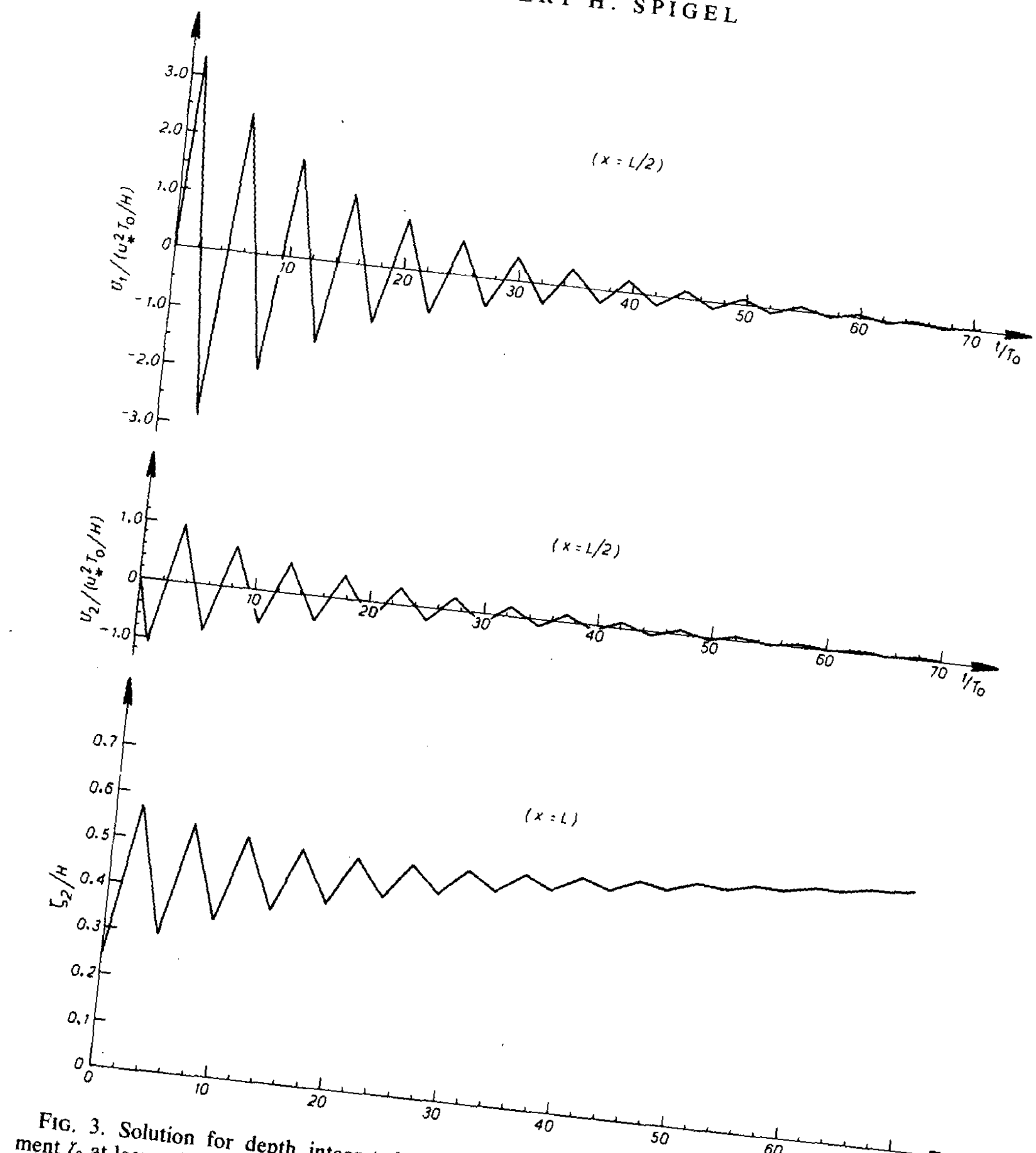

FIG. 3. Solution for depth integrated velocities U_1, U_2 at mid-basin, and interface displacement ζ_2 at leeward end of basin. Initial conditions are $h_1 = 12$ m, $h_2 = 38$ m, $g' = 7.8 \times 10^{-3}$ m s^{-2}, $u_*^2 = 2.5 \times 10^{-4}$ m^2 s^{-2}.

served period of ~12.7 h. By fitting their results to observations, Heaps and Ramsbottom estimated an e-folding time for wave decay $T_d = 55$ h. This gives a value for the decay modulus $\alpha_d = T_i/T_d = 0.25$, which was used in evaluating the solution illustrated in Fig. 3. The sawtooth character of Fig. 3 is due the linearization, and is part of the solution (see also Rao, 1967). Within the decay time T_d the mixed layer deepened 1.9 m and the wave period lengthened from 13.8 to 14.1 h. As is to be expected, these are perturbations on the main motion, and dominate the results only for $t \gg T_i$. In this case the time for complete vertical mixing $T_e = 1090$ h, much longer than the time for which the wind would remain steady under any circumstances. In the full event referred to, the wind changed on a time scale of T_i, and for most of the time was weaker than the value used here for u_*^2. No mention was made of mixed layer deepening in Heaps and Ramsbottom's discussion.

4. The quasi-steady solution

The solution tells us nothing explicitly about the baroclinic circulation (u_1', u_2') that remains after the wave motion (U_1, U_2) has decayed. However, the scaling and lowest order equations (31a)–(31d) provide important clues which allow us to get a solution easily for the circulation to $O(\epsilon^0)$. In the first place we know that pressures are hydrostatic, that to lowest order vertical velocities are zero, and the flow locally parallel. Obviously, this is not true at the end walls, but we neglect this region and concentrate on the much larger region some distance from the end walls. In the second place, the shear at the interface—τ_{zx}^{+} and τ_{zx}^{-}—enters only at higher order; to $O(\epsilon^0)$ the shear is zero at the interface. There is thus no way to drive a circulation in the bottom layer; to $O(\epsilon^0)$, $u_2' = 0$. Finally, we see from (31a,b) that on the long time scale the pressure gradients due to the setup of the free surface and interface are balanced by the wind stress. In physical variables, (31a,b) become (to lowest order and for $U_1 = U_2 = 0$ and hence $\tau_{zx}^{H} = 0$)

$$\frac{\partial \zeta_1}{\partial x} = \frac{-u_*^2}{g\bar{h}_1}, \tag{45a}$$

$$\frac{\partial \zeta_{12}}{\partial x} = -\left(\frac{\Delta\rho}{\rho}\right)^{-1}\frac{\partial \zeta_1}{\partial x} = \frac{1}{\mathrm{Ri}}, \tag{45b}$$

as the full solution predicts; $\bar{h}_1$, $\bar{h}_2$ are still functions of time.

Within the top layer the same balance must hold between pressure gradient and shear stress as expressed by (45a), which applies to the layer as a whole. Invoking an eddy viscosity as $\nu_e = 0.3u_*\bar{h}_1$, (see SI), the balance between shear and pressure gradient is

$$0 = g\frac{\partial \zeta_1}{\partial x} + 0.3u_*\bar{h}_1\frac{\partial^2 u_1}{\partial z^2} \tag{46a}$$

or

$$\frac{\partial^2 u_1}{\partial z^2} = \frac{u_*}{0.3\bar{h}_1^2}, \tag{46b}$$

with boundary conditions $\partial u_1/\partial z = -u_*/(0.3\bar{h}_1)$ at $z = 0$ and $\partial u_1/\partial z = 0$ at $z = h_1$. These conditions, together with the condition that

$$U_1 = \int_0^{h_1} u_1 dz = 0,$$

are sufficient to determine u_1 from (46) as

$$u_1 = \frac{u_*}{0.3}\left[\frac{1}{2}\left(\frac{z}{\bar{h}_1}\right)^2 - \frac{z}{\bar{h}_1} + \frac{1}{3}\right]. \tag{47}$$

The profile is thus parabolic, in effect a linear superposition of Poiseuille flow due to the pressure gradient of free-surface setup and of Couette flow due to free-surface shear stress. The velocity jump at the interface is $O(u_*)$, as predicted by SI, and thus has negligible effect on the shear production term in the entrainment model. The parabolic shape arises because of our use of a constant eddy viscosity for the layer. Although I am unaware of any comparable experimental result, the shape does not agree particularly well with that of Baines and Knapp (1965) for wind-driven barotropic circulation in a long rectangular tank. A parabolic velocity profile has also been derived by Hellström (1941) for the baroclinic circulation.

5. Summary and conclusions

A solution has been obtained for the initial value problem of a uniform wind stress suddenly applied to a two-layer rectangular basin. The solution incorporates the effects of mixed-layer deepening as well as the internal seiching and interface setup accompanying application of the wind stress, and is valid for the range of Ri given by (1). For this range of Ri, the solution is consistent with the scaling that was introduced initially and reinforces the following conclusions:

1) Interface displacements due to seiching and setup are small enough that mixed-layer deepening may be considered one-dimensional and uniform over the basin.

2) The energy source for mixed-layer deepening is stirring by the wind, which does work at the water surface at a rate proportional to u_*^3.

3) Shear production at the base of the mixed layer provides an insignificant amount of energy for the deepening process, both during the initial response when internal waves occur, as well as after wave decay when a baroclinic circulation remains in the top layer.

The solution breaks down as $\mathrm{Ri} \rightarrow L/h_1$ from above. For these smaller values of Ri, the solution shows that interface displacements are large, of the order of the top layer depth itself. The convective terms in the momentum equations (20a,b) are of the same order as the local acceleration and pressure terms. For these reasons the perturbation technique used to linearize the equation fails. In addition, velocity shear at the interface is large enough to have an important effect on the entrainment model, which can no longer be simplified as in (19).

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REFERENCES

Baines, W. D., and Knapp, D. J., 1965: Wind driven water currents. *ASCE J. Hydraul. Div.*, **91**, 205–221.

Bye, J. A. T., 1965: Wind driven circulation in unstratified lakes. *Limnol. Oceanogr.*, **10**, 451–458.

Fischer, H. B., E. J. List, R. Y. C. Koh, J. Imberger and N. H. Brooks, 1979: *Mixing in Inland and Coastal Waters*. Academic Press (in press).

Heaps, N. S., and A. E. Ramsbottom, 1966: Wind effects on the water in a narrow two layered lake. *Phil. Trans. Roy. Soc. London,* **A259,** 391–430.

Hellström, B., 1941: Wind effect on lakes and rivers. Bull. No. 41, Institution of Hydraulics at the Royal Institute of Technology, Stockholm.

Keulegan, G. R. L., 1959: Energy dissipation in standing waves in rectangular basins. *J. Fluid Mech.,* **6,** 33–50.

Kraus, E. B., and J. S. Turner, 1967: A one-dimensional model of the seasonal thermocline, II. General theory and its consequences. *Tellus,* **19,** 98–106.

Nayfeh, A., 1973: *Perturbation Methods.* Wiley, 425 pp.

Niiler, P. P., 1975: Deepening of the wind mixed layer. *J. Mar. Res.,* **33,** 405–422.

Rao, D. B., 1967: Response of a lake to a time dependent wind stress. *J. Geophys. Res.,* **72,** 1697–1708.

Sherman, F. S., J. Imberger and G. M. Corcos, 1978: Turbulence and mixing in stably stratified waters. *Annual Review of Fluid Mechanics,* Vol. 8, Annual Reviews, Inc., 141–193.

Slattery, J. C., 1972: *Momentum, Energy and Mass Transfer in Continua.* McGraw-Hill, 679 pp.

Spigel, R. H., and J. Imberger, 1979: The classification of mixed layer dynamics in lakes of small to medium size. *J. Phys. Oceanogr.* (accepted for publication).

Stoker, J. J., 1957: *Water waves.* Interscience, 567 pp.

Thorpe, S. A., 1973: Turbulence in stably stratified fluids: a review of laboratory experiments. *Bound.-Layer Meteor.,* **5,** 95–119.

Wu, J., 1975: Wind induced drift currents. *J. Fluid Mech.,* **68,** 49–70.

Zeman, O., and H. Tennekes, 1977: Parameterization of the turbulent energy budget at the top of the daytime atmospheric boundary layer. *J. Atmos. Sci.,* **34,** 111–123.