Inverse Modeling of the Action-Balance Equation. Part I: Source Expansion and Adjoint-Model Equations

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ABSTRACT

In this paper a series of numerical experiments is defined to explore the inverse modeling of the action-balance equation governing the evolution of the surface gravity wave field, using the adjoint data-assimilation model-optimization procedure of Thacker and Long. We begin by exploiting power series, functional power series, and a variety of physical and mathematical considerations to derive a systematic expansion of the source terms in this equation for the deep-water case. This expansion, which naturally incorporates a Thacker representation for the nonlinear transfer from wave–wave interactions, defines a set of dimensionless expansion coefficients to be determined by the inverse modeling and identifies the simplified cases to be investigated in the numerical experiments.

Dimensional analysis determines a natural scaling for each term in this expansion and suggests a general form for the whitecap dissipation term, which includes as a special case the form proposed by Hasselmann, determining the first-order contribution to his unknown spectrum-dependent coefficient to within a multiplicative spectrum-independent constant.

A general discussion of the evolution of the simplified cases reveals a striking tendency to concentrate action in a single band when whitecap dissipation has the Hasselmann form and nonlinear transfer is ignored.

A derivation of the adjoint-model equations is included for one of the simplified cases and a general discussion of the model-optimization procedure is given. In these equations, nonlinear transfer is mirrored by a term of similar form, with Thacker's nonlinear transfer coefficients replaced by a related set of adjoint coefficients and the triple product of spectral intensities replaced by a product of two spectral intensities and a Lagrange multiplier.

1. Introduction

The study described in this series of papers is part of an ongoing long-range program to parameterize the evolutionary dynamics of the surface gravity wave field by comparing synoptic wave observations in the Bight of Abaco with the predictions of a fully nonlinear two-dimensional wave model (Snyder et al. 1990, hereafter SNLdV). We begin by summarizing this program and discussing its relation to the present study.

Let $A(\mathbf{k}, \mathbf{x}, t)$ be the (local) action spectral density for vector wavenumber $\mathbf{k}$ at position $\mathbf{x}$ and time t. Then the evolution of A (the spectral evolution of the wave field) is governed by the action-balance equation (Hasselmann 1968; Hasselmann et al. 1973), which we summarize in the form

$$\frac{\partial A}{\partial t} = -P + S, \tag{1.1}$$

where P denotes propagation terms and

$$S = I + N - D - B + \cdots \tag{1.2}$$

denotes source terms. Here I, N, D, and B determine the time rate of change of action spectral density from atmospheric input, nonlinear wave–wave interactions, whitecapping, and bottom friction, respectively. If we know how these source terms depend on A and on other relevant observables and are given an initial state $A(\mathbf{k}, \mathbf{x}, 0)$, an appropriate set of boundary conditions, and a complete record of the observables for $t > 0$, we can, in principle, integrate (1.1) to predict $A(\mathbf{k}, \mathbf{x}, t)$ for all $\mathbf{k}$, $\mathbf{x}$ (in some domain), and $t > 0$.

One source term, the atmospheric input I, has been partially parameterized from simultaneous array observations of near-surface air pressure and surface ele-

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vation (Snyder et al. 1981). Because it is difficult to identify intermediate observables on which to base an estimate of the corresponding action transfer, however, direct experimental parameterization of D and B is problematic. Instead, one must base this estimate on observed changes in the action spectrum itself. Such an estimate is difficult because 1) small differences between large random variables are inherently difficult to measure reliably and 2) all source terms are operating simultaneously, so that the observed changes cannot be attributed to a single source term.

One way to disentangle the combined effects of the various source terms, smooth the statistical errors, and fix the parameterization of I, D, and B is to invert the problem by synoptically monitoring both the wave field and other relevant observables in an enclosed basin of variable depth, discovering appropriate parameterizations for these terms by comparing the resulting observations with the predictions of a computer model of the action-balance equation (1.1). It is just such a comparison that is the object of the ongoing SNLdV program.

This comparison involves:

1) Conduct of a pair of high-(observational-)density high-(directional-)resolution synoptic field experiments in a semienclosed section of the Little Bahama Bank (the Bight of Abaco) to generate an appropriate dataset for the comparison.

2) Development of an efficient and fully nonlinear computer model of the action-balance equation (1.1), using a technique for calculating the nonlinear transfer N first suggested by Thacker (1982).

3) Expansion of the source function S in terms of A and other relevant observables, including wind velocity $\mathbf{W}$, wind stress $\boldsymbol{\tau}$, and depth H.

4) Development of an efficient procedure for optimizing the coefficients of this expansion, minimizing the variance between observation and model prediction.

To effect an efficient optimization as in 4), SNLdV will use the adjoint procedure of Thacker and Long (1988). As a first step toward adapting Thacker and Long's procedure to optimize the wave-model expansion coefficients with respect to the real comparison dataset provided by the Bight of Abaco field experiments, we are carrying out and we describe in this series of papers a numerical experiment to investigate this procedure for a sequence of simplified wave models, using the models themselves to simulate comparison datasets. Overall objectives are to prove the optimization procedure for these cases (and, by extension, for the full model) and to uncover any problems that might hinder the subsequent optimization of the full model.

The base model for this numerical study is an explicit deep-water version of the third-generation wave model developed by SNLdV, incorporating a Thacker (1982) representation for the nonlinear transfer from wave–wave interactions. In each case, we fix N from this representation, assume some truncated expansion for the partial source function $I - D$, assign nominal values to the relevant expansion coefficients, and, for various meteorological forcing, simulate from the model a representative synoptic dataset. We then apply the adjoint inverse-modeling procedure of Thacker and Long and a conjugate-gradient search procedure to recover the assigned values for the expansion coefficients. The stability of this recovery is studied as a function of noise level, observation set, complexity of meteorological forcing, and number and type of expansion coefficients.

In this paper, we provide a framework for the numerical experiments by developing a systematic expansion of the source function S, defining a set of dimensionless expansion coefficients and identifying a sequence of simplified cases for these experiments. In section 2, we discuss the general considerations that define our expansion of action-balance source terms. In section 3, we use this expansion to transform equation (1.1) into a set of prognostic equations for the action spectral density in the various spectral bands, and identify four simplified cases for the numerical experiments. In all four cases, the atmospheric input I and whitecap dissipation D are assumed linear and quadratic, respectively, in A. In case 1, each spectral band evolves independently from the other bands. In cases 2, 3, and 4, the evolution of the various spectral bands is coupled through the quadratic dependence of D on A. In case 4, additional third-order coupling is provided by the inclusion of the nonlinear transfer from wave–wave interactions. In section 4, we examine the behavior of the prognostic equations in these simplified cases, encountering an evolutionary singularity when whitecap dissipation has Hasselmann form and nonlinear transfer is ignored (case 2). We further discuss this singularity in section 5. In section 6, we derive the adjoint-model equations for case 4 [with second-order Hasselmann (1974) whitecap dissipation and nonlinear transfer] and discuss the inverse-modeling procedure. We draw final conclusions in section 7.

Subsequent papers in this series will describe the numerical experiments, in which we attempt to recover assigned values for the expansion coefficients by fitting comparison datasets simulated from the model equations.

2. Source expansion

In this section we develop a general expansion of the source function S for the action-balance equation (1.1). We base this expansion on a number of (somewhat loose but, we believe, essentially correct) general physical and mathematical considerations, referring only somewhat tangentially to existing theoretical predictions for the individual source terms. We assume, however, that 1) the general form of (1.1) is correct, 2) Eq. (1.2) correctly identifies the physical nature of

the important interactions, and 3) the propagation terms P are known.

To simplify the discussion, we assume a fully enclosed deep-water basin (for which $B \equiv 0$) with fully absorbing boundaries and no current, initially at rest. We further assume that the atmospheric input I is determined by a single vector wind parameter, the anemometer wind velocity $\mathbf{W}$. Note that 1) this latter assumption does not fundamentally alter the structure of the expansion (only the number of parameters) and 2) instead of choosing $\mathbf{W}$ as primary wind parameter, we could as readily have chosen the wind stress $\boldsymbol{\tau}$ (or friction velocity $\mathbf{W}_*$).

We also assume that all interactions are local in $\mathbf{x}$ and t, that is, that the change in $A(\mathbf{k}, \mathbf{x}, t)$ at $\mathbf{x}$ and t depends only on the source terms evaluated at $\mathbf{x}$ and t. (Note that this assumption is implicit in a classical two-scale derivation of the action-balance equation.) We further assume that the dependence on $\mathbf{x}$ and t is implicit in the dependence of the source terms on $A(\mathbf{k}, \mathbf{x}, t)$ and on the forcing variable $\mathbf{W}(\mathbf{x}, t)$. It follows that we may write (1.1) in the form

$$\frac{D}{Dt} A(\mathbf{k}, \mathbf{x}, t) = S(\mathbf{k}, \mathbf{W}(\mathbf{x}, t); A(\mathbf{k}, \mathbf{x}, t)), \quad (2.1)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V}(\mathbf{k}) \cdot \nabla_{\mathbf{x}}.$$

$\mathbf{V} \equiv \nabla_{\mathbf{k}} \omega(k)$ is the group velocity. The frequency $\omega(k)$ is given by

$$\omega(k) \equiv (gk)^{1/2}, \quad (2.2)$$

where g is the acceleration of gravity.

The dependence of the interaction source function S on A is nonlocal in $\mathbf{k}$. To make this dependence explicit, we expand S in functional power series, obtaining to third order

$$\begin{aligned} S = {} & \alpha(\mathbf{k}, \mathbf{W}(\mathbf{x}, t)) \\ & + \int d^2k_1 \beta(\mathbf{k}, \mathbf{k}_1, \mathbf{W}(\mathbf{x}, t)) A(\mathbf{k}_1, \mathbf{x}, t) \\ & + \iint d^2k_1 d^2k_2 \gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{W}(\mathbf{x}, t)) \\ & \times A(\mathbf{k}_1, \mathbf{x}, t) A(\mathbf{k}_2, \mathbf{x}, t) \\ & + \iiint d^2k_1 d^2k_2 d^2k_3 \delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{W}(\mathbf{x}, t)) \\ & \times A(\mathbf{k}_1, \mathbf{x}, t) A(\mathbf{k}_2, \mathbf{x}, t) A(\mathbf{k}_3, \mathbf{x}, t) + \cdots, \end{aligned} \quad (2.3)$$

where the coefficients (kernels) α, β, γ, and δ are determined by the interaction physics. Consistent with Hasselmann (1974), we argue that, although some of this physics, in particular the physics of the whitecapping interaction, is locally strong and therefore cannot be expanded at the dynamical level, because this physics is weak in the mean, such a functional-power-series expansion is meaningful at the spectral level.

In principle, each term in this expansion may incorporate contributions from each of the physical interactions I, N, and D. In fact, each term in the expansion can be identified primarily with a single physical interaction, the fourth possibly with two interactions. The reason that we can make such an identification is that each physical interaction is characterized by a leading order below which it does not contribute and above which its contribution, resulting from higher-order nonlinearities within the interaction, is diminished. The atmospheric interaction I, involving turbulent and wave-induced air pressure fluctuations (Phillips 1957; Miles 1957), is intrinsically linear and thus contributes primarily to the zero order and first order terms. The whitecap dissipation interaction D, on the other hand, is intrinsically nonlinear; its principal contribution presumably begins at second order. It may also contribute significantly to third order (and above). We will allow for third-order dissipation in this discussion, but will not investigate this type of dissipation in the numerical computations. Similarly, because of energy and momentum conservation, the contribution of the nonlinear wave–wave interaction N begins at third order (Hasselmann 1962). Thus, we identify the first two terms in (2.3) primarily with I, the third term primarily with D, and the fourth term primarily with N (and D).

The corresponding contributions to the coefficients α, β, γ, and δ are restricted by the physics of these interactions and by the fact that A must be everywhere nonnegative. Because A could otherwise develop negative values, α must be everywhere nonnegative. In the Phillips (1957) theory, this property is guaranteed by the fact that α is proportional to the turbulent air pressure spectrum. Because D is a dissipative interaction and should therefore give nonpositive transfer for arbitrary $A \geq 0$, the magnitude of which should be a monotonic increasing function of all wave amplitudes, γ and the contribution of D to δ must be everywhere nonpositive. [It is possible that this interaction could redistribute action as well as dissipate it; in this case we would need to include additional terms for which this assumption is not satisfied. We here consider only purely dissipative whitecapping interactions, such as the Hasselmann (1974) interaction.] We may also reasonably assume that, whatever its precise nature, the whitecapping interaction D is independent of the airflow, and, therefore, γ and the contribution of D to δ are independent of $\mathbf{W}$. Similarly, we may assume that the contribution of N to δ is independent of $\mathbf{W}$. (These latter two assumptions are in fact equivalent to the truncation at zero order of subsequent expansions in powers of the air–water density ratio s. To simplify the notation, we present our results without a formal development of these expansions.) Following Miles

(1957), we further assume that the first-order contribution of the atmospheric interaction I to β is local in $\mathbf{k}$. It follows that β contains a delta function

$$\beta(\mathbf{k}, \mathbf{k}_1, \mathbf{W}) = \beta(\mathbf{k}, \mathbf{W})\delta(\mathbf{k}_1 - \mathbf{k}).$$

(Note, however, that this assumption precludes interaction with the turbulence of the airflow.) Because A could otherwise develop negative values (see the Appendix), γ also contains a delta function

$$\gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{W}) = -\gamma(\mathbf{k}, \mathbf{k}_1)\delta(\mathbf{k}_2 - \mathbf{k}),$$

$$\text{with} \quad \gamma(\mathbf{k}, \mathbf{k}_1) \geq 0.$$

Similarly, the contribution of D to δ also contains a delta function. These conclusions support Hasselmann's (1974) contention that the whitecap dissipation is quasi-linear in A. Hasselmann argues that any interaction that is weak in the mean must be quasi-linear. We argue simply that the dissipation must contain a factor of A in order to guarantee that A remain nonnegative for an arbitrary initial state.

Also, because of energy and momentum conservation, the contribution of N to δ contains the delta function product $\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k})\delta(\omega_1 + \omega_2 - \omega_3 - \omega)$ (Hasselmann 1962). Thus, δ is of the form

$$\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{W}) = -\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\delta(\mathbf{k} - \mathbf{k}_3)$$
$$+ \delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k})$$
$$\delta(\omega_1 + \omega_2 - \omega_3 - \omega), \quad \text{with} \quad \delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \geq 0,$$

the first term coming from D and the second from N. (To avoid a profusion of variable names, we use the same names to label entire sets of related functions α, β, γ, and δ. In accordance with the common convention, we also use the variable name δ to label the spike or delta function. The reader is asked to distinguish the various functions α, β, γ, and δ by the number and type of arguments.)

Rotational invariance implies that

$$\alpha(\mathbf{k}, \mathbf{W}) = \alpha(k, W, \varphi),$$
$$\beta(\mathbf{k}, \mathbf{W}) = \beta(k, W, \varphi),$$
$$\gamma(\mathbf{k}, \mathbf{k}_1) = \gamma(k, k_1, \varphi_1) \geq 0,$$
$$\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = \delta(k, k_1, k_2, \varphi_1, \varphi_2) \geq 0,$$

and

$$\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \delta(k, k_1, k_2, k_3, \varphi_1, \varphi_2, \varphi_3), \quad (2.4)$$

where

$$\varphi \equiv \vartheta_W - \vartheta, \quad \varphi_1 \equiv \vartheta_1 - \vartheta,$$
$$\varphi_2 \equiv \vartheta_2 - \vartheta, \quad \text{and} \quad \varphi_3 \equiv \vartheta_3 - \vartheta,$$

with

$$\vartheta_W \equiv \arg(\mathbf{W}), \quad \vartheta \equiv \arg(\mathbf{k}), \quad \vartheta_1 \equiv \arg(\mathbf{k}_1),$$
$$\vartheta_2 \equiv \arg(\mathbf{k}_2), \quad \text{and} \quad \vartheta_3 \equiv \arg(\mathbf{k}_3).$$

Rotational symmetry implies that

$$\alpha(k, W, -\varphi) = \alpha(k, W, \varphi),$$
$$\beta(k, W, -\varphi) = \beta(k, W, \varphi),$$
$$\gamma(k, k_1, -\varphi_1) = \gamma(k, k_1, \varphi_1),$$
$$\delta(k, k_1, k_2, -\varphi_1, -\varphi_2) = \delta(k, k_1, k_2, \varphi_1, \varphi_2),$$
$$\delta(k, k_1, k_2, k_3, -\varphi_1, -\varphi_2, -\varphi_3)$$
$$= \delta(k, k_1, k_2, k_3, \varphi_1, \varphi_2, \varphi_3). \quad (2.5)$$

Finally, dimensional analysis implies that

$$\alpha(k, W, \varphi) = k^{-4}s\alpha(\sigma, \chi),$$
$$\beta(k, W, \varphi) = \omega s\beta(\sigma, \chi),$$
$$\gamma(k, k_1, \varphi_1) = k^2\omega^2\gamma(\kappa_1, \chi_1),$$
$$\delta(k, k_1, k_2, \varphi_1, \varphi_2) = k^4\omega^3\delta(\kappa_1, \kappa_2, \varphi_1, \varphi_2),$$

and

$$\delta(k, k_1, k_2, k_3, \varphi_1, \varphi_2, \varphi_3)$$
$$= k^2\omega^3\delta(\kappa_1, \kappa_2, \kappa_3, \varphi_1, \varphi_2, \varphi_3), \quad (2.6)$$

where

$$\sigma \equiv \frac{kW}{\omega}, \quad \chi \equiv \frac{\mathbf{k}\cdot\mathbf{W}}{kW} = \cos\varphi,$$
$$\chi_1 \equiv \frac{\mathbf{k}\cdot\mathbf{k}_1}{kk_1} = \cos\varphi_1,$$

and

$$\kappa_1 \equiv \frac{k_1}{k}, \quad \kappa_2 \equiv \frac{k_2}{k}, \quad \text{and} \quad \kappa_3 \equiv \frac{k_3}{k}, \quad (2.7)$$

with

$$\delta(\kappa_1, \kappa_2, -\varphi_1, -\varphi_2) = \delta(\kappa_1, \kappa_2, \varphi_1, \varphi_2)$$

and

$$\delta(\kappa_1, \kappa_2, \kappa_3, -\varphi_1, -\varphi_2, -\varphi_3)$$
$$= \delta(\kappa_1, \kappa_2, \kappa_3, \varphi_1, \varphi_2, \varphi_3).$$

Note that we may use the dispersion relation (2.2) to represent each of the equations (2.6) in several equivalent ways, supressing either a dependence on k, ω, or g. We have chosen the latter representation, which also avoids fractional powers. Note also that the relations (2.6) are derived for $\mathbf{k}$ space. The equivalent description for (ω, ϑ) space involves a Jacobian, which goes as ωk^{-2}.

Strictly speaking, the dimensionless variable s should not appear as a factor in the first two of the equations (2.6), but should appear as an argument of the dimensionless functions $\alpha(\sigma, \chi)$, $\beta(\sigma, \chi)$, $\gamma(\kappa_1, \chi_1)$, $\delta(\kappa_1, \kappa_2, \varphi_1, \varphi_2)$, and $\delta(\kappa_1, \kappa_2, \kappa_3, \varphi_1, \varphi_2, \varphi_3)$. To simplify

the notation, we have in effect skipped a step and incorporated into (2.6) a power-series expansion of these functions in s, in each case retaining only the leading term. In the first two equations, this leading term is order 1 in s (because without an atmosphere, there is no atmospheric interaction); in the last three equations, it is order zero. Because s is very small, the neglect of higher-order terms in these expansions is of little consequence. Note that, as previously remarked, this truncation provides a justification for the assumption that γ and δ are independent of $\mathbf{W}$ (because without an atmosphere, the wind speed cannot influence either the whitecapping interaction or wave–wave interactions).

We next introduce a set of dimensionless model parameters (expansion coefficients) Q_m by further expanding the functions $\alpha(\sigma, \chi)$, $\beta(\sigma, \chi)$, and $\gamma(\kappa_1, \chi_1)$ with respect to their dimensionless arguments. We take as standard the truncated power-series expansions

$$\alpha(\sigma, \chi) = \sum_{m}^{M_\alpha} Q_m \sigma^{v_m+1} \chi^{\nu_m},$$

$$\beta(\sigma, \chi) = \sum_{m}^{M_\beta} Q_{(M_\alpha+m)} \sigma^{v_m+1} \chi^{\nu_m},$$

and

$$\gamma(\kappa_1, \chi_1) = \sum_{m}^{M_\gamma} Q_{(M_\alpha+M_\beta+m)} (\ln\kappa_1)^{v_m} \chi_1^{\nu_m}, \quad (2.8)$$

where the exponents v_m and ν_m are given by

$$v_m = v_{(M_\alpha+m)} = v_{(M_\alpha+M_\beta+m)}$$
$$= 0, 1, 0, 2, 1, 0, 3, \cdots,$$

and

$$\nu_m = \nu_{(M_\alpha+m)} = \nu_{(M_\alpha+M_\beta+m)}$$
$$= 0, 0, 1, 0, 1, 2, 0, \cdots.$$

The expansions for α and β are both assumed to contain a factor of σ, so that, for null wind, the corresponding atmospheric interactions vanish.

Possible expansions are not limited to power series. There are many alternatives, some capable of representing a full range of dependence in the limit $M_\alpha, M_\beta, M_\gamma \to \infty$, some not. Because we can obtain physically reasonable representations using fewer terms, it is from this latter group of alternatives that we select the representations for the simplified models considered in the present study. We emphasize that these choices are illustrative, not definitive. In particular, they represent only a first step toward defining the model expansion for the parent SNLdV program.

To help ensure a reasonable behavior for the simplified wave models, we represent α in the form

$$\alpha(\sigma, \chi) = \eta \sum_{m=1}^{M_\alpha} Q_m \sigma^m, \quad (2.9)$$

where

$$\eta \equiv \eta(\chi), \quad \text{for} \quad \chi > 0,$$
$$\text{and} \quad \eta \equiv 0, \quad \text{for} \quad \chi \leq 0. \quad (2.10)$$

This choice allows a turbulent input only if there is wind and then only in downwind directions, and it provides a standard directional taper (for example, $\eta(\chi) \equiv \chi^l$, with l some positive integer).

Similarly, to reflect the approximate structure of Miles (1957) theory (in the context of a single wind-parameter model) and to allow for a convenient generalization of the results of Snyder et al. (1981), we represent β in the form

$$\beta(\sigma, \chi) = \sum_{m=1}^{M_\beta} Q_{(M_\alpha+m)} \mu^m, \quad (2.11)$$

where

$$\mu \equiv \frac{\mathbf{k}\cdot\mathbf{W}}{\omega} - 1 = \sigma\chi - 1, \quad \text{for} \quad \sigma\chi > 1,$$
$$\text{and} \quad \mu \equiv 0, \quad \text{for} \quad \sigma\chi \leq 1. \quad (2.12)$$

This choice allows a wave-coherent input only for downwind directions and only for wave components with projected phase velocities less than the wind speed.

The above representations for α and β imply broad regions in the space of the variables σ and χ over which these coefficients vanish identically. Nonetheless, these representations allow for considerable variation in the physically important regions of this space and are clearly more efficient than the straight power-series alternative. In so far as this study is concerned, it does not matter precisely which representation we employ. What is important is that the simplified models reflect various fundamental aspects of the full model and properly exercise relevant aspects of the optimization procedure.

What remains is to specify a representation for γ (and δ). We set these parameters in the next section on a case-by-case basis.

3. Derivation of the prognostic equations

We discretize (2.1) with respect to the vector wavenumber $\mathbf{k}$ by introducing a piecewise-constant representation for the action spectral density A

$$A(\mathbf{k}, \mathbf{x}, t) = \sum_i A_i(\mathbf{x}, t) G_i(\mathbf{k}),$$

where the basis function G_i is unity inside the ith spectral band and vanishes outside this band. Multiplying (2.1) by G_i, dividing by the area of the ith band R_i, substituting from (2.3), and integrating over $\mathbf{k}$, we obtain a coupled set of simultaneous nonlinear first-order differential equations for the prognostic variables A_i

$$\frac{D_i}{Dt}A_i = \alpha_i + \beta_i A_i - (\sum_j \gamma_{ij}A_j)A_i - (\sum_{jk} \delta_{ijk}A_jA_k)A_i + \sum_{jkl} \delta_{ijkl}A_jA_kA_l + \cdots, \quad (3.1)$$

where

$$\frac{D_i}{Dt} \equiv \frac{\partial}{\partial t} + \frac{1}{R_i}\left(\int d^2k G_i(\mathbf{k})\mathbf{V}(\mathbf{k})\right)\cdot\nabla_{\mathbf{x}} \approx \frac{\partial}{\partial t} + \mathbf{V}(\mathbf{k}_i)\cdot\nabla_{\mathbf{x}},$$

$$\alpha_i(\mathbf{W}) \equiv \frac{1}{R_i}\int d^2k G_i(\mathbf{k})\alpha(\mathbf{k}, \mathbf{W}) \approx \alpha(\mathbf{k}_i, \mathbf{W}),$$

$$\beta_i(\mathbf{W}) \equiv \frac{1}{R_i}\int d^2k G_i(\mathbf{k})\beta(\mathbf{k}, \mathbf{W}) \approx \beta(\mathbf{k}_i, \mathbf{W}),$$

$$\gamma_{ij} \equiv \frac{1}{R_i}\int\int d^2k d^2k_1 G_i(\mathbf{k})G_j(\mathbf{k}_1)\gamma(\mathbf{k}, \mathbf{k}_1) \approx R_j\gamma(\mathbf{k}_i, \mathbf{k}_j) \geq 0,$$

$$\delta_{ijk} \equiv \frac{1}{R_i}\int\int\int d^2k d^2k_1 d^2k_2 G_i(\mathbf{k})G_j(\mathbf{k}_1)G_k(\mathbf{k}_2) \times \delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \approx R_jR_k\delta(\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_k) \geq 0,$$

and

$$\delta_{ijkl} \equiv \frac{1}{R_i}\int\int\int\int d^2k d^2k_1 d^2k_2 d^2k_3 G_i(\mathbf{k}) \times G_j(\mathbf{k}_1)G_k(\mathbf{k}_2)G_l(\mathbf{k}_3)\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k})\delta(\omega_1 + \omega_2 - \omega_3 - \omega) \quad (3.2)$$

(note that $\mathbf{k}_1, \mathbf{k}_2, \ldots$ denote integration variables, while $\mathbf{k}_i, \mathbf{k}_j, \ldots$ denote spectral bands).

The simple form of the prognostic equations (3.1) suggests a sequence of simplified models for consideration in the numerical experiments. In each case, consistent with the representations (2.9) and (2.11), we take

$$\alpha_i = k_i^{-4} s \eta_i \sum_{m=1}^{M_\alpha} Q_m \sigma_i^m$$

and

$$\beta_i = \omega_i s \sum_{m=1}^{M_\beta} Q_{(M_\alpha+m)}\mu_i^m,$$

with σ and χ defined by (2.7) and η and μ defined by (2.10) and (2.12). These parameters are averaged over the ith spectral band and are functions of $(\mathbf{x}, t)$ (through their dependence on $\mathbf{W}$). In each case, we also ignore third-order whitecap dissipation ($\delta_{ijk} = 0$) and, except in the last case, nonlinear transfer ($\delta_{ijkl} = 0$).

Case 1: Uncoupled second-order whitecap dissipation, no nonlinear transfer

We assume

$$\gamma(\mathbf{k}, \mathbf{k}_1) = \gamma(\mathbf{k})\delta(\mathbf{k}_1 - \mathbf{k}), \quad \text{with} \quad \gamma(\mathbf{k}) \geq 0,$$

and we allow $\gamma(\mathbf{k})$ to be determined from (2.6). We can in fact satisfy this equation only if $M_\gamma = 1$,

$$\gamma(\kappa_1, \chi_1) = Q_{(M_\alpha+M_\beta+1)}\delta(\kappa_1 - 1)\delta(\varphi_1),$$

and

$$\gamma(\mathbf{k}) = \gamma(k) = k^4\omega^2 Q_{(M_\alpha+M_\beta+1)} \quad \text{with} \quad Q_{(M_\alpha+M_\beta+1)} \geq 0,$$

[since $\delta(\kappa_1 - 1)\delta(\varphi_1) = k^2\delta(\mathbf{k}_1 - \mathbf{k})$]. It follows that $\gamma_{ij} = \gamma_i\delta_{ij}$, where δ_{ij} is the Kronecker delta and

$$\gamma_i \equiv \frac{1}{R_i}\int d^2k G_i(\mathbf{k})\gamma(\mathbf{k}) = k_i^4\omega_i^2 Q_{(M_\alpha+M_\beta+1)} \geq 0.$$

The governing equations (3.1) become

$$\frac{D_i}{Dt}A_i = \alpha_i + \beta_i A_i - \gamma_i A_i^2. \quad (3.3)$$

These equations are second-order nonlinear in the A_i and are uncoupled.

Note that the second-order uncoupled case is physically unrealistic because it implies that the total dissipation of a narrow spectral peak is unreasonably sensitive to the shape of this peak (K. Hasselmann, private communication). Our object in this numerical study, however, is not to directly model the real world, but to explore a series of simplified numerical cases of increasing physical and numerical complexity approaching the real-world SNLdV case (with realistic physics and real-data input). Our primary interest is in the practical application of Thacker and Long's inverse-modeling procedure to this progression of simplified cases. The uncoupled second-order case is a reasonable starting point for this exploration.

Case 2: Second-order Hasselmann (1974) whitecap dissipation, no nonlinear transfer.

We assume

$$\gamma(\mathbf{k}, \mathbf{k}_1) = \omega(k)^2\gamma(k_1), \quad \text{with} \quad \gamma(k_1) \geq 0. \quad (3.4)$$

The resulting dissipation has the general form proposed by Hasselmann (1974), with the unknown spectrum-dependent constant expressed as a linear functional of A. This functional, however, is constrained by the dimensional analysis. Comparing Eq. (3.4) with the middle equation (2.6), it is clear that one can simultaneously satisfy both equations only if $M_\gamma = 1$,

$$\gamma(\kappa_1, \varphi_1) = \kappa_1^2 Q_{(M_\alpha+M_\beta+1)}$$

and

$$\gamma(k_1) = k_1^2 Q_{(M_\alpha+M_\beta+1)}, \quad \text{with} \quad Q_{(M_\alpha+M_\beta+1)} \geq 0,$$

from which we conclude that the first-order contribution to Hasselmann's unknown constant must be of the form

$$\left(\int d^2k k^2 A(\mathbf{k}, \mathbf{x}, t)\right) Q_{(M_\alpha+M_\beta+1)},$$

where $Q_{(M_\alpha+M_\beta+1)}$ is a nonnegative spectrum-independent dimensionless constant. It follows that $\gamma_{ij} = \omega_i^2 \gamma_j$, with

$$\gamma_j \equiv \int d^2k G_j(\mathbf{k})\gamma(k) \approx \mathrm{k}_j^2 R_j Q_{(M_\alpha+M_\beta+1)} \geq 0.$$

The governing equations

$$\frac{D_i}{Dt} A_i = \alpha_i + \beta_i A_i - \omega_i^2 (\sum_j \gamma_j A_j) A_i, \quad (3.5)$$

are second-order nonlinear in the A_i and are coupled.

We note that if the width of the directional bands is uniform and the spacing of the wavenumber (frequency) bands is logarithmic, $R_j = Ck_j^2$, where C is a dimensionless constant. Absorbing this constant into the expansion coefficient $Q_{(M_\alpha+M_\beta+1)}$, we obtain

$$\gamma_j \approx k_j^4 Q_{(M_\alpha+M_\beta+1)}.$$

We also note in passing that similar arguments applied to the third-order contribution of D to a Hasselmann-type dissipation also constrain the possible dependence of $\delta(\kappa_1, \kappa_2, \varphi_1, \varphi_2)$ on the arguments κ_1 and κ_2.

Case 3: Weakly coupled second-order whitecap dissipation, no nonlinear transfer

We assume $\gamma(\mathbf{k}, \mathbf{k}_1)$ to be given by (2.6), with[1]

$$\gamma(\kappa_1, \chi_1) = (\sum_m^{M_\gamma} Q_{(M_\alpha+M_\beta+m)} (\kappa_1 - 1)^{2\nu_m} (\chi_1 - 1)^{2\nu_m})^{-1}.$$

For $Q_{(M_\alpha+M_\beta+m)} > 0$, the resulting dissipative coupling is strongest for wave components with similar vector wavenumbers. In the limit $Q_{(M_\alpha+M_\beta+m)} \to \infty$, for all $m \neq 1$, we recover the uncoupled case 1 model. We have in general

$$\gamma_{ij} \approx k_i^2 \omega_i^2 R_j \gamma(\kappa_{ij}, \chi_{ij}),$$

where

$$\kappa_{ij} \equiv \frac{k_j}{k_i} \quad \text{and} \quad \chi_{ij} \equiv \frac{\mathbf{k}_i \cdot \mathbf{k}_j}{k_i k_j}.$$

The governing equations

$$\frac{D_i}{Dt} A_i = \alpha_i + \beta_i A_i - (\sum_j \gamma_{ij} A_j) A_i \quad (3.6)$$

are second-order nonlinear in the A_i and are coupled.

Again, if the width of the directional bands is uniform and the spacing of the wavenumber (frequency) bands is logarithmic, $R_j = Ck_j^2$. Absorbing C into the expansion coefficients $Q_{(M_\alpha+M_\beta+m)}$, we obtain

$$\gamma_{ij} \approx k_i^2 \omega_i^2 k_j^2 \gamma(\kappa_{ij}, \chi_{ij}).$$

Case 4: Second-order whitecap dissipation, nonlinear transfer

We assume $\gamma(\mathbf{k}, \mathbf{k}_1)$ defined as in case 1, 2, or 3 and $\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ defined by the Hasselmann (1962) theory for nonlinear wave–wave interactions. It follows that

$$\delta_{ijkl} = \frac{1}{R_i} T_{ijkl},$$

where R_i is defined as before and the T_{ijkl} are the (hybrid) Thacker (1982) coefficients estimated by SNLdV (1990) and Snyder et al. (1992). It follows that

$$\frac{D_i}{Dt} A_i = \alpha_i + \beta_i A_i - (\sum_j \gamma_{ij} A_j) A_i + \frac{1}{R_i} \sum_{jkl} T_{ijkl} A_j A_k A_l. \quad (3.7)$$

The governing equations (3.7) are third-order nonlinear in A_i and are coupled.

4. General discussion of the prognostic equations for cases 1–4

In all cases considered in section 3, and in their extensions employing third-order dissipation, the governing prognostic equations have the important property that, regardless of the sign of the β_i, $A_i(\mathbf{x}, t) \geq 0$, for all i, $\mathbf{x}$, and t. This property is a consequence of two conditions built into these equations (so that this property would apply) and a third condition characterizing the nonlinear transfer term. These are

1) $\alpha_i(\mathbf{x}, t) \geq 0$, for all i, $\mathbf{x}$, and t.

2) The middle terms on the right-hand side of Eq. (3.1) (the wave-induced contribution to the atmospheric input and the second- and third-order whitecap dissipation terms) contain a factor of A_i.

3) Because of the fundamental nature of the nonlinear quadruplet interaction, T_{ijkl} (and thus δ_{ijkl}) can be negative only if j, k, or l is equal to i (in which case this interaction again contains a factor of A_i).

Proof: If A_i is ever positive, it cannot subsequently become negative without passing through $A_i = 0$. But, if all negative terms on the right-hand side of (3.1) contain the factor A_i, then, for $A_i = 0$,

$$\frac{D_i}{Dt} A_i \geq 0.$$

It follows that A_i cannot become negative.

[1] One of several possible algorithms resulting in weak coupling.

A second property of these equations is that they drive the system toward a (bounded) equilibrium solution. This second property is a consequence of

1) the first property (whereby $A_i \geq 0$, for all i, $\mathbf{x}$, and t),
2) the dissipative and quasi-linear character of the whitecapping interaction (whereby D_i/A_i is monotonic-increasing in the A_j), and
3) the conservative character of the nonlinear interaction.

Proof: Putting aside nonlinear transfer, the positive terms on the right-hand side of (3.1), which cause A_i to grow, are at most linear in the A_i. The negative terms are dominated by a dissipation quasi-linear in A_i with a coefficient which is monotonic increasing in the A_j. Thus, as the A_i become larger, the balance between the positive and negative terms, initially positive, must tend toward zero. The addition of nonlinear transfer, which is third order in the A_i but does not contain terms with powers of a single A_i beyond second (which in the equation for A_i are all negative), readjusts the details of this balance, but leaves the total action $\Sigma_i A_i$ the same.

To illustrate this second property, we consider first the second-order homogenous uncoupled case. Here each spectral band evolves independently and is governed by an equation of the form (3.3), with $A_i(t) \geq 0$, $\alpha_i(\mathbf{W}) \geq 0$, $\beta_i(\mathbf{W})$, $\gamma_i \geq 0$, $\mathbf{W}(t)$, and

$$\frac{D_i}{Dt} A_i = A_i'(t)$$

all independent of $\mathbf{x}$. Figure 1 shows the right-hand side of this equation $S_i(\mathbf{W}, A_i)$ as a function of A_i. This function has two (wind-dependent) roots, one positive (stable) and one negative (unstable)

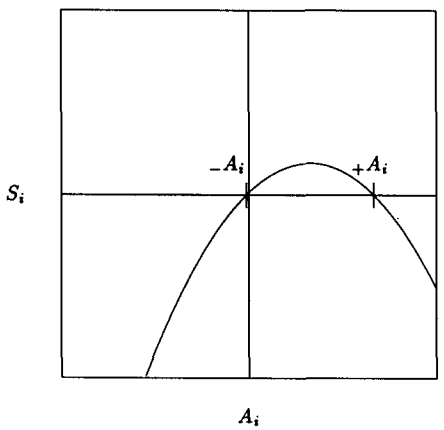

FIG. 1. Interaction source function S_i as a function of action spectral density A_i. Uncoupled whitecap dissipation. No nonlinear transfer.

$$A_i = {}_{\pm}A_i \equiv \frac{\beta_i \pm \sqrt{\beta_i^2 + 4\alpha_i\gamma_i}}{2\gamma_i}. \qquad (4.1)$$

For constant α_i and β_i (constant wind velocity), the spectral intensity A_i stays entirely in the range $0 \leq A_i \leq {}_{+}A_i$ and tends toward ${}_{+}A_i$. If the wind is a function of time, ${}_{+}A_i$ changes, causing A_i to follow. In terms of the figure, the state of the ith band (as defined by A_i and S_i) is represented by a point on the curve. This point moves along the curve, and, if the wind is not constant, the curve itself also moves. Starting for example from (0, 0), following a sudden onset of wind, the point moves with the curve to $(0, \alpha_i)$ and then to the right along the curve, continuing until $S_i = 0$ at $({}_{+}A_i, 0)$. A subsequent change in wind initiates a new progression along the curve (in either direction, depending on the sign of S_i) toward the new equilibrium $({}_{+}A_i, 0)$. If the wind dies, this progression returns the point to the new equilibrium at (0, 0).

Let $\mathbf{A}(t) \equiv [A_i(t)]$ be the state vector for the system of Eqs. (3.3), (3.5), or (3.6). In the multidimensional space of this vector, a point represents a possible state of the system and a moving point describes a possible evolution of this system. This evolution is determined by the dependence of S_i on $\mathbf{W}$ and $\mathbf{A}$, in particular, by the location of the (time-dependent) root loci (4.1). In the uncoupled case, these root loci are hyperplanes perpendicular to the axes, subdividing the space into multiple (time-dependent) regions, within which the signs of the S_i are invariant. In particular, $S_i \geq 0$, for all i, throughout the region defined by ${}_{-}A_i \leq A_i \leq {}_{+}A_i$, for all i, and in the principal operational subregion defined by $0 \leq A_i \leq {}_{+}A_i$, for all i. Furthermore, the multiple positive-root hyperplanes $A_i = {}_{+}A_i$ intersect in a single point ${}_{+}\mathbf{A}(t) = [{}_{+}A_i(t)]$ toward which the system tends. Here $S_i = 0$ and $A_i \geq 0$, for all i, and the resulting equilibrium is stable with respect to all A_i. Changes in wind move the root hyperplanes along the axes, changing ${}_{+}\mathbf{A}$, and moving the system toward the new equilibrium (see Fig. 1). A dying wind shrinks the principal operational subregion to a null region at the origin and drives the system toward the equilibrium ${}_{+}\mathbf{A} = 0$.

The first panel of Fig. 2 shows schematically the evolution of a two-band uncoupled system following the sudden onset of a constant wind. The four root loci, $A_1 = {}_{\pm}A_1$ and $A_2 = {}_{\pm}A_2$, are in this case straight lines perpendicular to the A_1 and A_2 axes, respectively, and divide the (A_1, A_2) plane into nine regions. More importantly, the two positive-root loci divide the first quadrant of the (A_1, A_2) plane into four regions and intersect in a single equilibrium point, toward which the system tends and about which the evolution is stable.

The argument for the second-order homogeneous coupled case is similar. Here the evolution of the bands is governed by a set of simultaneous equations of the form (3.6), where, as before, $A_i(t) \geq 0$, $\alpha_i(\mathbf{W}) \geq 0$,

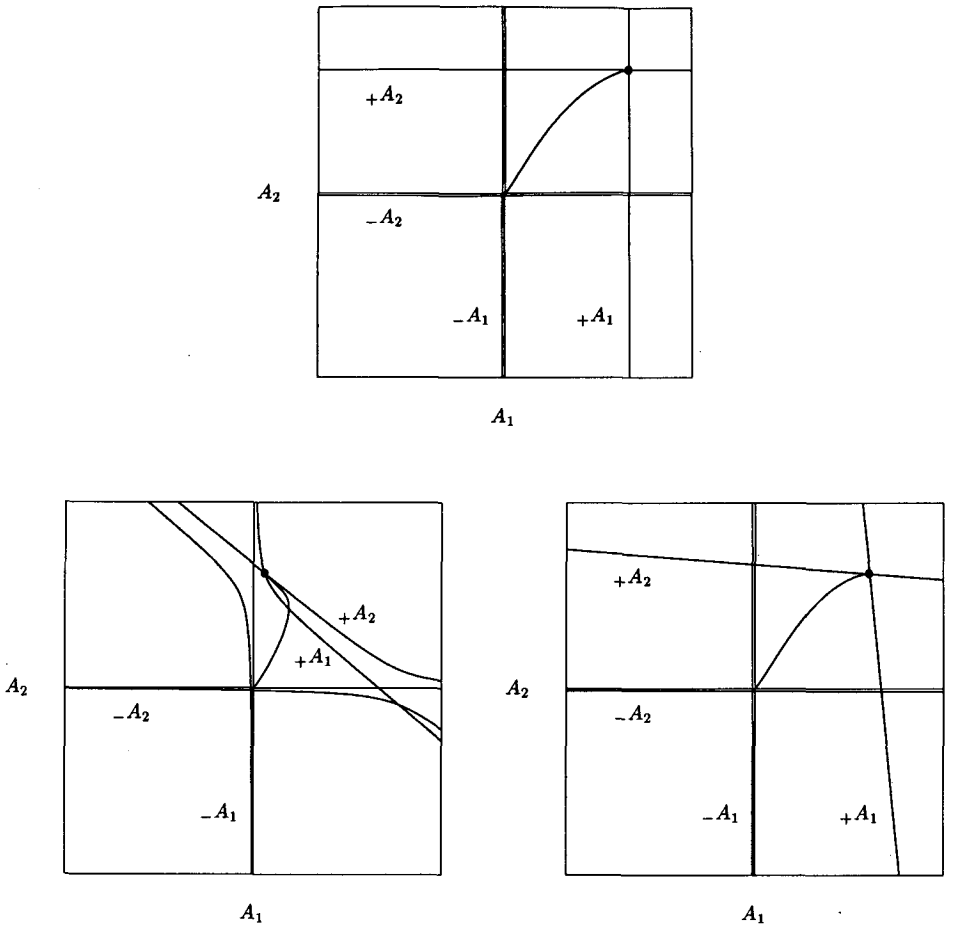

FIG. 2. Evolution of homogeneous two-band system. Upper panel is second-order uncoupled case. Lower left panel is second-order coupled Hasselmann (1974) case. Lower right panel is second-order weakly coupled case. Diagrams are highly schematic, with α orders of magnitude larger than in the real world.

$\beta_i(\mathbf{W})$, $\gamma_i \geq 0$, $\mathbf{W}(t)$, and $A_i'(t)$ are independent of $\mathbf{x}$. In this case, $S_i(\mathbf{W}, \mathbf{A})$ again has two roots

$$A_i = {}_{\pm}A_i \equiv \frac{\beta_i - \sum_{j\neq i} \gamma_{ij}A_j \pm \sqrt{(\beta_i - \sum_{j\neq i} \gamma_{ij}A_j)^2 + 4\omega_i^2 \alpha_i \gamma_{ii}}}{2\omega_i^2 \gamma_{ii}},$$

but, because these roots depend on A_j other than A_i, the corresponding hypersurfaces in the space of the state vector $\mathbf{A}$ are no longer hyperplanes. Nonetheless, several qualitative features of the uncoupled system are retained. In particular, the positive-root hypersurface is everywhere to the positive side of the $A_i = 0$ hyperplane ($_+A_i \geq 0$) and the negative-root hypersurface is everywhere to the negative side of this hyperplane ($_-A_i \leq 0$). Each hypersurface is single valued in the A_j, $j \neq i$ and extends to infinity in all directions. Between the ith root hypersurfaces, S_i is positive; elsewhere, this derivative is negative. The multiple positive-root hypersurfaces intersect in a single equilibrium point toward which the system tends and about which the evolution of the system is stable. If the wind is a function of time, the root hypersurfaces and their intersection change, moving the system toward a new equilibrium.

The second and third panels of Fig. 2 show schematically the evolution of a two-band coupled system following the sudden onset of a constant wind. In the second panel, γ_{ij} is chosen as in case 2, $\gamma_{ij} = \omega_i^2 \gamma_j$ (Hasselmann dissipation). Here the positive-root loci are qualitatively quite different from those of the uncoupled case (first panel). Instead of being essentially perpendicular to one another, these loci, over a significant portion of their range, are essentially parallel. In consequence, the intersection of these loci defines an equilibrium with one A_i close to its uncoupled value and the other A_i close to zero. In evolving toward this equilibrium, starting from a null state, both bands start off growing in more or less normal fashion. As soon as the system crosses the closest of the nearly parallel positive-root loci, however, the time derivative of the

corresponding A_i becomes negative. Thereafter, this band decays while the other band continues to grow.

Extension of this case to a full set of spectral bands implies a similar qualitative evolution. The positive-root hypersurfaces are again essentially parallel over a significant portion of their ranges. The resulting equilibrium concentrates the action in a single band (belonging to the outer positive-root hypersurface), with little action in all remaining bands. The reasons for this singular behavior and its implications are explored further in the next section.

In the third panel of Fig. 2, γ_{ij} is chosen as in case 3, and the $Q_{(M_\alpha+M_\beta+m)}$ are chosen to approximate the uncoupled case (rapid dropoff for $j \neq i$). As might be expected, the behavior of this case is qualitatively similar to that of the uncoupled case.

The argument for a system with second-order whitecap dissipation and nonlinear transfer, governed by (3.7), is less transparent. The gross picture, however, cannot be very different from that described above. Because the coefficients T_{ijkl} can have at most two repeated indices (if there were a third, energy and momentum conservation would require that the fourth index be the same, in which case the spectral product would vanish identically), the right-hand side of (3.7) is still only quadratic in A_i. It can further be shown that in the ith equation (3.7), those nonlinear transfer terms which are quadratic in A_i have a negative Thacker coefficient T_{ijkl}. Thus, the dependence of the time derivative S_i on $\mathbf{W}$ and $\mathbf{A}$ is qualitatively identical to that of the previous case. In particular, this derivative vanishes on two root hypersurfaces lying on opposite sides of the $A_i = 0$ hyperplane. Between these hypersurfaces, this derivative is positive; elsewhere it is negative. The multiple positive-root hypersurfaces again intersect in a single (time-dependent) equilibrium point toward which the system tends and about which the evolution of this system is stable.

Note, however, that, for Hasselmann dissipation, the resulting case 4 equilibrium would be expected to differ significantly from the case 2 equilibrium. (It would not be singular.)

5. Discussion of the case 2 singularity

We encountered in the previous section a singularity in the evolution of the spectrum, when dissipation has Hasselmann form and nonlinear transfer is ignored. We next discuss this singularity in more detail.

To simplify the argument, we assume for the moment that the α_i, which we know to be small, are in fact vanishingly small. We ignore the question of how such a system starts up from a null state and seek an equilibrium solution. Setting the right-hand side of (3.5) equal to zero gives

$$\beta_i A_i - \omega_i^2 \Big(\sum_j \gamma_j A_j\Big) A_i = A_i\Big(\beta_i - \omega_i^2 \Big(\sum_j \gamma_j A_j\Big)\Big) = 0, \quad \text{for all } i. \tag{5.1}$$

It follows that either

$$A_i = 0 \quad \text{or} \quad \sum_j \gamma_j A_j = \omega_i^{-2} \beta_i. \tag{5.2}$$

Because the left-hand side of this latter equation is independent of i, this equation can apply in at most one band (say, the ith band). In the remaining bands, we must satisfy (5.1) by setting $A_j = 0$. Thus for each i we obtain the possible equilibrium solution

$$A_i = \omega_i^{-2} \gamma_i^{-1} \beta_i,$$

and

$$A_j = 0, \quad \text{for} \quad j \neq i.$$

Only one of these possible solutions, however, is stable (that for which $\omega_i^{-2} \beta_i$ is a maximum). (Also the solution $A_j = 0$, for all j, is unstable.) Throughout the hyperframe equivalent of the first quadrant, where $A_i > 0$, for all i, the resulting positive-root loci are segments of the hyperplanes defined by the second of equations (5.2) and are strictly parallel.

It is important to notice that this singularity is probably not the result of the specific representation that we have chosen for Hasselmann's unknown constant (i.e., a linear functional of the action spectrum). It may in fact be a property of all such representations. To see this we note that, analogous to (5.1), arbitrary functional representation of this constant gives

$$A_i(\beta_i - \omega_i^2 \gamma(\mathbf{A})) = 0, \quad \text{for all } i,$$

where $\gamma(\mathbf{A})$ is some arbitrary function of the A_i. Once again, we have, for all i, either $A_i = 0$ or

$$\gamma(\mathbf{A}) = \omega_i^{-2} \beta_i. \tag{5.3}$$

But $\gamma(\mathbf{A})$ is independent of i. Therefore, the second condition can be applied in at most one band; in the remaining bands we must set $A_j = 0$. The hypersurfaces defined by (5.3) for various i are no longer hyperplanes, but they are still in some sense parallel. Furthermore, the band for which (5.3) applies is again that band corresponding to the outermost hypersurface, that is, that band for which $\omega_i^{-2} \beta_i$ is a maximum.

Note also that a Hasselmann-type interaction using other powers of ω_i [such as investigated by Komen et al. (1984) or by Janssen (1990)] should also exhibit this same singularity when nonlinear transfer is ignored.

One might wonder why such a simple yet striking effect has as yet gone unremarked. We conjecture that the reason for this is that reported wave-model computations employing the Hasselmann (1974) form for whitecap dissipation (Komen et al. 1984, and numerous WAM model computations) have also simultaneously included some form of nonlinear transfer (either EXACT-NL or the discrete-interaction WAM approximation). Presumably this nonlinear transfer so dominates the evolutionary dynamics (by redistributing the action away from the singular band) that any

tendency for dissipation to produce a singular result is effectively lost.

It might be argued that such a juxtaposition of nonlinear interactions is only proper, that the presence of locally strong nonlinear effects (whitecapping) presupposes that the wave field has already evolved to the point where weak nonlinear interactions must be significant. Our response is that it is nonetheless appropriate to consider the behavior of systems without nonlinear transfer, and, if the remaining physics is acceptable, the evolution of such systems should be physically reasonable. Also, because the nonlinear interactions require energy and momentum conservation, we can imagine perhaps unrealistic but not inconceivable circumstances under which dissipation might operate in the absence of nonlinear transfer, even in the real ocean. For example, suppose we are able to create (mechanically) a sea consisting of two comparably energetic bands for which there exist no allowable nonlinear interactions (for example, two bands with distinct wavenumbers but identical directions). Further, suppose that the band for which $\omega^{-2}\beta$ is largest has been created close to its equilibrium of $\omega^{-2}\gamma^{-1}\beta$. Now let the system go. In the near term, the dynamics will be dominated by wave-induced atmospheric input and by whitecap dissipation. The band close to equilibrium may decay somewhat initially, but will then approach equilibrium. The other band will decay rapidly.

The question is whether it is physically reasonable for the whitecap dissipation, in the absence of nonlinear transfer, to single out a particular band, causing it to evolve toward a reasonable equilibrium, while simultaneously suppressing the remaining bands.

To answer this question, we consider first the case of a homogeneous system with linear dissipation and no nonlinear transfer. The prognostic equations for such a system are of the form

$$\frac{\partial A_i}{\partial t} = S_i = \alpha_i + (\beta_i - \gamma_i)A_i, \qquad (5.4)$$

where the β_i are functions of $\mathbf{W}$ and the γ_i are constant. Each band grows or decays, depending primarily on whether the atmospheric input coefficient β_i or the dissipation coefficient γ_i is greater, and each band retains its growing or decaying character, as long as the wind remains constant. Even though the evolution of this system is typically unstable (has at least one unstable band), this evolution is physically reasonable. Neighboring bands behave in similar fashion. They grow or decay depending upon whether atmospheric input or dissipation is greater. Because both terms are strictly linear in A_i, the character of the evolution within a given band is independent of the A_j.

The evolution of a homogeneous second-order case 2 Hasselmann system is governed by the prognostic equations (3.5), which are of the form (5.4), with

$$\gamma_i = \omega_i^2 \gamma(\mathbf{A}), \text{ where } \gamma(\mathbf{A}) \equiv \Big(\sum_j k_j^2 R_j A_j\Big) Q_M.$$

For constant wind, β_i is constant; γ_i, however, is not constant but increases with the A_i. If the system starts from a null state, the A_i are initially small, so that typically $\beta_i > \gamma_i$, and atmospheric input exceeds dissipation; thus, the bands grow. As the A_i get larger, so does the dissipation coefficient $\gamma(\mathbf{A})$. Eventually, as, one by one, the resulting dissipation coefficients γ_i become larger than the corresponding atmospheric input coefficients β_i and dissipation exceeds atmospheric input, the character of the evolution in these bands changes from growth to decay. This process continues until there is only one growing band left. Unlike the linear system, however, this system is stable and cannot evolve indefinitely. As previously discussed, it eventually achieves an equilibrium in which the action is large in the ith band (the last growing band) and small in the remaining bands. Near equilibrium, the dissipation coefficient $\gamma(\mathbf{A})$ goes as A_i. The dissipation in the ith band goes as A_i^2 and is essentially uncoupled from the remaining bands, while the dissipation in the remaining bands goes as $A_i A_j$ and is strongly coupled to the ith band.

Similar remarks apply to the more general Hasselmann case, where $\gamma(\mathbf{A})$ is some more general function of the A_i.

What property of the Hasselmann interaction is ultimately responsible for the evolutionary singularity? We believe this to be an interesting question to which we do not have a definitive answer. We speculate, however, that what may be involved is the degree of dissipative coupling between bands. Clearly, if there is no coupling or the coupling is weak (as in case 1 or case 3), each band can come to its own (reasonable) equilibrium more or less independent of the other bands. On the other hand, if the dissipation in one band depends strongly on the spectral level of another band, then the growth of this second band may imply a dissipation in the first band which cannot be matched by the atmospheric input. Such is the case in the second-order example above, where the dissipation coefficient eventually becomes effectively independent of all but one A_i.

A second possibility would be that this singularity, like Hasselmann's theory, is somehow implied by the patchy character of the whitecapping interaction.

In any event, we conclude that, despite its singularity, the evolution of a system with Hasselmann dissipation and no nonlinear transfer, discussed in terms of the relative balance between atmospheric input and dissipation, is physically reasonable. If such a system were realized in nature, we believe it would behave as predicted. What is missing is a more definitive identification of that property of the Hasselmann interaction which ultimately gives rise to the singularity and, perhaps concurrently, some intuitive explanation as to why

nature should prefer a whitecapping mechanism which develops this singularity.

We note that the present development, based on some very simple general considerations, yields a general form (2.6) for the second-, third-, and (by extension) higher-order contributions to the whitecap dissipation. This general form allows a full range of dissipative coupling between spectral bands, from the uncoupled case to a variety of strongly coupled cases, and contains the Hasselmann prediction as a special case. More importantly, the optimization of this form, as proposed by SNLdV, provides a systematic means to either verify this prediction, with its implied singularity, or to find some alternative.

6. Derivation of the adjoint-model equations for case 4

In this section, we complete the discretization of the action-balance equation (2.1) by introducing into the prognostic equations finite-difference representations for the derivatives of A with respect to $\mathbf{x}$ and t. We then derive the corresponding case 4 adjoint-model equations (with case 2 dissipation) and discuss the inverse-modeling procedure.

The source expansion introduced in sections 2 and 3 defines a set of expansion coefficients which control the evolution of the wave spectrum through Eq. (3.1). The SNLdV parent program seeks to adjust these coefficients to achieve a best fit between model prediction and synoptic field observations. To determine optimal coefficients, we need first to define some measure of the fit between model prediction and observation. In specifying this cost function or variance of fit V, we have several choices. The most intriguing of these choices is to make this definition in terms of the primary statistical observables, that is, the cross correlations between wave-array-sensor pairs or the Fourier transforms of these cross correlations (the cross spectra). Such a definition would be expected to reduce the error of the overall computation by short circuiting the intermediate directional-spectrum analysis, essentially building this analysis into the optimization (incidentally ensuring a directional spectrum that is everywhere nonnegative). For the purposes of this discussion, however, we assume that the directional-spectrum analysis is performed as part of the preliminary data analysis and consider a second less-fundamental but more-transparent definition of variance in terms of the prognostic variables A_i. To simplify the notation we ignore possible error correlations between the various bands. We define

$$V([Q_m]) \equiv \frac{1}{2} \sum_{irn} w_{irn}(A_i(\mathbf{x}_r, t_{rn}, [Q_m]) - a_i(\mathbf{x}_r, t_{rn}))^2,$$

where the A_i and a_i are the modeled and observed prognostic variables representing the action spectral density in the ith band (A_i from the model equations and a_i from the directional-spectrum analysis), the Q_m are the expansion parameters, and the w_{irn} are a set of weights reflecting relative confidence limits for the observed spectral estimates, or their relative dynamical significance, or both. The sum over $i = 1, 2, \cdots, I$ is over all bands in the spectral representation, the sum over $r = 1, 2, \cdots, R$ is over all wave-observation stations, and the sum over $n = 1, 2, \cdots, N(r)$ is over all times t_{rn} for which an observed estimate $a_i(\mathbf{x}_r, t_{rn})$ is defined.

The task that we face is to minimize such a cost function with respect to the unknown parameters Q_m. If the cost function were quadratic in these parameters, we could, using a conjugate-gradient search procedure and estimating each component of the gradient (with respect to the parameters) as a forward difference, determine the minimum in at most $M(M + 1)$ model runs, where M is the total number of parameters. The cost function is, however, more highly nonlinear. Thus 1) there is a possibility of more than one relative minimum and, more significantly, 2) minimization of the cost function will involve an iterative search.

If the number of unknown parameters M is small and the dynamical situation simple, the computational demands of such a straightforward search may prove acceptable. Using a forward difference to estimate the components of the gradient, Monbaliu (1991) has in fact successfully tuned a two-parameter (dissipation and atmospheric input) version of a one-dimensional wave model to fit the JONSWAP results of Hasselmann et al. (1973). If, however, M is not so small and the dynamical situation requires a two-dimensional wave model, these demands may be prohibitive.

An effective search procedure for the case of large M is described by Thacker and Long (1988). Their method is an efficient implementation of the so-called "adjoint method" initially proposed by Marchuk (1974) and subsequently explored by a number of other researchers (Lewis and Derber 1985; Le Dimet and Talagrand 1986; Courtier and Talagrand 1987). Developed originally to enable the assimilation of data into several ocean circulation models, Thacker and Long's scheme can readily be turned to the optimization problem at hand. In essence, this scheme provides a recipe for estimating all M components of the gradient of the cost function from the equivalent of two (as compared with $M + 1$) model runs. It achieves this reduction by attaching to the present wave model an adjoint wave model that is driven by the misfit between model computation and observation and that runs backward in time, yielding an estimate for the full gradient of the cost function. Combined with a second procedure (preconditioning) for improving the efficiency of the conjugate-gradient descent algorithm (Hasdorff 1976), this scheme promises to greatly re-

duce the number of model computations required to optimize the fit.

Consider the evolution of the wave field in a closed deep-water basin initially at rest, with second-order Hasselmann (1974) whitecap dissipation, null current, and fully absorbing boundaries. The prognostic equations governing this evolution are of the form (3.7), with

$$\gamma_{ij} = \omega_i^2 \gamma_j \quad \text{and} \quad \gamma_j = k_j^2 R_j Q_M,$$

where

$$M \equiv M_\alpha + M_\beta + 1,$$

and, as before, R_j is the area of the jth spectral band. Let $\mathbf{V}_i$ be the mean group velocity of the ith spectral band, and let $\mathbf{n}(\mathbf{x})$ be the outward-pointing normal to the boundary of the basin. Then we have, for $t > 0$, at all interior points and at all boundary points for which $\mathbf{k}_i \cdot \mathbf{n} \geq 0$,

$$\frac{\partial A_i}{\partial t} + \mathbf{V}_i \cdot \nabla_{\mathbf{x}} A_i = k_i^{-4} s \eta_i \sum_{m=1}^{M_\alpha} Q_m \sigma_i^m + \omega_i s\Big(\sum_{m=1}^{M_\beta} Q_{(M_\alpha+m)} \mu_i^m\Big) A_i - \omega_i^2 \Big(\sum_j k_j^2 R_j A_j\Big) Q_M A_i + \frac{1}{R_i} \sum_{jkl} T_{ijkl} A_j A_k A_l, \quad (6.1)$$

while, for $t = 0$, at all points, and, for $t > 0$, at all boundary points for which $\mathbf{k}_i \cdot \mathbf{n} < 0$,

$$A_i = 0. \quad (6.2)$$

Let $\mathbf{x}_r$, $r = 1, 2, \cdots, R$, and $t_n = n\Delta t$, $n = 0, 1, \cdots, N$, define the spatial and temporal grid points of the model computation, and let the observations a_i be defined at a subset of these grid points. Then, for $A_{irn} \equiv A_i(\mathbf{x}_r, t_n, [Q_j])$ and $a_{irn} \equiv a_i(\mathbf{x}_r, t_n)$, V takes the form

$$V = \frac{1}{2} \sum_{irn} w_{irn} (A_{irn} - a_{irn})^2,$$

where the sum over rn now runs over all model grid points (spatial and temporal), with $w_{irn} \equiv 0$, for all rn for which $a_i(\mathbf{x}_r, t_n)$ is undefined.

We may now write down explicit finite-difference representations for (6.1) and (6.2). For $n > 0$, and $\mathbf{x}_r$ an interior point, or $n > 0$, and $\mathbf{x}_r$ a boundary point for which $\mathbf{k}_i \cdot \mathbf{n}(\mathbf{x}_r) \geq 0$, we have

$$E_{irn} \equiv A_{irn} - A_{ir(n-1)} + \sum_s D_{irs} A_{is(n-1)} - \Delta t F_{ir(n-1)} = 0, \quad (6.3)$$

where

$$F_{irn} \equiv k_i^{-4} s \eta_{irn} \sum_{m=1}^{M_\alpha} Q_m \sigma_{irn}^m + \omega_i s\Big(\sum_{m=1}^{M_\beta} Q_{(M_\alpha+m)} \mu_{irn}^m\Big) A_{irn} - \omega_i^2 \Big(\sum_j k_j^2 R_j A_{jrn}\Big) Q_M A_{irn} + \frac{1}{R_i} \sum_{jkl} T_{ijkl} A_{jrn} A_{krn} A_{lrn}.$$

Here σ_{irn}, η_{irn}, and μ_{irn} are the values of σ_i, η_i, and μ_i at $(\mathbf{x}_r, t_n)$; D_{irs} is a sparse matrix whose precise form depends on the finite-difference representation of $\nabla_{\mathbf{x}} A_i$.

Similarly, for $n = 0$, or for $n > 0$ and $\mathbf{x}_r$ a boundary point for which $\mathbf{k}_i \cdot \mathbf{n}(\mathbf{x}_r) < 0$, we have simply

$$E_{irn} \equiv A_{irn} = 0. \quad (6.4)$$

Following Thacker and Long, we treat the finite-difference equations (6.3) and (6.4) as constraints on the minimization of V, enlarging the set of free parameters to include the A_{irn} and a corresponding set of Lagrange multipliers λ_{irn}, and defining a Lagrange function

$$L([A_{irn}], [\lambda_{irn}], [Q_m]) \equiv V + \sum_{irn} \lambda_{irn} E_{irn}. \quad (6.5)$$

The minimum of V with respect to the parameters Q_m corresponds to a stationary (saddle) point of the Lagrange function. To obtain this point we demand that the derivative of this function with respect to each of its arguments be zero. The derivative of this function with respect to λ_{irn} is simply

$$\frac{\partial L}{\partial \lambda_{irn}} = E_{irn}.$$

Setting this derivative equal to zero recovers the model equations (6.3) and (6.4). The derivative of the Lagrange function with respect to A_{irn} is more complex (because of the nonlinear terms) but is nonetheless tractable. For $n < N$ and $\mathbf{x}_r$ an interior point, or for $n < N$ and $\mathbf{x}_r$ a boundary point for which $\mathbf{k}_i \cdot \mathbf{n}(\mathbf{x}_r) \geq 0$, we have

$$\frac{\partial L}{\partial A_{irn}} = w_{irn}(A_{irn} - a_{irn}) + \lambda_{irn} - \lambda_{ir(n+1)} + \sum_s D_{isr} \lambda_{is(n+1)} - \Delta t G_{irn},$$

where

$$G_{irn} \equiv \omega_i s\Big(\sum_{m=1}^{M_\beta} Q_{(M_\alpha+m)} \mu_{irn}^m\Big) \lambda_{ir(n+1)} - \omega_i^2 \Big(\sum_j k_j^2 R_j A_{jrn}\Big) Q_M \lambda_{ir(n+1)}$$

$$- k_i^2 R_i \Big(\sum_j \omega_j^2 \lambda_{jr(n+1)} A_{jrn}\Big) Q_M + \sum_{jkl} T^{\dagger}_{ijkl} \lambda_{jr(n+1)} A_{krn} A_{lrn},$$

with

$$T^{\dagger}_{ijkl} \equiv \frac{1}{R_j} (T_{jikl} + T_{jkil} + T_{jkli}).$$

Setting this derivative equal to zero gives the adjoint equation

$$w_{irn}(A_{irn} - a_{irn}) + \lambda_{irn} - \lambda_{ir(n+1)} + \sum_s D_{isr} \lambda_{is(n+1)} - \Delta t G_{irn} = 0. \quad (6.6)$$

For $n = N$, or for $n < N$ and $\mathbf{x}_r$ a boundary point for which $\mathbf{k}_i \cdot \mathbf{n}(\mathbf{x}_r) < 0$, we have simply

$$\frac{\partial L}{\partial A_{irn}} = w_{irn}(A_{irn} - a_{irn}) + \lambda_{irn} = 0. \quad (6.7)$$

Finally, the derivatives of the Lagrange function with respect to the Q_m are

$$\frac{\partial L}{\partial Q_m} = -\Delta t \sum_{irn} \lambda_{irn} \frac{\partial F_{ir(n-1)}}{\partial Q_m}, \quad (6.8)$$

with

$$\frac{\partial F_{irn}}{\partial Q_m} = k_i^{-4} s \eta_{irn} \sigma^m_{irn}, \quad m = 1, 2, \cdots, M_\alpha,$$

$$\frac{\partial F_{irn}}{\partial Q_{(M_\alpha + m)}} = \omega_i s \mu^m_{irn} A_{irn}, \quad m = 1, 2, \cdots, M_\beta,$$

and

$$\frac{\partial F_{irn}}{\partial Q_M} = -\omega_i^2 \Big(\sum_j k_j^2 R_j A_{jrn}\Big) A_{irn}. \quad (6.9)$$

In (6.8) [and in (6.10) below], the sum over *irn* excludes all points for which $n = 0$ as well as boundary points $\mathbf{x}_r$ for which $\mathbf{k}_i \cdot \mathbf{n}(\mathbf{x}_r) < 0$. Setting these derivatives equal to zero gives the corresponding equations

$$\sum_{irn} k_i^{-4} \eta_{irn} \sigma^m_{irn} \lambda_{irn} = 0, \quad m = 1, 2, \cdots, M_\alpha,$$

$$\sum_{irn} \omega_i \mu^m_{irn} \lambda_{irn} A_{ir(n-1)} = 0, \quad m = 1, 2, \cdots, M_\beta,$$

and

$$\sum_{ijrn} \omega_i^2 k_j^2 R_j \lambda_{irn} A_{ir(n-1)} A_{jr(n-1)} = 0. \quad (6.10)$$

Equations (6.3), (6.4), (6.6), (6.7), and (6.10) together determine the A_{irn}, λ_{irn}, and Q_m for which the Lagrange function is stationary and the cost function is a minimum.

To solve this system of equations we adopt the following iterative procedure:

1) We make a guess for the best-fit parameters Q_m.

2) Using 1), we solve the (model) equations (6.3) and (6.4) for the A_{irn}. Note that these equations are independent of the λ_{irn} and are readily solved by stepping forward in time, starting from $n = 0$.

3) Using 1) and 2), we solve the (adjoint model) equations (6.6) and (6.7) for the λ_{irn}. Note that these equations are forced by the misfits $w_{irn}(A_{irn} - a_{irn})$ and by a rather complicated inhomogeneity involving the parameters Q_m, the present values of the A_{irn}, and the future values of the λ_{irn}, and they are readily solved by stepping backward in time, starting from $n = N$.

4) Using 1), 2), and 3), we estimate the gradient of the Lagrange function with respect to the parameters Q_m from (6.8) and (6.9).

5) Employing a conjugate-gradient descent algorithm, we estimate improved values for our guess for the parameters Q_m.

Given a reasonable initial guess, repeated iteration of steps 2) through 5) converges to the best-fit result [and satisfies equations (6.10)].

What is important to notice is that, despite the profusion of independent variables that Thacker and Long's Lagrange-multiplier scheme introduces into the computation, despite the nonlinearity of the original model equations, and despite the budding complexity of the source-function expansion that we have built into the example, we are led in the end to a recipe for determining the expansion parameters which is computationally straightforward and considerably more efficient than the brute-force alternative. Furthermore, the adjoint equations, which are the heart of the scheme, are in principle as readily solved as the model equations themselves. In particular, the evaluation of nonlinear transfer in the model equations is mirrored in the adjoint equations by the evaluation of a term of similar form in which the Thacker coefficients T_{ijkl} are replaced by a related set of coefficients $T^{\dagger}_{ijkl}$, which like the T_{ijkl} are mostly zero, and the triple product of prognostic variables $A_{jrn}A_{krn}A_{lrn}$ is replaced by the triple product $\lambda_{jr(n+1)}A_{krn}A_{lrn}$, in which one of the prognostic variables is replaced by a Lagrange multiplier. Note that a formal piecewise-constant Thacker representation for the nonlinear transfer is particularly well suited to the formulation of the adjoint problem.

7. Conclusions

1) Using power series, functional power series, dimensional analysis, and a variety of physical and mathematical considerations, we derive a systematic expansion of the action-balance source function for the deep-water case.

2) This expansion identifies a series of simplified cases for some numerical experiments to explore the inverse modeling of this equation, using the adjoint

data-assimilation technique of Thacker and Long (1988). These experiments will be described in succeeding parts to this series.

3) The expansion also lays the foundation for the more complete expansion of this equation required by an ongoing program to parameterize this equation by comparing its predictions with synoptic wave observations (Snyder et al. 1990).

4) The expansion naturally incorporates a Thacker (1982) representation for the nonlinear transfer from wave–wave interactions.

5) The dimensional analysis determines a natural scaling for each term in this expansion.

6) This dimensional analysis also suggests a general form for the whitecap dissipation that includes as a special case the form proposed by Hasselmann (1974) and determines the first-order contribution to his unknown spectrum-dependent coefficient (to within a multiplicative spectrum-independent constant).

7) We provide an alternative justification for Hasselmann's contention that the whitecap interaction is quasi-linear in A. We argue simply that this term must contain a factor of A in order that A remain nonnegative for all conceivable evolutions.

8) A general discussion of the evolution of the simplified cases suggests that under quite general conditions the prognostic variables A_i, representing the action densities of the various spectral bands, progress toward a stable bounded equilibrium.

9) This discussion reveals a striking tendency of the Hasselmann form for the whitecap dissipation to concentrate action in a single band. In numerical experiments with this dissipation and no nonlinear transfer, this tendency totally dominates the qualitative development of the spectrum. This tendency has not been remarked previously, as it is masked by the nonlinear-transfer interaction.

10) Geometrically, this tendency results from a dissipative coupling for which the loci on which the source functions S_i vanish (in the positive multidimensional space of the prognostic variables A_i) are roughly parallel over a significant portion of their range.

11) Physically, this tendency may be related to the degree of dissipative coupling between spectral bands. Cases consistent with the dimensional analysis, for which there is either no dissipative coupling between bands or this coupling is weak are free of this tendency.

12) We derive the adjoint wave-model equations for a closed deep-water basin, using a Thacker (1982) representation for the nonlinear transfer and assuming second-order Hasselmann (1974) whitecap dissipation, null current, and fully absorbing boundaries.

13) In these equations, nonlinear transfer is mirrored by a term of similar form with the Thacker coefficients replaced by a related set of adjoint coefficients and the triple product of spectral intensities replaced by a product of two spectral intensities and a Lagrange multiplier.

14) The SNLdV program is well suited to investigating the form of the whitecapping interaction (and other interactions).

15) The adjoint data-assimilation model-optimization procedure of Thacker and Long provides an appropriate means for carrying out this program.

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APPENDIX

Quasi Linearity of the Dissipation Source Terms

In this Appendix, we provide a justification for our contention that the kernel $\gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$ of Eq. (2.3) and the contribution of D to the kernel $\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ of this same equation must contain delta functions. (We have dropped the dependence on $\mathbf{W}$.) We proceed from the basic assumption that these coefficients must be everywhere nonpositive (otherwise, the interaction would not be "dissipative") and demand that the resulting dynamics not allow A to become negative. We further demand this not only of the real world, but of all physically conceivable worlds, including a simpler homogeneous world with whitecap dissipation the only physical interaction, evolving from an arbitrary initial state.

Consider first the case of a single initial wave component with arbitrary spectral amplitude A_1 and vector wavenumber $\mathbf{k}_1$. The corresponding second-order dissipation at vector wavenumber $\mathbf{k}$ is

$$\gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_1)A_1^2;$$

γ must be either negative or zero. But, for $\mathbf{k}$ distinct from $\mathbf{k}_1$, γ cannot be negative because $A(\mathbf{k})$, initially zero, would become negative. It follows that

$$\gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_1) = 0. \tag{A.1}$$

Next consider the case of two initial wave components with arbitrary spectral amplitudes A_1 and A_2 and vector wavenumbers $\mathbf{k}_1$ and $\mathbf{k}_2$. It follows from (A.1) that, for $\mathbf{k}$ distinct from $\mathbf{k}_1$ and $\mathbf{k}_2$, the second-order dissipation at vector wavenumber $\mathbf{k}$ is

$$2\gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)A_1A_2.$$

Again, because $A(\mathbf{k})$ would otherwise become negative,

$$\gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 0. \tag{A.2}$$

We summarize (A.1) and (A.2) by concluding that, for arbitrary $\mathbf{k}$, $\mathbf{k}_1$, and $\mathbf{k}_2$, $\mathbf{k}$ distinct from $\mathbf{k}_1$ and $\mathbf{k}_2$,

$$\gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 0. \tag{A.3}$$

Similarly it can be shown that, for arbitrary $\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$, $\mathbf{k}$ distinct from $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$,

$$\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = 0. \qquad (A.4)$$

It follows that both γ and δ must contain delta functions (or vanish altogether).

A less cumbersome complementary proof that these kernels contain delta functions is provided by K. Hasselmann (private communication). Consider an initial spectrum that is everywhere positive except at vector wavenumber $\mathbf{k}$, where this spectrum is zero. The kernels γ and δ are everywhere nonpositive. Thus, if, for any $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ distinct from $\mathbf{k}$, either $\gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$ or $\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ were nonzero, the rate of change of $A(\mathbf{k})$ would be negative and $A(\mathbf{k})$ would become negative. Therefore, $\gamma(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$ can be nonzero only if $\mathbf{k}$ is equal to either $\mathbf{k}_1$ or $\mathbf{k}_2$, and $\delta(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ can be nonzero only if $\mathbf{k}$ is equal to either $\mathbf{k}_1$, $\mathbf{k}_2$, or $\mathbf{k}_3$. Consequently, these kernels must contain delta functions.

REFERENCES

Courtier, P., and O. Tallagrand, 1987: Variational assimilation of meteorological observations with the adjoint vorticity equation—Part II. Numerical results. *Quart. J. Roy. Meteor. Soc.*, **113**, 1329–1347.

Hasdorff, L., 1976: *Gradient Optimization and Nonlinear Control*, Wiley and Sons, 264 pp.

Hasselmann, K., 1962: On the nonlinear energy transfer in a gravity-wave spectrum. Part 1: General Theory. *J. Fluid Mech.*, **12**, 481–500.

——, 1968: Weak interaction theory of ocean waves. *Basic Develop. Fluid Dyn.*, **2**, 117–182.

——, 1974: On the spectral dissipation of ocean waves due to white-capping. *Bound.-Layer Meteor.*, **6**, 107–127.

——, T. P. Barnett, E. Bouws, H. Carlson, D. E. Cartwright, K. Enke, J. A. Ewing, H. Gienapp, D. E. Hasselmann, P. Kruseman, A. Meerburg, P. Müller, D. J. Olbers, K. Richter, W. Sell, and H. Walden, 1973: Measurements of wind-wave growth and decay during the Joint North Sea Wave Project (JONSWAP). *Dtsch. Hydrogr. Z.*, **A8**, 12, 1–95.

Janssen, P. A. E. M., 1990: Quasi-linear theory of wind wave generation applied to wave forecasting.

Komen, G. T., S. Hasselmann, and K. Hasselmann, 1984: On the existence of a fully developed wind-sea spectrum. *J. Phys. Oceanogr.*, **14**, 1271–1285.

Le Dimet, F. X., and O. Talagrand, 1986: Variational algorithms for analysis and assimilation of meteorological observations: Theoretical aspects. *Tellus*, **38A**, 97–110.

Lewis, J. M., and J. C. Derber, 1985: The use of adjoint equations to solve a variational adjustment problem with advective constraints. *Tellus*, **37A**, 309–322.

Marchuk, G. I., 1974: The numerical solution of problems of atmospheric and oceanic dynamics. Gidrometeoizdat, Leningrad (translated from Russian by Rainbow Systems, Inc., Alexandria, Virginia), 387 pp.

Miles, J. W., 1957: On the generation of waves by shear flows. *J. Fluid Mech.*, **3**, 185–204.

Monbaliu, J., 1991: Wind driven seas: Optimal parameter choice for the wind input term. *Proc. of the Eighth Int. Conf. on Wind Engineering*, London, Ontario, Canada.

Phillips, O. M., 1957: On the generation of waves by turbulent wind. *J. Fluid Mech.*, **2**, 417–445.

——, 1958: The equilibrium range in the spectrum of wind generated waves. *J. Fluid Mech.*, **4**, 426–434.

Snyder, R. L., F. W. Dobson, J. A. Elliott, and R. B. Long, 1981: Array measurements of atmospheric pressure fluctuations above surface gravity waves. *J. Fluid Mech.*, **102**, 1–59.

——, W. L. Neu, R. B. Long, and W. J. P. de Voogt, 1990: A long-range program to parameterize the two-dimensional evolution of the surface-gravity-wave field. Nova University Tech. Rep., 90–2, 30 pp.

——, W. C. Thacker, K. Hasselmann, S. Hasselmann, and G. Barzel, 1993: Implementation of an efficient scheme for calculating nonlinear transfer from wave–wave interactions. *J. Geophys. Res.*, in press.

The SWAMP group, 1985: *Ocean Wave Modeling*, Plenum Press, 256 pp.

Thacker, W. C., 1982: Some computational problems of oceanography. *Proc. Fourth Int. Conf. on Finite Elements in Water Resources*, Springer-Verlag.

——, and R. B. Long, 1988: Fitting dynamics to data. *J. Geophys. Res.*, **93**(C2), 1227–1240.