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Statistical distribution of wave-surface elevation for second-order random directional ocean waves in finite water depth

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Abstract

Based on the second-order random wave solutions of water wave equations in finite water depth, a statistical distribution of the wave-surface elevation is derived by using the characteristic function expansion method. It is found that the distribution, after normalization of the wave-surface elevation, depends only on two parameters. One parameter describes the small mean bias of the surface produced by the second-order wave-wave interactions. Another one is approximately proportional to the skewness of the distribution. Both of these two parameters can be determined by the water depth and the wave-number spectrum of ocean waves. As an illustrative example, we consider a fully developed wind-generated sea and the parameters are calculated for various wind speeds and water depths by using Donelan and Pierson spectrum. It is also found that, for deep water, the dimensionless distribution reduces to the third-order Gram–Charlier series obtained by Longuet-Higgins [J. Fluid Mech. 17 (1963) 459]. The newly proposed distribution is compared with the data of Bitner [Appl. Ocean Res. 2 (1980) 63], Gaussian distribution and the fourth-order Gram–Charlier series, and found our distribution gives a more reasonable fit to the data.

Keywords: Statistical distribution; Wave surface elevation; Second-order random waves; Water depth; Wave-number spectrum

1. Introduction

The non-Gaussian behaviour of sea surface elevation distribution has well been recognized in many field measurements in deep to shallow water, as described in many handbooks and textbooks in physical oceanography and coastal engineering (e.g., Herbich, 1990; Goda, 2000). The distribution of measured sea surface elevations can differ noticeably from the Gaussian distribution, especially when the measurements are taken in shallow water (Bitner, 1980; Thompson, 1980) or when the measurements represent steep waves in relatively deep water (Longuet-Higgins, 1963; Hudspeth and Chen, 1979; Huang and Long, 1980). Shallow water waves and steep deep water waves tend to have high narrow crests and broad flat troughs, that lead to non-Gaussian sea surface elevation distributions.

The nonlinear behaviour of sea surface elevation distribution has a potentially important application in

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coastal and ocean engineering. For example, better predictions of wave force on circular pilings have been obtained with statistics based on nonlinear waves with non-Gaussian distribution than with statistics based on linear waves with Gaussian distribution law (Hudspeth and Chen, 1979; Song et al., 2000). The effect of nonlinear wave shapes on wave crest statistics and its implications for design of offshore structures have been examined by Jahns and Wheeler (1973).

To describe the nonlinear characteristics, a great deal of work has been carried out to develop proper expression for this non-Gaussian statistical distribution of wave-surface elevation. However, most of the studies are limited to deep water case (Longuet-Higgins, 1963; Huang and Long, 1980; Sun and Ding, 1994; Srokosz, 1998) due to the complication of bottom effects, although most of the practical applications of the wave statistical properties are related to engineering activities which are invariantly located in water of finite depth. Bitner (1980) first extended the analysis of Longuet-Higgins (1963) for deep water to water of finite depth, but no attempt was made to parameterize the skewness or the kurtosis in terms of the spectrum in finite depth. A straightforward Gram–Charlier expansion has been used by Bitner (1980). The skewness and the kurtosis appeared in the distribution were determined from observations. Later, Huang et al. (1983) proposed a distribution by using third-order expansion of Stokes waves in water of finite depth. Unfortunately, their results are only applicable to the narrowband unidirectional case and cannot be generalized to the broadband mutidirectional wave field case, which is more commonly found in nature.

Although previous works have led to a better understanding on the statistical distribution of wave surface elevation, there does not exist an analytical model and a simple method for the determination of the parameters involved in the distribution in term of the spectrum of ocean waves for random directional sea in finite water depth. Further work is thus needed to develop a theoretical distribution with parameters, which can be easily determined. In this paper, we use the second-order random wave theory in finite water depth and the characteristic function expansion method to derive a new expression for the statistical distribution of the wave-surface elevation. It will be shown that the distribution of the dimensionless wave-surface elevation derived in this paper depends only on two parameters, which can be subsequently determined by the wave-number spectrum of ocean waves and the water depth.

2. Second-order random wave model in finite water depth and calculations of moments

Consider the three-dimensional flow of water in ocean with finite uniform depth. Assuming that the flow is inviscid, incompressible, irrotational and the pressure is constant at the free surface, the governing equations in the Cartesian coordinates are

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0 \quad -h\leq z\leq\eta(x,y,t), \tag{1}$$

$$\frac{\partial\varphi}{\partial z}=\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\eta}{\partial y}\frac{\partial\varphi}{\partial y} \quad z=\eta(x,y,t), \tag{2}$$

$$g\eta+\frac{\partial\varphi}{\partial t}+\frac{1}{2}(u^2+v^2+w^2)=0 \quad z=\eta(x,y,t), \tag{3}$$

$$\frac{\partial\varphi}{\partial z}=0 \quad z=-h, \tag{4}$$

where the horizontal coordinate axes are fixed on the still water level, the z-axis is along the vertical direction with positive direction upwards, φ and η are three-dimensional velocity potential and surface elevation, u, v and w are the velocity components, respectively, in the x, y and z directions, h is the water depth, and g is the gravitational acceleration. Using the perturbation technique, the second-order random wave solutions of Eqs. (1)–(4) can be derived. For directional finite-depth waves, the wave-surface elevation is given by (see Sharma and Dean, 1993, which is a re-introduction of the previous work of Sharma and Dean, 1979)

$$\eta=\eta(\mathbf{x},t)=\eta_1+\eta_2, \tag{5}$$

with

$$\eta_1 = \sum_{i=1}^{\infty} a_i \cos\psi_i, \tag{6}$$

$$\begin{aligned}\eta_2 = \frac{1}{4}\sum_{i,j=1}^{\infty} a_i a_j \Bigg\{ & \left[\frac{D_{ij}^{-} - (\mathbf{k}_i \cdot \mathbf{k}_j + R_i R_j)}{\sqrt{R_i R_j}} + (R_i + R_j)\right] \cos(\psi_i - \psi_j) \\ & + \left[\frac{D_{ij}^{+} - (\mathbf{k}_i \cdot \mathbf{k}_j - R_i R_j)}{\sqrt{R_i R_j}} + (R_i + R_j)\right] \times \cos(\psi_i + \psi_j) \Bigg\},\end{aligned} \tag{7}$$

where $\mathbf{x}$ is the position vector at the still water level, t denotes the time, η_1 and η_2 are, respectively, the first and second order solutions of the water wave equations, $R_i = k_i \tanh k_i h$, $k_{ij}^{\pm} = |\mathbf{k}_i \pm \mathbf{k}_j|$, $\psi_i = \mathbf{k}_i \cdot \mathbf{x} - \omega_i t + \varepsilon_i$, a_i denotes the amplitude of the ith wave component, ω_i is the angular frequency in rad/s, $\mathbf{k}_i$ and $\mathbf{k}_j$ are the horizontal wavenumber vectors with the moduli k_i and k_j, respectively, ε_i denotes the independent random phase uniformly distributed over the interval $(0,2\pi)$ and

$$D_{ij}^{-} = \begin{cases} \dfrac{\left(\sqrt{R_i}-\sqrt{R_j}\right)\left[\sqrt{R_j}\left(k_i^2-R_i^2\right)-\sqrt{R_i}\left(k_j^2-R_j^2\right)\right]}{\left(\sqrt{R_i}-\sqrt{R_j}\right)^2-k_{ij}^{-}\tanh k_{ij}^{-}h} + \dfrac{2\left(\sqrt{R_i}-\sqrt{R_j}\right)^2\left(\mathbf{k}_i\cdot\mathbf{k}_j+R_iR_j\right)}{\left(\sqrt{R_i}-\sqrt{R_j}\right)^2-k_{ij}^{-}\tanh k_{ij}^{-}h} & (i \neq j), \\ 0 & (i = j) \end{cases} \tag{8}$$

$$D_{ij}^{+} = \frac{(\sqrt{R_i}+\sqrt{R_j})\left[\sqrt{R_j}(k_i^2-R_i^2)+\sqrt{R_i}(k_j^2-R_j^2)\right]}{(\sqrt{R_i}+\sqrt{R_j})^2 - k_{ij}^{+}\tanh k_{ij}^{+}h} + \frac{2(\sqrt{R_i}+\sqrt{R_j})^2(\mathbf{k}_i\cdot\mathbf{k}_j - R_iR_j)}{(\sqrt{R_i}+\sqrt{R_j})^2 - k_{ij}^{+}\tanh k_{ij}^{+}h}. \tag{9}$$

As Forristall (2000) has pointed out, the positive interaction terms given by Eq. (9) occur at the sum of the frequencies of the interacting wave components. They produce the sharpening of the crests and flattening of the troughs that we associate with the second-order Stokes waves. The negative interaction terms given by Eq. (8) occur at the difference of the frequencies of the first-order wave components. These interactions give the setdown of the water level under wave groups. Thus, the above second-order expansion of the sea surface can capture the effects of wave steepness, water depth, and directional spreading with no approximations other than the truncation of the expansion at the second order.

From Eq. (7), we have

$$\begin{aligned}\eta_2 &= \frac{1}{4}\sum_{i,j=1}^{\infty} a_i a_j \Bigg\{ \left[\frac{D_{ij}^{-} + D_{ij}^{+} - 2\mathbf{k}_i\cdot\mathbf{k}_j}{\sqrt{R_iR_j}} + 2(R_i+R_j)\right]\cos\psi_i\cos\psi_j \\ &\qquad + \left[\frac{D_{ij}^{-} - D_{ij}^{+} - 2R_i\cdot R_j}{\sqrt{R_iR_j}}\right]\sin\psi_i\sin\psi_j \Bigg\} \\ &= \eta_{21} + \eta_{22},\end{aligned} \tag{10}$$

with

$$\eta_{21} = \sum_{i,j=1}^{\infty} a_i a_j \alpha_{ij} \cos\psi_i \cos\psi_j, \tag{11}$$

$$\eta_{22} = \sum_{i,j=1}^{\infty} a_i a_j \beta_{ij} \sin\psi_i \sin\psi_j, \tag{12}$$

$$\alpha_{ij} = \frac{D_{ij}^{-} + D_{ij}^{+} - 2\mathbf{k}_i\cdot\mathbf{k}_j}{4\sqrt{R_iR_j}} + \frac{1}{2}(R_i + R_j), \tag{13}$$

$$\beta_{ij} = \frac{D_{ij}^{-} - D_{ij}^{+} - 2R_i\cdot R_j}{4\sqrt{R_iR_j}}. \tag{14}$$

From Eqs. (13), (14), (8) and (9), we obtain

$$\alpha_{ii} + \beta_{ii} = R_i - \frac{k_i^2 + R_i^2}{2R_i} = \frac{1}{2R_i}(R_i^2 - k_i^2). \tag{15}$$

For deep-water waves ($h \to -\infty$), α_{ij} and β_{ij} become (Hu and Zhao, 1993)

$$\alpha_{ij} = \frac{1}{2}\left[\frac{B_{ij}^{-} + B_{ij}^{+} - \mathbf{k}_i \cdot \mathbf{k}_j}{\sqrt{k_i k_j}} + k_i + k_j\right], \tag{16}$$

$$\beta_{ij} = \frac{1}{2}\frac{B_{ij}^{-} - B_{ij}^{+} - k_i k_j}{\sqrt{k_i k_j}}, \tag{17}$$

where

$$B_{ij}^{-} = \frac{(\sqrt{k_i} - \sqrt{k_j})^2(\mathbf{k}_i \cdot \mathbf{k}_j + k_i k_j)}{(\sqrt{k_i} - \sqrt{k_j})^2 - |\mathbf{k}_i - \mathbf{k}_j|}, \tag{18}$$

$$B_{ij}^{+} = \frac{(\sqrt{k_i} + \sqrt{k_j})^2(\mathbf{k}_i \cdot \mathbf{k}_j - k_i k_j)}{(\sqrt{k_i} + \sqrt{k_j})^2 - |\mathbf{k}_i + \mathbf{k}_j|}. \tag{19}$$

Noting that tanh $k_i h \approx 1$, Eq. (15) becomes

$$\alpha_{ii} + \beta_{ii} = 0. \tag{20}$$

Suppose η_1 is of the order of δ, where δ represents a small quantity, then η_{21}, η_{22} and η_2 are of the order of δ^2. Following the analysis by Sun and Ding (1994), the nth power of $\eta = \eta_1 + \eta_2$ must be evaluated correctly at least to the order of δ^{n+1}, as the wave surface elevation is to be approximated accurately to the second order. Thus, we have

$$\eta^n = \eta_1^n + n\eta_1^{n-1}\eta_2 = \eta_1^n + n\eta_1^{n-1}(\eta_{21} + \eta_{22}). \tag{21}$$

The time average of the terms on the right hand side of Eq. (21) can be calculated directly as follows

$$\overline{\eta_1^{2N+1}} = 0, \quad \overline{\eta_1^{2N-1}\eta_2} = 0, \quad \overline{\eta_1^{2N}} = E^N(2N-1)!!, \tag{22}$$

$$\overline{\eta_1^{2N}\eta_{21}} = \left[A + A_{12}\left(\frac{2}{E}\right)N\right]E^N(2N-1)!!, \tag{23}$$

$$\overline{\eta_1^{2N}\eta_{22}} = BE^N(2N-1)!!, \tag{24}$$

where the over-bar denotes average with respect to time, N is a natural number, the double exclamation mark represents the double factorial operator and

$$E = \frac{1}{2}\sum_{i=1}^{\infty} a_i^2 = \int \Psi(\mathbf{k})\mathrm{d}\mathbf{k}, \tag{25}$$

$$A = \frac{1}{2}\sum_{i=1}^{\infty} \alpha_{ii}a_i^2 = \int \alpha(\mathbf{k},\mathbf{k})\Psi(\mathbf{k})\mathrm{d}\mathbf{k}, \tag{26}$$

$$B = \frac{1}{2}\sum_{i=1}^{\infty} \beta_{ii}a_i^2 = \int \beta(\mathbf{k},\mathbf{k})\Psi(\mathbf{k})\mathrm{d}\mathbf{k}, \tag{27}$$

$$A_{12} = \frac{1}{4}\sum_{i,j=1}^{\infty} \alpha_{ij}a_i^2 a_j^2 = \int\int \alpha(\mathbf{k},\mathbf{k}')\Psi(\mathbf{k})\Psi(\mathbf{k}')\mathrm{d}\mathbf{k}d\mathbf{k}', \tag{28}$$

in which $\Psi(\mathbf{k})$ is the wave-number spectrum of sea waves corresponding to the first order (linear) approximation. From Eqs. (21)–(24), we have

$$\overline{\eta^{2N+1}} = \overline{\eta_1^{2N+1}} + (2N+1)\overline{\eta_1^{2N}(\eta_{21} + \eta_{22})} = \frac{(2N+1)!}{N!}\left(\frac{E}{2}\right)^N \times \left[A + A_{12}\left(\frac{2}{E}\right)N + B\right], \tag{29}$$

$$\overline{\eta^{2N}} = \overline{\eta_1^{2N}} + 2N\overline{\eta_1^{2N-1}\eta_2} = \frac{(2N)!}{N!}\left(\frac{E}{2}\right)^N. \tag{30}$$

It is noted that the surface elevation given by Eq. (5) has a mean bias

$$\bar{\eta} = A + B, \tag{31}$$

which is produced by the second-order correction to the Gaussian sea. Although this bias is negligible for infinite water depth as shown below, it is a small negative number for shallow water and intermediate water depth as shown in the examples of Section 4. The negative mean bias describes the setdown of the mean water level caused by second-order wave–wave interactions. The variance σ^2 of η is

$$\sigma^2 = \overline{\eta^2} - \overline{\eta}^2 = E - (A+B)^2. \tag{32}$$

3. Statistical distribution of wave surface elevation

Let $f(\eta)$ denote the probability density of η, then the characteristic function of η is

$$\varphi(it) = \int_{-\infty}^{+\infty} f(\eta)\exp(i\eta t)\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\langle \eta^n \rangle, \tag{33}$$

where the angled brackets $\langle \cdot \rangle$ represents the expectation operator. Assuming that the process of sea waves is ergodic, one has

$$\langle \eta^n \rangle = \overline{\eta^n}. \tag{34}$$

Substituting Eqs. (34), (29) and (30) into Eq. (33), we obtain an explicit form of the characteristic function $\varphi(it)$, namely

$$\varphi(it) = \exp\left(-\frac{E}{2}t^2\right)\Big[1 + (A+B)(it) + A_{12}(it)^3\Big]. \tag{35}$$

From Eqs. (33) and (35), the probability density of η can be obtained through an inverse Fourier transformation, namely

$$\begin{aligned} f(\eta) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(it)\exp(-i\eta t)\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi E}}\left[1 - (A+B)\frac{\partial}{\partial \eta} - A_{12}\frac{\partial}{\partial \eta^3}\right] \times\exp\left(-\frac{1}{2E}\eta^2\right) \\ &= \frac{1}{\sqrt{2\pi E}}\left[1 + (A+B)\frac{\eta}{E} + A_{12}\left(\frac{\eta^3}{E^3} - \frac{3\eta}{E^2}\right)\right] \times\exp\left(-\frac{1}{2E}\eta^2\right). \end{aligned} \tag{36}$$

To compare with the Gaussian distribution and some of the existent non-Gaussian distributions, the above distribution $f(\eta)$ is normalized by introducing the dimensionless wave-surface elevation $\xi=(\eta - \bar{\eta})/\sigma$ as follows

$$\begin{aligned} P(\xi) = \sigma f\Big(\sigma\xi + A + B\Big) = \frac{1}{\sqrt{2\pi}}\sqrt{1-C^2} &\times \Big\{1 + C\Big(\sqrt{1-C^2}\xi + C\Big) \\ &+ D\Big[\Big(\sqrt{1-C^2}\xi + C\Big)^3 - 3\Big(\sqrt{1-C^2}\xi + C\Big)\Big]\Big\} \\ &\cdot \exp\Big\{-\frac{1}{2}\Big(\sqrt{1-C^2}\xi + C\Big)^2\Big\}, \end{aligned} \tag{37}$$

where

$$C = \frac{A+B}{\sqrt{E}}; \qquad D = \frac{A_{12}}{E^{3/2}}. \tag{38}$$

Generally, distribution (37) will not be Gaussian. These discrepancies are usually expressed in terms of the skewness $\lambda_3 = \overline{\xi^3}$ and the kurtosis $\lambda_4 = \overline{\xi^4} - 3$. The skewness is a statistical measure of the vertical asymmetry of the sea surface exemplified by the sharp crests and rounded troughs of gravity waves. The kurtosis represents a degree of peakness of the distribution when the normal distribution is taken as a reference. From Eqs. (29)–(32) and the relation $\overline{\xi^n} = \frac{1}{\sigma^n}\overline{(\eta - \bar{\eta})^n}$, we found that the normalized distribution $P(\xi)$ has the mean equal to 0, the standard deviation equal to 1, and the skewness λ_3 and kurtosis λ_4 of the following values

$$\begin{aligned} \lambda_3 &= (1-C^2)^{-3/2}(6D + 2C^3); \\ \lambda_4 &= -6C(1-C^2)^{-2}(4D + C^3). \end{aligned} \tag{39}$$

Thus, the distribution of the dimensionless wave-surface elevation under the second-order approximation has a nonzero skewness and nonzero kurtosis. Distribution (37) depends only on two parameters, C and D. Generally, the parameter C is a small negative number as shown in the examples of following section, which describes the setdown of the mean water level caused by second wave–wave interactions. The parameter D

is approximately proportional to the skewness of the distribution. The distribution becomes increasingly skewed as the parameter D increases. The influence of the parameter D on distribution (37) with $C=0$ is shown in Fig. 1.

From Eqs. (38), (25), (26), (27) and (28), we know that the parameters C and D can be determined by the wavenumber spectrum $\Psi(\mathbf{k})$ and the water depth h. For deep water ($h \to -\infty$), from Eqs. (38), (39), (20), (26) and (27), we have

$$C = 0; \qquad \lambda_4 = 0; \qquad D = \frac{1}{6}\lambda_3. \tag{40}$$

Thus, the probability density of the dimensionless wave-surface elevation in deep water reduces to

$$P(\xi) = \frac{1}{\sqrt{2\pi}}\left[1 + \frac{1}{6}\lambda_3(\xi^3 - 3\xi)\right]\exp\left(-\frac{1}{2}\xi^2\right), \tag{41}$$

which is exactly the Gram–Charlier series obtained by Longuet-Higgins (1963) and Huang and Long (1980) truncated at the third-order cumulants and depends only on the parameter λ_3 or D. These show that the skewness of the dimensionless distribution of wave-surface elevation in deep water is obviously affected by second-order wave–wave interactions and kurtosis depends the third or higher order wave–wave interactions.

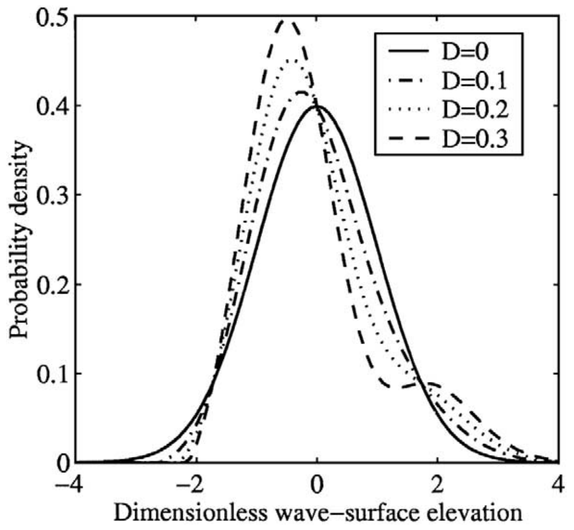

Fig. 1. The influence of the parameter D on distribution (37) with $C=0$.

4. Determination of the distribution parameters C and D

To determine the parameters C and D involved in the distribution, we need to know the two-dimensional wavenumber spectrum $\Psi(\mathbf{k})$ of ocean waves corresponding to the first order (linear) approximation. Cieslikiewicz and Gudmestad (1993) have shown that $\Psi(\mathbf{k})$ can be approximated by the observed spectrum. For a fully developed wind-generated sea, if we consider only waves with low wavenumber up to 10 times the spectral peak, we can use the low-wave number part of the wavenumber spectrum proposed by Donelan and Pierson (1987), thus

$$\begin{aligned}\Psi(\mathbf{k}) = \Psi(k,\vartheta) &= \frac{0.00162 \times U_{10}}{k^{3.5}g^{0.5}} \\ &\quad \times \exp\left(-\frac{g^2}{k^2(1.2U_{10})^4}\right)\cdot 1.7^{\Gamma}\cdot\mu\left(\frac{k}{k_{\mathrm{p}}}\right) \\ &\quad \cdot \mathrm{sech}^2\left[\mu\left(\frac{k}{k_{\mathrm{p}}}\right)\vartheta\right],\end{aligned} \tag{42}$$

where $\Psi(\mathbf{k})\mathrm{d}\boldsymbol{k} = \boldsymbol{k}\Psi(k,\vartheta)\mathrm{d}k\mathrm{d}\vartheta$ $(0<k<\infty, -\pi<\vartheta<\pi)$, U_{10} is the effective natural wind speed at 10 m above the still water level, ϑ is the wave direction relative to the wind (the direction of wind is assumed to be along the x-axis) and

$$\Gamma = \exp\left\{-1.22\left[\frac{1.2U_{10}k^{0.5}}{g^{0.5}} - 1\right]^2\right\}, \tag{43}$$

$$\mu\left(\frac{k}{k_{\mathrm{p}}}\right) = \begin{cases} 1.24 & 0 < k/k_{\mathrm{p}} < 0.31 \\ 2.61(k/k_{\mathrm{p}})^{0.65} & 0.31 < k/k_{\mathrm{p}} < 0.9. \\ 2.28(k_{\mathrm{p}}/k)^{0.65} & 0.9 < k/k_{\mathrm{p}} < 10 \end{cases} \tag{44}$$

The peak of the spectrum is given by

$$k_{\mathrm{p}} = \frac{g}{(1.2U_{10})^2}. \tag{45}$$

Using Eqs. (13), (15), (25), (26), (27), (28) and (38), we have

$$E = \int\int k\Psi(k,\vartheta)\mathrm{d}k\mathrm{d}\vartheta, \tag{46}$$

$$C = \frac{1}{2\sqrt{E}} \times \int\int\left[\tanh kh - \frac{1}{\tanh kh}\right]k^2\Psi(k,\vartheta)\mathrm{d}k\mathrm{d}\vartheta, \tag{47}$$

$$D = \frac{1}{E^{3/2}}\int\int\int\int F(k,k',\vartheta,\vartheta')kk' \times \Psi(k,\vartheta)\Psi(k',\vartheta')\mathrm{d}k\mathrm{d}k'\mathrm{d}\vartheta\mathrm{d}\vartheta', \tag{48}$$

where

$$F(k,k',\vartheta,\vartheta') = \frac{(\sqrt{R}-\sqrt{R'})D^- + (\sqrt{R}+\sqrt{R'})D^+ - 2\mathbf{k}\cdot\mathbf{k}'}{4\sqrt{RR'}} + \frac{1}{2}(R+R'), \tag{49}$$

in which

$$D^-(\mathbf{k},\mathbf{k}',h) = [\sqrt{R'}(k^2-R^2) - \sqrt{R}(k'^2-R'^2) + 2(\sqrt{R}-\sqrt{R'})(\mathbf{k}\cdot\mathbf{k}' + RR')] / [(\sqrt{R}-\sqrt{R'})^2 - |\mathbf{k}-\mathbf{k}'|\tanh|\mathbf{k}-\mathbf{k}'|h], \tag{50}$$

$$D^+(\mathbf{k},\mathbf{k}',h) = [\sqrt{R}(k'^2-R'^2) + \sqrt{R'}(k^2-R^2) + 2(\sqrt{R}+\sqrt{R'})(\mathbf{k}\cdot\mathbf{k}' - RR')] / [(\sqrt{R}+\sqrt{R'})^2 - |\mathbf{k}+\mathbf{k}'|\tanh|\mathbf{k}+\mathbf{k}'|h], \tag{51}$$

$$R = k\tanh kh, \qquad R' = k'\tanh k'h, \quad \mathbf{k} = (k\cos\vartheta, k\sin\vartheta), \quad \mathbf{k}' = (k'\cos\vartheta', k'\sin\vartheta'). \tag{52}$$

The integrals in Eqs. (46)–(48) can then be determined by using Eq. (42) and the Gaussian quadrature for multiple integrals. Considering the expansion (5) only valid for weakly nonlinear cases, the calculations are performed for various wind speeds $U_{10} \leq 10$ m/s and water depths $h \geq 5$ m. The numerical results shown that E (directly proportional to the total energy of ocean waves) increases with the wind speed U_{10}. The values of E are 0.033, 0.127 and 0.529 m^2 for wind speeds $U_{10}=5$, 7 and 10 m/s, respectively. For $U_{10}=10$ m/s, the corresponding significant wave height $H_s \approx 4\sqrt{E} \approx 2.9$ m, and the waves have already been subjected to the depth-limited breaking process when $h=5$ m. Thus, our calculations only limited to $h=5$ m. The calculated values of parameter C are small negative numbers. Their absolute values increase with the wind speed and decrease with the increase of the water depth and tend to be zero as the water depth h become very large. The smaller the wind speed U_{10}, the faster C tends to 0. The values of parameter D also increase with the wind speed and decrease with the increase of the water depth. The values of C and D for $U_{10}=$ 5, 7 and 10 m/s are given, respectively, in Tables 1 and 2.

It can be noted that the parameter D tends to be a constant which almost does not increase with the wind speed anymore for large water depth. One possible explanation to account for this unexpected trend is due to the wavenumber spectrum used in our calculations. But, the parameter D is generally not equal to zero even in deep water. This shows that the effect of nonlinearity on the distribution of wave-surface elevation cannot be neglected, especially when the water depth is small and the wind speed is high. To clearly illustrate the effect of nonlinearity and water depth on the distribution of the dimensionless wave-surface

Table 1
The values of C for various wind speeds U_{10} and water depths h

h (m)	$U_{10}=5$ m/s	$U_{10}=7$ m/s	$U_{10}=10$ m/s
5	− 0.004351	− 0.018972	− 0.057413
7	− 0.001622	− 0.009737	− 0.035474
10	− 0.000442	− 0.004124	− 0.019676
20	− 0.000013	− 0.000407	− 0.004351
50	0.0	− 0.000002	− 0.000166
100	0.0	0.0	− 0.000003

Table 2
The values of D for various wind speeds U_{10} and water depths h

h (m)	$U_{10}=5$ m/s	$U_{10}=7$ m/s	$U_{10}=10$ m/s
5	0.029113	0.058268	0.272760
7	0.026284	0.037089	0.123158
10	0.025383	0.028845	0.060298
20	0.025181	0.025361	0.029113
50	0.025180	0.025183	0.025229
100	0.025180	0.025183	0.025183

elevation, the distributions for $U_{10}=10$ m/s, $h=5$, 7, 10 and 50 m are graphed in Fig. 2.

From Tables 1 and 2, the corresponding values of skewness λ_3 and kurtosis λ_4 can be easily obtained by Eq. (39). As an example, we present the results of λ_3 and λ_4 in Table 3 for $U_{10}=10$ m/s. The values of λ_3 shown in Table 3 approximately agree with the results of the field measurements listed in Herbich's (1990) Handbook. However, the values of λ_4 in Table 3 are much smaller than that obtained by the measurements as listed in the Herbich's Handbook. For example, $\lambda_3=0.74$ and $\lambda_4=0.105$ for $U_{10}=10$ m/s (the corresponding significant wave height $H_s=2.9$ m) at $h=7.0$ m as shown in Table 3, and measured skewness and kurtosis are 0.81 and 0.63, respectively, as listed in Herbich's (1990) Handbook for the significant wave height $H_s=2.71$ m at $h=7.0$ m. The main reason for the small kurtosis values calculated by Eq. (39) are possibly caused by the order of approximation in the wave profile used in Eq. (5). A more

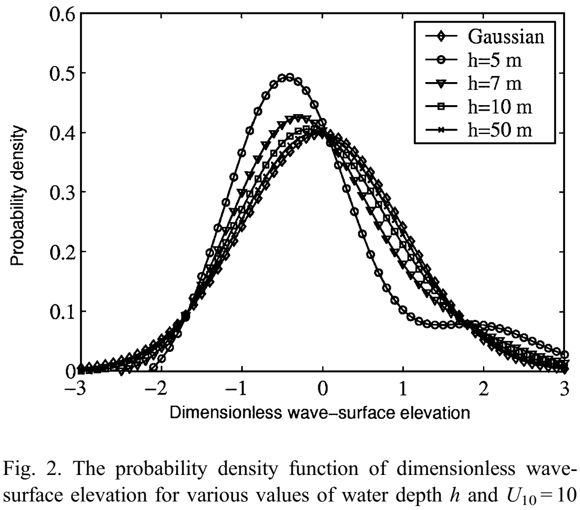

Fig. 2. The probability density function of dimensionless wave-surface elevation for various values of water depth h and $U_{10}=10$ m/s.

Table 3
The values of λ_3 and λ_4 for wind speed $U_{10}=10$ m/s and various water depths h

	$h=5$ m	$h=7$ m	$h=10$ m	$h=20$ m	$h=50$ m
λ_3	1.6443	0.7403	0.3620	0.17468	0.1514
λ_4	0.3783	0.1051	0.0285	0.0030	0.0001

reasonable kurtosis can be obtained by considering the higher-order wave–wave interactions.

5. Comparisons with data and previous estimates

Figs. 3 and 4 compared our distributions with the field observations reported by Bitner (1980) for two cases that have the most prominent deviation from Gaussian. In these two figures, we also plotted Gaussian distribution and normalized fourth-order Gram–Charlier approximations based on the parameters reported by Bitner (1980). Distribution (37) plotted in the figures have the form (41) with λ_3 given by Bitner (1980) since $C=0$ for these two cases. The

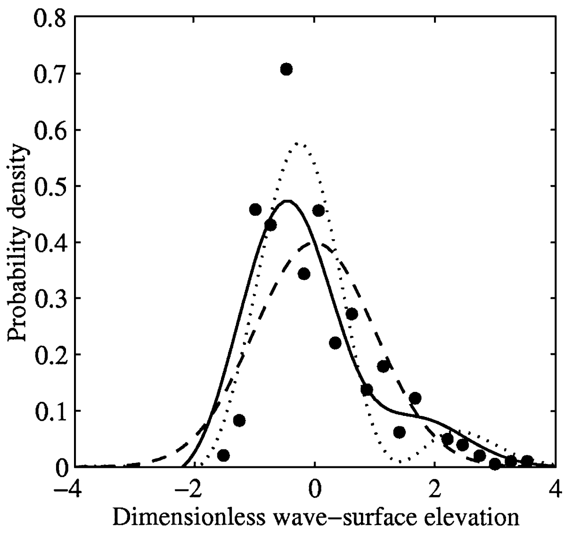

Fig. 3. Comparison of normalized distribution of surface elevation with observation by Bitner (1980) for shallow water experiment: sample 177, post 2, $h=2.2$ m, Oct. 15, 1974, 7:00 p.m. Circled points are the experimental data from Bitner (1980). Solid curve is distribution (37) with $C=0$, $D=\lambda_3/6$ and $\lambda_3=0.500$. Dotted curve is the fourth-order Gram–Charlier approximation (53) with $\lambda_3=0.500$ and $\lambda_4=6.443$. Dashed curve is Gaussian distribution.

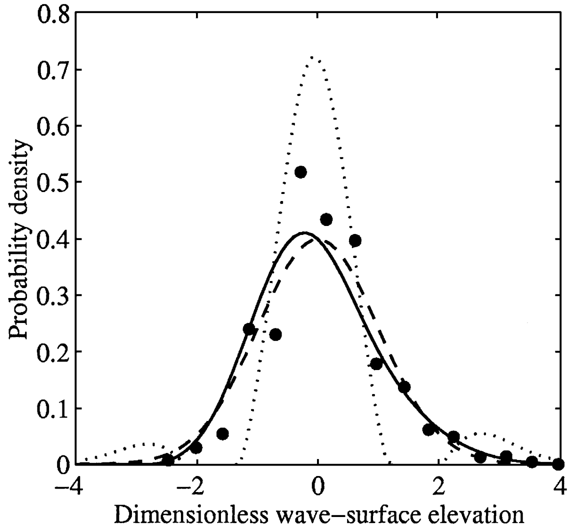

Fig. 4. As in Fig. 3, but for the experimental data of Bitner: sample 226, post 5, $h = 4.0$ m, Oct. 27, 1974, 7:00 p.m. with $\lambda_3 = 1.500$ and $\lambda_4 = 2.814$. Note that the negative values of the fourth-order Gram–Charlier series have been taken as 0.

normalized fourth-order Gram–Charlier approximations plotted in the figures are

$$P_{\mathrm{GC}}(\xi) = \frac{1}{\sqrt{2\pi}}\left[1 + \frac{1}{6}\lambda_3(\xi^3 - 3\xi) + \frac{1}{24}\lambda_4(\xi^4 - 6\xi^2 + 3)\right]\exp\left(-\frac{1}{2}\xi^2\right), \quad (53)$$

where $\xi = (\eta - \bar{\eta})/\sigma$, $\lambda_3 = \overline{\xi^3}$ and $\lambda_4 = \overline{\xi^4} - 3$. Skewness λ_3 and kurtosis λ_4, which appeared in Eq. (53), have the same value as reported by Bitner (1980). The values λ_3 and λ_4 are, respectively, 0.500 and 6.443 for the case of Fig. 3 (see Fig. 5 of Bitner, 1980; the corresponding significant wave height $H_s = 0.74$ m and $h = 2.2$ m) and $\lambda_3 = 1.500$, $\lambda_4 = 2.814$ for the case of Fig. 4 (see Fig. 7 of Bitner, 1980; the corresponding significant wave height $H_s \approx 1.19$ m and $h = 4.0$ m). Hence, the values of skewness do not necessarily increase as the wave train approaches a shore and it depends on the stage of wave transformation process.

Although the data collected for these two cases were very close to the plunging breaker line (see Bitner's paper) where the second-order expansion (Eq. (5)) may not be a good model, the comparison still appears quite favourable. The highest point in Fig. 3 seems quite anomalous. (This is probably caused by the use of too many classes. By combining the frequencies of two adjacent classes, a much smooth frequency distribution is obtained). This point contributes to setting of the fourth-order Gram–Charlier approximation by Bitner at artificially higher positions, while leaving many data points below the density curve. The curve predicted by Eq. (37), however, passes through the data cluster. As the depth increases, the present theoretical result shows definitely better agreement in Fig. 4. Huang and Long (1980) also compared their proposed distributions with the data of Bitner (1980) for the above two cases. Their results of the comparisons are much similar to ours although our distributions are different from that proposed by Huang and Long (1980). It should also be emphasised that the fourth-order Gram–Charlier approximation (Eq. (53)) will become not real on the two ends as for the above two cases that have a large λ_4 (see Figs. 3 and 4). From Figs. 3 and 4, it is noted that our distribution (37) gives a more reasonable fit to the data than the results by using Gaussian distribution or the fourth-order Gram–Charlier approximation.

Comparisons made by Srokosz (1998) between the third-order Gram–Charlier series and the wave data measured on a reservoir show a reasonable fit of the third-order Gram–Charlier approximation for deep water waves. This also affirms our results since distribution (37) reduces to the third-order Gram–Charlier series for very large water depth h as stated in the Section 3.

6. Conclusions

The theoretical nonlinear distribution of the wave-surface elevation has been derived based on the second-order random wave model in finite water depth. The dimensionless form of the distribution depends on two parameters C and D as stated in the text. Both the parameter C and the parameter D can be determined by the wavenumber spectrum and the water depth. As an illustrative example, we consider a fully developed wind-generated sea and the two parameters are calculated by using Donelan and Pierson spectrum for different wind speeds and different water depths. From the results obtained, it is noted that:

(1) The values of parameter C are small negative numbers. Their absolute values increase with the wind

speed and decrease with the increase of the water depth and tend to be zero as the water depth h becomes very large. The smaller the wind speed U_{10}, the faster C tends to 0. The values of parameter D also increase with the wind speed and decrease with the increase of the water depth. However, the parameter D is generally not equal to zero even in deep water. So, the effect of nonlinearity on the distribution of the wave-surface elevation must be considered, especially when the water depth is small and the wind speed is high.

(2) For deep water, the distribution of the dimensionless wave-surface elevation reduces to the third-order Gram–Charlier series obtained by Longuet-Higgins (1963) and it depends only on the parameter D or the skewness λ_3.

We also compared our results with the data of Bitner (1980), the Gaussian distribution and the fourth-order Gram–Charlier series, and found that our distribution gives a more reasonable fit to the data.

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References

Bitner, E.M., 1980. Nonlinear effects of the statistical model of shallow-water wind waves. Appl. Ocean Res. 2, 63–73.

Cieslikiewicz, W., Gudmestad, O.T., 1993. Stochastic characteristics of orbital velocities of random water waves. J. Fluid Mech. 255, 275–299.

Donelan, M.A., Pierson, W.J.P., 1987. Radar scattering and equilibrium ranges in wind-generated waves with application to scatterometry. J. Geophys. Res. 92, 4971–5029.

Forristall, G.Z., 2000. Wave crest distribution: observation and second-order theory. J. Phys. Oceanogr. 30, 1931–1943.

Goda, Y., 2000. Random sea and design of maritime structures, 2nd edn. Advanced series on ocean engineering, vol. 15. World Scientific, Singapore.

Herbich, J.B. (Ed.), 1990. Handbook of coastal and ocean engineering. Wave phenomena and coastal structure. Chapter 4: Random waves and spectra (by Y. Goda), vol. 1. Gulf Publishing Company, Houston, TX, pp. 201–204.

Hu, S.-L.J., Zhao, D., 1993. Non-Gaussian properties of second-order random waves. J. Eng. Mech. ASCE 119, 344–364.

Huang, N.E., Long, S.R., 1980. An experimental study of the surface elevation probability distribution and statistics of wind generated wave. J. Fluid Mech. 101, 179–200.

Huang, N.E., Tung, C.C., Yuan, Y., Bliven, L.F., 1983. A non-Gaussian statistical model for surface elevation of nonlinear random wave fields. J. Geophys. Res. 88, 7597–7606.

Hudspeth, R.T., Chen, M.C., 1979. Digital simulation of nonlinear random waves. J. Waterw., Port, Coastal, Oceanic Div., ASCE 105 (WW1), 67–85.

Jahns, H., Wheeler, J., 1973. Long-term wave probabilities based on hindcasting of severe storms. J. Petrol. Technol., 473–486.

Longuet-Higgins, M.S., 1963. The effect of non-linearities on statistical distributions in the theory of sea waves. J. Fluid Mech. 17, 459–480.

Sharma, J.N., Dean, R.G., 1979. Development and evaluation of a procedure for simulating a random directional second order sea surface and associated wave forces. Ocean Eng. Rep. 20, University of Delaware.

Sharma, J.N., Dean, R.G., 1993. Second order directional wave kinematics in shallow water. Proceeding of the Second International Symposium, waves'93, pp. 165–179.

Song, J.B., Wu, Y.H., Wiwatanapataphee, B., 2000. Probability distribution of random wave forces in weakly nonlinear random waves. Ocean Eng. 27 (12), 1391–1405.

Srokosz, M.A., 1998. A new statistical distribution for the surface elevation of weakly nonlinear water waves. J. Phys. Oceanogr. 28, 149–155.

Sun, F., Ding, P.X., 1994. A statistical distribution of surface elevation for nonlinear random sea waves and its physical explanation. Science (China) (series B) 37, 1401–1408.

Thompson, E.F., 1980. Shallow water surface wave elevation distributions. J. Waterw., Port, Coastal, Oceanic Div., ASCE 106 (WW2), 285–291.