Waves and Currents

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1. Waves without currents.

Much progress in analyzing the interaction between surface waves and currents has come from assuming that the waves behave locally like plane-waves. That is, the solution for waves in still water with a flat bottom is applied, modified only to account for uniform flow (i.e., translation in x or y). This assumption is typically applied in two opposite limits: i) separate regions of uniform flow and flat bottoms, connected at a thin vertical boundary where suitable matching conditions can be derived; and ii) flow and topography varying slowly compared to the time and length scales of the waves, so the errors are bounded (and quantifiably small).

In view of this, it's worthwhile to review the plane-wave solution for surface waves, so the quantities of interest (and the potential shortcomings) are more or less clear.

Notation is as follows: vector locations and velocities are

$$\mathbf{x} = (x,y,z) \qquad \text{and}$$

$$\mathbf{u} = (u,v,w).$$

Differentiation is denoted several ways:

$$\partial_x \equiv \frac{\partial}{\partial x},\ \partial_y \equiv \frac{\partial}{\partial y},\ \partial_z \equiv \frac{\partial}{\partial z}$$

$$\nabla \equiv (\partial_x, \partial_y, \partial_z)$$

$$\nabla^2 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2$$

$$\nabla_H^2 \equiv \partial_x^2 + \partial_y^2 \qquad \text{and}$$

$$\nabla_k \equiv (\partial_{k_x}, \partial_{k_y}),$$

where z is positive upwards, zero at the mean surface level, and $-h$ at the bottom. The horizontal directions are x and y. Subscripts i or j each take the values x and y.

1.1. Linear solution, dispersion relation.

We assume irrotational, inviscid, homogeneous water, and use a velocity potential:

$$\mathbf{u} = \nabla\phi.$$

In the interior we assume incompressible fluid obeying Bernoulli's law:

$$\nabla^2\phi = 0,$$

$$\partial_t\phi + \tfrac{1}{2}\left|\nabla\phi\right|^2 + \rho^{-1}P + gz = F(t),$$

where $F(t)$ is independent of (x,y,z), and so has no effect on the velocities (but it may affect the overall mean pressure P). Here we can take $F(t) = constant$. The boundary conditions are

$$\partial_z\phi = 0 \text{ at } z = -h \qquad \text{and}$$

$$\rho^{-1}P = -T\nabla_H^2\zeta \text{ at } z = \zeta,$$

where T is surface tension over density.

Now Taylor expand from z=0 and linearize:

$$\rho^{-1}P \approx g\zeta - T\nabla_H^2\zeta \text{ at } z=0, \qquad \text{so}$$

$$\partial_t\phi + (g - T\nabla_H^2)\zeta \approx 0 \text{ there.}$$

We look for solutions of the form $\sin(\mathbf{k}\bullet\mathbf{x}-\sigma t+\delta) \equiv \sin\chi$ (here δ allows arbitrary phase):

$$\nabla^2\phi = (\partial_z^2 - k^2)\phi = 0,$$

where $k \equiv |\mathbf{k}|$. This implies solutions proportional to $e^{\pm kz}$. From the bottom b.c., $\partial_z\phi = 0 \propto \sinh k(h+z)$, so let

$$\phi = B\cosh k(h+z)\sin\chi$$

To relate this to the surface elevation amplitude, $\zeta = a\cos\chi$ at $z = 0$, use the surface kinematic boundary condition

$$\partial_t\zeta + \partial_x\phi\,\partial_x\zeta - \partial_z\phi = 0.$$

So, to lowest order,

$$\zeta \approx \int_{t_0}^{t} \partial_z\phi\Big|_{z=0} dt = B(k/\sigma)\sinh kh\cos\chi.$$

Then

$$a = B(k/\sigma)\sinh kh, \text{ or } B = ac\,\mathrm{csch}\,kh$$

where $c \equiv \sigma/k$ is the phase speed of the waves. Thus,

$$\phi = ac\,\frac{\cosh k(h+z)}{\sinh kh}\sin\chi.$$

Finally, substituting these into the linearized surface boundary condition yields

$$\sigma^2 = (gk + Tk^3)\tanh kh,$$

the dispersion relation with no mean flow. It is not straightforward to continue the expansion of the surface boundary condition to higher order. In deep water, only the small parameter ak enters, and Stokes' expansion is appropriate. In shallow water, trouble can arise if either a/h or a/k^2h^3 is not small (**REF??**). Physically, in shallow water the waves tend to evolve steadily into "shock fronts" or "bores." Various methods for toying with the nonlinear surface condition have been devised, leading to such things as the KdV equations describing solitons.

1.2. Stokes' drift and Wave Momentum.

Next, consider an average over the phase of the waves:

$$\overline{(\,)} \equiv (2\pi)^{-1}\int_0^{2\pi} (\,)\, d\delta.$$

Stokes pointed out that, even with $\overline{u}(z)$=0, the waves induce movement of water parcels (or "Lagrangian drift"). Define a purely oscillatory displacement field, due to the waves, as a function of the position **x** a particle would have in the absence of waves [*Andrews and McIntyre* 1978]:

$$\xi(\mathbf{x}) \equiv \int_{t_0}^{t} \mathbf{u}(\mathbf{x},t)\, dt,$$

choosing t_0 so that $\overline{\xi} \equiv 0$. Let the wave be travelling in the *x*-direction, and let $\xi \equiv (\eta, 0, \zeta)$. Then the Stokes' drift as a function of position (depth) is

$$\begin{aligned} u^s(\mathbf{x}) &= \overline{u(\mathbf{x}+\xi)} - \overline{u(\mathbf{x})} \\ &\approx \overline{\zeta\, \partial_z u} + \overline{\eta\, \partial_x u} = \overline{\zeta\, \partial_z u} - \overline{u\, \partial_x \eta} \\ &= \overline{\zeta\, \partial_z u} + \overline{u\, \partial_z \zeta} = \partial_z(\overline{\zeta u}) \end{aligned}$$

(using $\nabla \bullet \xi = 0$ and $\overline{\eta u} = 0$). The total mass flux due to the waves, rotating coordinates so the wave direction is arbitrary, is then

$$\mathbf{M} \equiv \overline{\int_{-h}^{0} \rho \mathbf{u}^s(z)\, dz} \approx \rho\, \overline{\zeta \mathbf{u}}\,|_{z=0}$$

per unit area of surface. For surface waves, this also describes the net momentum associated with the wave propagation, so there is no confusion in referring to this also as the "wave momentum" (but see *Andrews and McIntyre* [1978] for another point of view on this). From the above solution,

$$\overline{\zeta \mathbf{u}} = \mathbf{k}\, \frac{kB^2}{4\sigma} \sinh 2k(h+z) \qquad \text{and so}$$

$$\begin{aligned} \mathbf{u}^s(z) &= \mathbf{k}\, \frac{k^2 B^2}{2\sigma} \cosh 2k(h+z) \\ &= \mathbf{k}\sigma^{-1}\left(\overline{u^2} + \overline{w^2}\right) \end{aligned}$$

Note that in the shallow water limit, $u^s \rightarrow$ *constant*. It does not vanish at $z=-h$: the ηu_x term dominates in this case. Also, this solution does not take into account additional boundary layer streaming shown to occur by [*Longuet-Higgins* 1953].

1.3. Energy

The energy density of the wavetrain is the sum of kinetic and potential energies, $E \equiv K+V$. The potential energy, including the stretching potential of surface tension, is (to lowest order)

$$\begin{aligned} V &= \overline{\int_0^{\zeta} \rho g z\, dz} + \rho T\left[\overline{\left(1+|\nabla\zeta|^2\right)^{1/2}} - 1\right] \\ &\approx \tfrac{1}{2}\rho\left(g+Tk^2\right)\overline{\zeta^2} = \tfrac{1}{4}\rho k B^2 \sinh 2kh \end{aligned}$$

and the kinetic energy is

$$\begin{aligned} K &= \tfrac{1}{2}\rho\overline{\int_{-h}^{\zeta} (u^2+w^2)\, dz} \approx \tfrac{1}{2}\rho\int_{-h}^{0}\left(\overline{u^2}+\overline{w^2}\right) dz \\ &= \tfrac{1}{4}\rho k^2 B^2 \int_{-h}^{0} \cosh 2k(h+z)\, dz \\ &= \tfrac{1}{4}\rho k B^2 \sinh 2kh \end{aligned}$$

so the net energy is

$$E = \tfrac{1}{2}\rho k B^2 \sinh 2kh = c\,|\mathbf{M}| = \rho(g+Tk^2)\overline{\zeta^2}.$$

1.4. Radiation stress; momentum flux.

Waves carry momentum **M** along at the group velocity, $\mathbf{c}^g \equiv \nabla_k \sigma$. This momentum flux, together with a wave-induced pressure term, form the "radiation stress" defined by *Longuet-Higgins and Steward,* [1964]: S_{ij} is defined as the total excess flux of *i* momentum in the *j* direction in the presence of the waves, compared to that their absence (here *i* and *j*take the values *x* or *y*, denoting horizontal components of S, etc.).

For convenience, let the waves be aligned with the *x*-axis. Then there is no velocity in the *y*-direction, and the surface slopes only in the *x*-direction, at an angle $\theta \equiv \arctan(\partial_x \zeta)$. The transverse (*y*) flux of transverse wave momentum results from both the wave-induced pressure and surface tension acting across an increased length of surface:

$$\begin{aligned} S_{yy} &\equiv \overline{\int_{-h}^{\zeta} (p-p^m)\, dz} + \rho T(1-\overline{\sec\theta}) \\ &\approx \overline{\int_{-h}^{\zeta} (p-p^m)\, dz} - \tfrac{1}{2}\rho T\overline{(\partial_x \zeta)^2}. \end{aligned}$$

The net flux of *x*-momentum in the *x*-direction due to the waves includes a contribution from the horizontal velocity, while the horizontal component of surface tension is reduced due to the angle of the surface:

$$\begin{aligned} S_{xx} &\equiv \overline{\int_{-h}^{\zeta} (p-p^m+\rho u_x^2)\, dz} + \rho T(1-\overline{\cos\theta}) \\ &= S_{yy} + \int_{-h}^{\zeta} \overline{\rho u^2}\, dz + \rho T(\overline{\sec\theta} - \overline{\cos\theta}) \\ &\approx S_{yy} + \int_{-h}^{0} \overline{\rho u^2}\, dz + \rho T\overline{(\partial_x \zeta)^2}. \end{aligned}$$

The other two components are $S_{xy} = S_{yx} = 0$. This diagonal form for S_{ij} is easily rotated to accommodate arbitrary wave directions.

The mean pressure $\overline{p}$ is not the same as the pressure which would exist without the waves, $p^m \equiv \rho g z$. On the average, the vertical momentum flux must be just enough to hold up the weight of water above (LHS64):

$$\overline{p(z)} + \rho\overline{w^2(z)} = \rho g z \equiv p^m(z)$$

(at second order, the surface tension contribution averages out). So the mean pressure contributes

$$\begin{aligned} \int_{-h}^{\zeta} (\overline{p}-p^m)\, dz &\approx \int_{-h}^{0} -\rho\overline{w^2\, dz} \\ &= -\tfrac{1}{2}\rho k B^2(\sinh 2kh - 2kh). \end{aligned}$$

Near the surface, the fluctuating part of the pressure is

$$\rho^{-1} p(z) = -gz - \partial_t \phi$$

$$= -gz + (g - T\nabla_H^2)\zeta - \zeta\,\partial_z \partial_t \phi + \ldots$$

so, to lowest order, the fluctuating pressure and tension terms together contribute

$$\rho^{-1}\overline{\int_0^\zeta p\,dz} - \tfrac{1}{2}T\overline{(\partial_x \zeta)^2} \approx \tfrac{1}{2}g\overline{\zeta^2} - T\overline{\zeta\partial_x^2\zeta} - \tfrac{1}{2}T\overline{(\partial_x\zeta)^2}$$

$$= \tfrac{1}{2}(g + Tk^2)\overline{\zeta^2} = \tfrac{1}{8}kB^2 \sinh 2kh$$

to S_{yy}. The resulting total simplifies to

$$S_{yy} = h\left(\frac{Ek}{\sinh 2kh}\right) = h\left(\tfrac{1}{2}\rho\left(\overline{u^2} - \overline{w^2}\right)\right) \equiv hJ$$

(note that $\left(u^2 - w^2\right)$ stays constant with depth). The remaining terms in S_{xx} reduce to (after some algebra)

$$\int_{-h}^{0} \rho\overline{u^2}\,dz + \rho T\overline{(\partial_x\zeta)^2} = M_x c_x^g$$

where $c_x^g \equiv \partial_{k_x}\sigma$ is the group velocity and $\mathbf{M}$ is the wave momentum, as defined above (here, both $\mathbf{M}$ and $\mathbf{c}^g$ are aligned with the x-axis). The depth distribution of the term $P_i c_j^g$ is simply that of $\mathbf{M}$; i.e., like $\mathbf{u}^s(z)$.

Rotating these results to arbitrary orientation yields

$$S_{ij} = M_i c_j^g + hJ\,\delta_{ij}$$

where δ_{ij} is the Kronecker delta function. Note that, because $\mathbf{M}$ and $\mathbf{c}^g$ are parallel for surface waves, this form is symmetric. In deep water, the pressure-like term hJ is negligible, leaving just the first term [*Garrett and Smith* 1976].

1.5. Other solutions.

It was mentioned in the beginning that small scale changes in the depth or current are often treated by matching at a boundary between two uniform regions. For this matching, other solutions of the basic equations are often required. In particular, there is a class of solutions which are oscillatory in depth and decay exponentially in one or both horizontal directions. Picking a set of functions which obey the top and bottom conditions, and which are oscillatory in one direction (y) but decay exponentially in the direction perpendicular to the boundary (x), one obtains an infinite set of solutions of the form [*Miles* 1967]

$$\Psi_n(z) = \left(\frac{2k_n}{2k_n h + \sin 2k_n h}\right)^{1/2} \cos\left(k_n(h+z) + qy - \sigma t\right) e^{\pm r_n x},$$

where the k_n are solutions of

$$gk_n \tan k_n h = -\sigma^2,$$

and

$$r_n = \left(k_n^2 + q^2\right)^{1/2}.$$

These are constructed to be orthonormal over the z interval 0 to $-h$. Note that these modes effectively propagate up or down at some angle on the vertical plane $x = constant$. These modes are needed to match across a step [*Miles* 1967] or a vortex sheet [*Evans* 1975, *Smith* 1983, *Smith* 1987].

The current or depth change can be such that no free wave exists on one side of the boundary (i.e., the y wavenumber p is too large to admit a real solution for k in the regular dispersion relation). In this case, the appropriate primary solution in that region is exponential in both x and z, and there is total reflection. To my knowledge, no one has ever used the solutions which are oscillatory in depth but exponential in both x and y.

2. The influence of currents on waves.

2.1. Uniform flow

What changes are necessary to adapt the above to waves on a uniform current? Since the flow is inviscid, there is no change in (e.g.) the bottom conditions. However, such things as the encounter frequency and the energy with waves vs. without do change. With a uniform flow velocity $\mathbf{U}$, the new quantities (primed) can be written in terms of the old ones; e.g.,

$$\omega \equiv \sigma' = \sigma + \mathbf{k}\cdot\mathbf{U}, \qquad \text{and}$$

$$E' = E + \mathbf{M}\cdot\mathbf{U}.$$

At this point it is convenient to introduce "wave action" A, defined so that

$$E = \sigma A \text{ and } \mathbf{M} = \mathbf{k}A$$

It is an easy matter to verify that the action is invariant:

$$E' = \omega A' = (\sigma + \mathbf{k}\cdot\mathbf{U})A' = \sigma A' + \mathbf{M}'\cdot\mathbf{U}.$$

Now, since we know the momentum $\mathbf{M}$ is invariant, it follows that A' must be the same as A. Of course this in no way proves that action is conserved with changes in the flow. In general, for example, one cannot conserve both the wave momentum and wave action. However, the conservation of action can be shown to hold for a very wide class of problems, including most surface wave problems [e.g., *Hayes* 1970, *Whitham,* 1974]. In these cases, conservation of wave action is a tool which helps in determining the exchanges of momentum and energy between the waves and the mean flow. A rule of thumb is that action is conserved when the phase of the waves can be changed without changing the interaction. To violate this rule requires an interaction which is confined in both time and space, such as a stone dropping into a pond (which generally creates new wave action).

Next consider the radiation stress. Let the total velocity be $\mathbf{u}' = \mathbf{U} + \mathbf{u}$. Then we obtain

$$S_{ij}' \equiv \overline{\int_{-h}^{\zeta}(p + \rho u_i' u_j')\,dz} - \int_{-h}^{0}(p^m + \rho U_i U_j)\,dz$$

$$= S_{ij} + U_i M_j + U_j M_i$$

where we have assumed that the uniform mean flow $\mathbf{U}$ is independent of depth. It is often more convenient to leave S_{ij}

split up into the intrinsic value plus the two explicit advection terms, as in the second line above.

Finally, note that the S_{yy} term can be written in terms of wave action in the form

$$hJ = hA\frac{\partial\sigma}{\partial h}, \text{ or } J = A\partial_h\sigma.$$

2.2. Slowly varying currents and depth.

At last we address the problem of waves propagating in a slowly varying environment. Two classes of problems arise: i) With no reflection, the "WKB" approximation applies. In this case, conservation of action is sufficient to determine the outcome. ii) When reflection can occur, information in addition to conservation of action is required, to determine the amount of reflection.

First, the solution should be self-consistent; this leads to kinematic conditions. The phase function χ must be continuous, and so

$$\nabla\times\mathbf{k} = 0, \text{ and} \tag{2.2.1}$$

$$\partial_t\mathbf{k} + \nabla\omega = \partial_t\mathbf{k} + \nabla(\sigma + \mathbf{k}\cdot\mathbf{U}) = 0, \tag{2.2.2}$$

where the "intrinsic frequency" σ is a predetermined function of $(\mathbf{k},\mathbf{x},t)$ or, in the present case, of $(\mathbf{k},h,g,T)$.

The mean flow field must also be a self-consistent solution of the appropriate equations, which may in general be rotational. To make the problem tractable, we assume the mean flow to be large scale; i.e., assume σ is much larger than both $\nabla\times\mathbf{U}$ and $\nabla_{\mathrm{H}}\cdot\mathbf{U}$.

Evaluation of the effect of currents on waves is facilitated by conservation of wave action [*Bretherton and Garrett* 1968, *Hayes* 1970, *Whitham* 1974]. In the absence of generation or dissipation, this takes the form

$$\partial_t A + \nabla_{\mathrm{H}}\cdot\left(A(\mathbf{U}+\mathbf{c}^g)\right) = 0. \tag{2.2.3}$$

The proof with broadest applicability is provided by *Whitham* [1974], who demonstrates that action conservation holds whenever the Lagrangian density can be described in a quadratic form. This principle holds for unimodal incident waves; note that, in general, the waves vary in wavenumber and frequency over $\mathbf{x}$ and t. To extend this to packets of waves, with finite spatial extent, consider a sum of components over some finite area of the wavenumber plane, sufficiently small that the group velocity doesn't vary significantly compared to the changes induced by the varying medium.

It is often useful to consider the evolution of the spectral density of action, N. Conceptually, for a "wave packet" as alluded to above, the action density within the packet (in terms of both $\mathbf{k}$ and $\mathbf{x}$) would be $N(\mathbf{k}, \mathbf{x}, t) \propto A/b$, where b is an element of area in $\mathbf{k}$-space representing the (2-dimensional) "bandwidth" of the packet, surrounding the center value $\mathbf{k}$, which is itself a function of $(\mathbf{x}, t)$. The wavenumber evolution (2.2.2) affects both the center wavenumber $\mathbf{k}$ and the bandwidth b of the packets. For an elemental change in $\mathbf{k}$ (to $\mathbf{k} + \mathbf{dk}$, say), (2.2.2) yields

$$\partial_t(\mathbf{dk}) + \nabla_{\mathrm{H}}\left(\mathbf{dk}\cdot(\mathbf{U}+\mathbf{c}^g)\right) = 0 \tag{2.2.4}$$

This corresponds to keeping the number of waves in a given packet constant as the total size and orientation varies. A convenient measure of b is given by the cross product of two such elemental displacements from $\mathbf{k}$; e.g., let b be the z-component of $\mathbf{dk}_x\times\mathbf{dk}_y$. Its then simple to show that "bandwidth flux" is conserved along rays:

$$\partial_t b + \nabla_{\mathrm{H}}\left(b(\mathbf{U}+\mathbf{c}^g)\right) = 0. \tag{2.2.5}$$

Again, this corresponds to keeping the number of wave crests in the packet constant. Using $N' \propto A/b$, (2.2.3) and (2.2.5) combine to yield

$$d_t N' \equiv \left(\partial_t + (\mathbf{U}+\mathbf{c}^g)\cdot\nabla\right)N' = 0, \tag{2.2.6}$$

where d_t is the "ray-tracing" or "packet-following" total derivative. The packet following action density N' is constant (not conserved) along rays. But note that N' is the action density at the varying wavenumber $\mathbf{k}(\mathbf{x},t)$ of the packet, so (2.2.6) is Eulerian in space and time but Lagrangian in wavenumber. To convert back to a fixed (Eulerian) wavenumber, we must account for the variation in wavenumber along a wave ray. This leads to a general equation for the evolution of action density, $N(\mathbf{k},\mathbf{x},t)$, at a fixed wavenumber $\mathbf{k}$:

$$\left(d_t + (d_t\mathbf{k})\cdot\nabla_k\right)N = 0. \tag{2.2.7}$$

In practice, it is important to recall that the limits of integration for this spectrum of waves now become functions of the medium. In addition, 2.2.7 implies that the directional form also must vary. At times this can lead to confusion.

It is appealing to put this into a more symmetric form (though not necessarily more useful) using the "wave rays," $\mathbf{x}(t)$ (let $\partial_t\mathbf{x}=0$ and $\partial_j x_i=\delta_{ij}$):

$$d_t\mathbf{x} \equiv \mathbf{U}+\mathbf{c}^g. \tag{2.2.8}$$

Then (2.2.7) can be written

$$\left(\partial_t + (d_t\mathbf{x})\cdot\nabla_H + (d_t\mathbf{k})\cdot\nabla_k\right)N = 0. \tag{2.2.9a}$$

Finally, to account for growth and dissipation, one can add a "Miles-like" growth term GN to the right side of (2.2.7), and subtract a dissipation term of the same form (say), $-DN$:

$$\left(\partial_t + (d_t\mathbf{x})\cdot\nabla_H + (d_t\mathbf{k})\cdot\nabla_k\right)N = (G-D)N. \tag{2.2.9b}$$

In general, D must conceal some additional dependence on N, so that a stable equilibrium level is defined. Alternatively, the equivalent term $(G\text{-}D)A$ can be placed on the right hand side of 2.2.3.

3. The influence of waves on currents.

Once the waves and currents have been specified adequately (i.e., to first order in ak for the fluctuating parts, and to second order for the mean quantities), the net effect on the mean flow due to the interaction with the waves may be evaluated. The exposition roughly follows *Garrett* [1976], with extension to include finite depth and surface tension. The total vertically-integrated momentum budget is derived, and divided into mean and wave quantities. The wave momentum budget is deduced from the equations for action and wavenumber conservation. By subtracting the waves' momentum budget from the total, the net effect of the waves on the mean (larger scale) momentum budget is deduced.

For incompressible, inviscid flow in a non-rotating frame of reference, the horizontal momentum equation can be written

$$\partial_t(\rho u_i)+\partial_j(\rho u_i u_j+p\delta_{ij})+\partial_z(\rho u_i w)=0, \tag{3.1}$$

where subscripts *i, j* refer to the two horizontal dimensions, u_i is horizontal velocity, ρ is the density of the water, and p is pressure. The vertical components *z* and *w* are treated separately from the horizontal ones, to facilitate vertical integration. Here and throughout, the summation convention is used: repeated indices are summed over the two horizontal components. The kinematic boundary conditions at the free surface ζ and bottom -*h* can be written:

$$\partial_t\zeta+u_i\partial_i\zeta-w=0 \text{ at } z=\zeta, \tag{3.2}$$

$$\partial_t h+u_i\partial_i h+w=0 \text{ at } z=-h, \tag{3.3}$$

Here, *z* is positive upwards as before, but -*h* may now be a material fluid boundary below which the wave motion is negligible (in deep water), or may be the actual bottom. For example, in deep water the "wave layer" between the surface and -*h* may be thin compared to other motions of interest.

Vertical integration of (3.1) results in

$$\begin{aligned}\partial_t\left(\int_{-h}^{\zeta}\rho u_i\,dz\right)+\partial_j\left(\int_{-h}^{\zeta}(\rho u_i u_j+p\delta_{ij})\,dz\right)\\ = p\partial_i\zeta\,\Big|_{z=\zeta}+p\partial_i h\,\Big|_{z=-h}+\tau_i^s+\tau_i^b\end{aligned} \tag{3.4}$$

The terms on the right are from the boundary conditions. For example, the surface pressure term can be regarded as supplying input to the waves from wind. The pressure working on the mean flow is generally subsumed into the stress terms.

Next, the flow is separated into mean and wave components: $u_i=\overline{u_i}+u_i'$. The vertical particle displacements $\zeta(z)$ from reference position *z* are defined throughout the fluid as before (with a kinematic condition like eq. 3.2), but are also divided into $\overline{\zeta}$ and ζ' (the mean flow can also involve surface displacements). The pressure is separated into a part, p^m, which would exist without the waves, and a wave-part, p^w. As noted above, p^w is not necessarily zero. Averaging 3.4 over the waves, and dividing into the wave and mean quantities as described, we find

$$\begin{aligned}&\partial_t T_i+\partial_j\left(\int_{-h}^{\overline{\zeta}}(\rho\overline{u}_i\overline{u}_j+p^m\delta_{ij})\,dz\right)+\\ &\partial_t M_i+\partial_j\left(S_{ij}+M_iU_j^a+M_jU_i^a\right)=GM_i+\tau_i^m\end{aligned} \tag{3.5}$$

where

$$T_i\equiv\int_{-h}^{\overline{\zeta}}\rho\overline{u}_i\,dz \tag{3.6}$$

is the mass transport (momentum) of the mean flow,

$$M_i\equiv\overline{\int_{-h}^{\zeta}\rho u_i'\,dz} \tag{3.7}$$

is the intrinsic wave momentum,

$$S_{ij}\equiv\overline{\int_{-h}^{\zeta}(\rho u_i'u_j'+p^w\delta_{ij})dz} \tag{3.8}$$

is the intrinsic radiation stress (as above), and

$$U_i^a\equiv M_j^{-1}\int_{-h}^{\overline{\zeta}}\rho\overline{u}_i\,\partial_z\overline{(\zeta' u_j')}\,dz \tag{3.9}$$

is the horizontal advection velocity of the waves, defined by a wave-weighted integral of the mean flow. This last allows us to extend the results to mean flows which vary weakly with depth. For many problems, the value of U_i^a arising from either component of u_j' and M_j is the same, as long as the component chosen isn't zero, so that this definition of U_i^a is unambiguous (but see discussion below). The term GM_i is the surface pressure-working on the waves, represented here as a Miles-like growth rate, and the term τ_i^m represents the net stress (top and bottom) exerted directly on the mean flow.

Some discussion of the dispersion relation for waves in the presence of vertical shear should help clarify what is meant by "advection velocity" and "intrinsic group velocity" (which is needed for evaluation of S_{ij}). For example, *Valenzuela* [1976] has solved for wave phase velocities in opposing directions as eigenvalues of the linearized equations for wave-like perturbations of two-dimensional shear-flow along the air/water interface. The average real part of the phase velocity of the two oppositely directed wave solutions is a convenient definition of the "advection velocity": moving with this speed, an observer sees waves propagating in opposite directions with equal phase speeds for equal wavelengths. Likewise, the phase-speeds (and group velocities) seen by such an observer are defined as "intrinsic" to waves of the given wavelength. In the small shear limit, the intrinsic frequency approaches the classical value given above (except we shall allow g', the apparent gravity including vertical acceleration by the mean flow, to replace *g*). As the shear (windspeed) increases, the intrinsic frequency and phase-speed decrease. This can be regarded as largely the result of a "Bernoulli effect" on the pressure field: low pressures induced over the crests and highs over the troughs act to reduce the net restoring force due to gravity and surface tension. Both the slowing of the intrinsic phase speeds and the value of the advection velocity of the waves were convincingly verified in wave-tank experiments by *Plant and Wright* [1980]. Over the range of conditions covered in that experiment, the advection velocity defined by eq. 3.9 is also found to be in agreement, although a rigorous relationship between this and the values derived by *Valenzuela [1976]* or by *Plant and Wright* [1980] has not been explored. For yet another view on this subject, see *Henyey et al.* [1988]. In any case, eq. 3.9 is adopted here, and this advection velocity is assumed to apply in the wave action equation as well as for the momentum. In deep water, eq. 3.9 reduces to

$$U_j^a\approx 2k\int_{-h}^{\overline{\zeta}}\overline{u}_j(z-\overline{\zeta})e^{2k(z-\overline{\zeta})}dz \tag{3.10}$$

[*Stewart and Joy* 1974]. Since U_j^a is a function of the wavenumber k, the variation with k should be incorporated into the group velocity, $\mathbf{c}_i^g = \partial_{k_i}\sigma + k_j \partial_{k_i} U_j^a$. Although the phase-speeds of oppositely directed waves are equal in the intrinsic frame (as defined here), the group velocities are different. In the presence of vertical shear, there is no frame in which both phase-speed and group velocities are symmetric. As noted by *Henyey, et al.* [1988], resonant interactions could be sensitive to the modification of group velocity: In the weak interaction limit, transfer rates are proportional to $(\partial_k c^g)^{-1}$, and the modified c^g is a much flatter function of k on the gravity side of the gravity-capillary minimum for downwind travelling waves.

Evolution of the wave momentum can be evaluated using the identity $\mathbf{M} = A\mathbf{k}$, as noted above. Here, we use the non-spectral form for action, since we wish to deduce the total effect on the mean flow. The wavenumber equation can be re-written

$$\partial_t k_i + (c_j^g + U_j^a)\partial_j k_i = -k_j \partial_i U_j^a - \partial_m \sigma \, \partial_i m \qquad (3.11)$$

where the sum over variables m accounts for variations in the medium other than advection that affect the dispersion relation, such as depth, apparent gravity, or surface tension. Combining this with action, the wave momentum is governed by

$$\begin{aligned} &\partial_t M_i + \partial_j\left(M_i (c_j^g + U_j^a) \right) = \\ &(G - D) M_i - M_j \partial_i U_j^a - A \partial_m \sigma \partial_i m \end{aligned} \qquad (3.12)$$

Subtracting this from the total momentum budget, the net effect on the mean momentum is found to be

$$\partial_t T_i + \partial_j\left(\int_{-h}^{\bar{\zeta}} (\rho \bar{u}_i \bar{u}_j + p^m \delta_{ij})\, dz \right) = F_i^w + \tau_i^m \qquad (3.13)$$

where the "wave force" F_i^w is given by

$$\begin{aligned} F_i^w = {} & M_i D + M_j\left(\partial_i U_j^a - \partial_j U_i^a\right) - U_i^a \partial_j M_j \\ & -(h + \bar{\zeta})\partial_i J + A\, \partial_m \sigma \, \partial_i m \end{aligned} \qquad (3.14)$$

as identified by *Garrett* [1976] for deep water gravity waves (top line only). Here, use was made of the identity $J = A\partial_h \sigma$, and an "$m=h$" term was combined with the hJ term of the radiation stress. Additional variations in the medium (vertical acceleration, surface tension modifications, etc.) are still subsumed into m.

Finally, the surface kinematic condition (3.2) is Taylor-expanded in ζ' about $\bar{\zeta}$ and then averaged, leading to [*Hasselmann* 1971]:

$$\partial_t \bar{\zeta} + \bar{u}_j \partial_j \bar{\zeta} - w = -\rho^{-1} \partial_j M_j \text{ at } z = \bar{\zeta}. \qquad (3.15)$$

Physically, the variations in wave-induced mass-flux act as sources and sinks of fluid at mean surface, $\bar{\zeta}$. It is worth pointing out that for nonzero $\bar{\zeta}$, the mass source at the surface contributes to the potential energy of the mean flow. This is the essential point raised by *Hasselmann* [1971] in his refutation of the "maser mechanism" for long-wave growth [*Longuet-Higgins* 1969]. This virtual mass source at the surface can be important elsewhere as well. For waves of varying amplitude, the "forced long wave" discussed by *Longuet-Higgins and Stewart* [1962, 1964] can be regarded as arising entirely from this virtual mass source, as we shall see.

Alternatively, when this potential energy transfer is not important, the top and bottom conditions can be combined into an equation for conservation of total mass:

$$\partial_t\left(\int_{-h}^{\bar{\zeta}} \rho \, dz \right) + \partial_j T_j + \partial_j M_j = 0 \qquad (3.16)$$

4. An example: forced long waves.

Consider the problem described by *Longuet-Higgins and Stewart* [1962; henceforth LHS62] concerning the mean motion induced by groups of waves propagating into still water. The analysis here most resembles "method 2" of that paper.

In this problem, the mean current remains irrotational, so the term $\mathbf{M}\times(\nabla\times\mathbf{U}^a)$ is zero. Also, assume that U^a/c^g and $\bar{\zeta}/h$ are smaller than $(ak)^2$, so all the terms in $\mathbf{F}^w$ involving U^a are small compared to the "pressure stress" term, $h\nabla J$. Line up the x-axis with $\mathbf{M}$, so that subscripts x and y are unnecessary (e.g., $M \equiv M_x \equiv \mathbf{M}$).

Consider first the case where the forced wave is a shallow water wave. Then $\bar{u} \equiv U$ is uniform from $\bar{\zeta}$ to -h, which is assumed to be constant. To lowest order in the mean flow variations, the momentum balance becomes

$$\partial_t(hU) + \partial_x(hg\bar{\zeta}) = h\partial_x(J/\rho), \qquad (4.1)$$

where the mean pressure is assumed to be hydrostatic: $p^m = g\bar{\zeta}$. We seek a solution propagating with the wave group velocity c^g, so ∂_t can be replaced by $-c^g\partial_x$. This results in

$$c^g U - g\bar{\zeta} = J/\rho, \qquad (4.2)$$

where the constant of integration has been chosen so that $U = \bar{\zeta} = 0$ when there are no waves (this is just equation 3.20 of LHS62). The first order mass conservation equation is

$$hU - c^g \bar{\zeta} = -M/\rho, \qquad (4.3)$$

where the constant of integration is again zero. Combining these equations,

$$U = -\frac{g\, M/\rho + c^g\, J/\rho}{gh - (c^g)^2}, \text{ and} \qquad (4.4)$$

$$\bar{\zeta} = -\frac{c^g\, M/\rho + h\, J/\rho}{gh - (c^g)^2}, \qquad (4.5)$$

as in LHS62.

Now consider the case where the forced wave is not a shallow water wave. Since the length of the groups must be somewhat longer than that of the individual waves, it seems safe to assume in this case that the waves comprising the group are deep-water waves. In fact, let's assume that the waves are confined to a thin surface layer, $\bar{\zeta}$ to $-h$, within which the forced wave velocity is uniform: $\bar{u}(-h) \approx \bar{u}(0) \equiv U$. Note also that $J=0$ for deep water waves. Then

$$c^g U - g\bar{\zeta} \approx 0. \qquad (4.6)$$

Integrating the mass conservation from the actual bottom at z=–H,

$$-c^g\bar{\zeta}H + \int_{-H}^{\bar{\zeta}} \bar{u}\, dz + M/\rho = constant \equiv 0 \tag{4.7}$$

(again choosing the frame of reference in which $\bar{u}=\bar{\zeta}=0$ when M=0). As in LHS62, we observe that the above admits a simple solution if the groups force a simple harmonic long wave with wavenumber Δk (say). In this case, the forced motion has the same general form as a free wave solution; i.e., $\bar{u}$ is proportional to $\cosh(z\Delta k)$. Let θ be defined so that

$$\int_{-H}^{0} \bar{u}\, dz = \left(\frac{\tanh(H\Delta k)}{H\Delta k}\right) UH \equiv \theta UH, \tag{4.8}$$

and we recover the equivalent of equation (3.19) or (3.29) of LHS62 (except that 3.29 has a misplaced theta; also, the velocity in LHS62 is given in terms of the vertical average, $|\bar{u} \equiv \theta U$):

$$U \approx -\frac{gM/\rho}{\theta gH - (c^g)^2} \tag{4.9}$$

and

$$\bar{\zeta} \approx -\frac{c^g\, M/\rho}{\theta gH - (c^g)^2}. \tag{4.10}$$

Note that J and indeed all of $\mathbf{F}^w$ is negligible here. The long forced wave results entirely from the mass source/sink at the surface as the waves vary in size.

Finally, consider the general case. To obtain a simple answer, we need only assume that the vertical structure of the forced response in $\bar{u}$ and p^m is the same (as is the case for a simple harmonic forced wave, and indeed for any surface-wave like potential flow response). Then define $U \equiv \bar{u}(z{=}0)$, and define theta such that

$$\int_{-h}^{\bar{\zeta}} \bar{u}\, dz \equiv \theta(h+\bar{\zeta})U. \tag{4.11}$$

Then by assumption we also have

$$\int_{-h}^{\bar{\zeta}} p^m dz = \theta(h+\bar{\zeta})g\bar{\zeta}. \tag{4.12}$$

To lowest order in the mean quantities, the momentum equation becomes

$$\partial_t(\theta hU) + \partial_x(\theta hg\bar{\zeta}) = -(h/\rho)\partial_x J \tag{4.13}$$

or, using $\partial_t \to -c^g$, integrating (with c^g and h assumed to be independent of x), and dividing by $\theta\, h$, this becomes

$$c^g U - g\bar{\zeta} = J/\rho\theta. \tag{4.14}$$

(Again, integration constants are chosen so there is no motion in the absence of waves.) Mass conservation yields

$$\partial_t\bar{\zeta} + \partial_x(\theta hU) \approx -\rho^{-1}\partial_x M \tag{4.15}$$

or

$$c^g\bar{\zeta} - \theta hU = M/\rho. \tag{4.16}$$

Combining these, we find

$$U = -\frac{g\, M/\rho + c^g\, J/\rho\theta}{\theta gh - (c^g)^2} \tag{4.17}$$

and

$$\bar{\zeta} = -\frac{c^g\, M/\rho + h\, J/\rho}{\theta gh - (c^g)^2}. \tag{4.18}$$

It is seen that this combines and generalizes the previous results: for a simple harmonic forced wave, θ is given by 4.8, and J=0; for the shallow water case, θ=1 and 4.4 and 4.5 are recovered. It appears that this can be extended to arbitrarily shaped groups by Fourier-expanding the forcing terms M and J, and (to lowest order in $\bar{u}$ and $\bar{\zeta}$) summing the results.

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