THE RANGE OF EXISTENCE OF RAYLEIGH AND STONELEY WAVES

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1. *Introduction.*—The problem of the range of existence of surface waves in a single superficial layer has been the subject of many publications. As the period equation of Rayleigh waves in a layer is rather complicated an algebraical analysis of this equation, which is of course necessary to solve this problem, has only been carried out in two limiting cases, namely $T/L \gtrsim 0$ and $T/L \gtrsim \infty$ (where T = the thickness of the layer and $L/2\pi$ = the length of transverse waves in that layer).

Many years ago Bromwich * determined the influence of a thin layer on the Rayleigh waves in the underlying medium, thereby assuming compressibility of both media. These calculations have been extended by many writers to compressible materials of different kinds (limestone, clay, granite, etc. †).

The alternative case $T/L \gtrsim \infty$ was investigated by Love ‡ for incompressible media and afterwards by Stoneley § without this assumption. If $T/L \gtrsim \infty$ the period equation splits up into two equations: the Rayleigh equation of the upper medium and the equation for the Stoneley wave system. This second equation can only be solved for certain values of the material constants of the two media; this wave system is therefore not always possible. Now it is important to know exactly in which circumstances the Stoneley wave system can exist, as the wave systems which occur in a layer of finite thickness are closely connected with the systems existing when $T/L \gtrsim \infty$.

2. *The range of existence of Stoneley waves.*—Both Love and Stoneley stated that it was only possible for such a wave system to exist if Wiecherts' condition ($\beta = \sqrt{\mu/\rho} = \beta' = \sqrt{\mu'/\rho'}$) was almost fulfilled. The conclusion of these authors however was not stated precisely; a general algebraical treatment of the Stoneley wave equation shows that these waves can exist for every value of μ/μ', ρ/ρ' which lies between the two curves drawn in Figs. 1 and 2. The equations of these curves are:

$$(\mu/\mu')^2\{(2-\omega)^2 - 4\sqrt{(1-\nu\omega)(1-\omega)}\} - (\mu/\mu')\{2(2-\omega) - 4\sqrt{(1-\nu\omega)(1-\omega)} + \omega\sqrt{(1-\nu')(1-\omega)}\} + \{1 - \sqrt{(1-\nu\omega)1-\omega)}\} = 0 \qquad (1)$$

for curve A, and

$$(\mu/\mu')^2\{1 - \sqrt{(1-1/\omega)(1-\nu'/\omega)}\} - (\mu/\mu')\{2(2-1/\omega) - 4\sqrt{(1-1/\omega)(1-\nu'/\omega)} + 1/\omega\sqrt{(1-\nu)(1-1/\omega)}\} + \{(2-1/\omega)^2 - 4\sqrt{(1-1/\omega)(1-\nu'/\omega)}\} = 0 \qquad (2)$$

* T. J. Bromwich, *Proc. Lond. math. Soc.*, **30**, 1899.
† A. W. Lee, *Geophys. Suppl.*, **3**, 86, 1932.
‡ A. E. H. Love, *Some problems of Geodynamics*, p. 165, 1911.
§ R. Stoneley, *Proc. Roy. Soc.* A, **106**, 421, 1927.

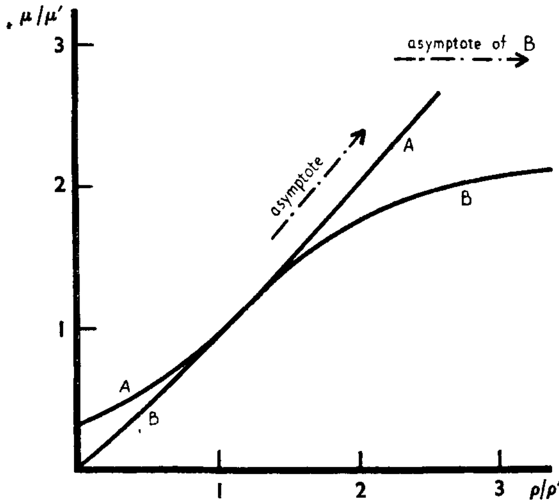

FIG. 1.—*Stoneley boundary curves* ($\lambda/\mu = \lambda'/\mu' = 1$).

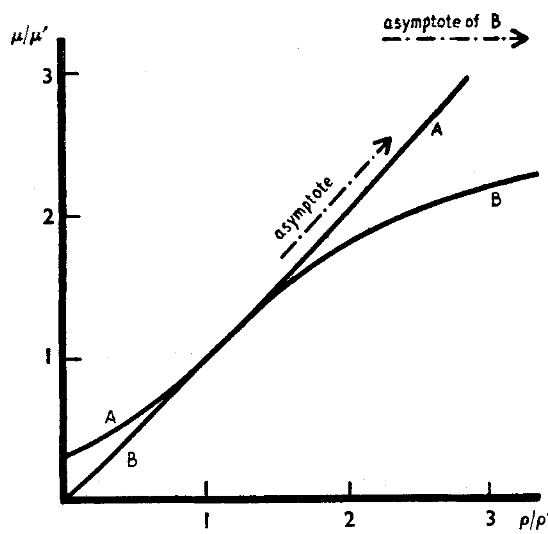

FIG. 2.—*Stoneley boundary curves* ($\lambda = \lambda' = \infty$).

for curve B; where $\omega = (\beta'/\beta)^2$, $\nu = \mu/(\lambda + 2\mu)$ (layer) and $\nu' = \mu'/(\lambda' + 2\mu')$ (subjacent medium).

In Tables I and II some solutions of these equations are given, assuming $\lambda/\mu = \lambda'/\mu' = 1$ (Table I) or $\lambda = \lambda' = \infty$ (Table II).

Curve A has an asymptote determined by the equation

$$(2 - \omega)^2 - 4\sqrt{(1 - \omega)(1 - \nu\omega)} = 0,$$

which is the Rayleigh equation of the upper medium; this asymptote is therefore $\mu/\mu' : \rho/\rho' = 0{\cdot}9194^{-2}$ or $1{\cdot}1831$ in Fig. 1 and $\mu/\mu' : \rho/\rho' = 1{\cdot}0958$ in Fig. 2.

TABLE I — $\lambda/\mu = \lambda'/\mu' = 1$ | TABLE II — $\lambda = \lambda' = \infty$

Table I Curve A ($\omega<1$) μ/μ'	ρ/ρ'	Table I Curve B ($\omega>1$) μ/μ'	ρ/ρ'	Table II Curve A ($\omega<1$) μ/μ'	ρ/ρ'	Table II Curve B ($\omega>1$) μ/μ'	ρ/ρ'
0·3435	0	0·118	0·131	0·3090	0	0·219	0·288
0·355	0·0319	0·544	0·549	0·327	0·0327	0·483	0·488
0·367	0·0776	1·388	1·402	0·334	0·0635	1·427	1·442
0·383	0·115	1·752	1·946	0·362	0·130	1·508	1·571
0·398	0·155	1·939	2·394	0·392	0·200	1·803	1·982
0·422	0·215	2·129	3·092	0·426	0·273	1·974	2·350
0·443	0·266	2·259	3·765	0·464	0·348	2·156	2·874
0·469	0·323	2·371	4·649	0·507	0·426	2·347	3·667
0·516	0·418	2·515	6·447	0·555	0·505	2·548	4·997
0·571	0·514	2·613	8·71	0·663	0·637	2·701	7·68
0·7204	0·713	2·706	12·9	0·701	0·694	2·990	15·8
1·840	1·821	2·820	31·4	2·072	2·051	3·054	30·9
8·49	7·64	2·912	∞	4·56	4·38	3·236	∞

Curve B has an asymptote parallel to the ρ/ρ' axis, given by

$$\mu/\mu' = \frac{2\nu' + \sqrt{1-\nu} + \sqrt{5-\nu+4\nu'\sqrt{1-\nu}}}{1+\nu'};$$

this becomes $\mu/\mu' = 2{\cdot}9117$ if $\lambda/\mu = \lambda'/\mu'$ and $\mu/\mu' = 3{\cdot}2360$ if $\lambda = \lambda' = \infty$.

The two curves are closely connected with each other; if we change ρ, λ, μ and ρ', λ', μ' into each other equations (1) and (2) will also interchange. Curve A has therefore the same connection with the upper medium as curve B has with the subjacent medium; in view of the physical aspect of this problem this relation is evident.

It must be remarked that these curves have also been obtained by Sezawa and Kanai * who solved the Stoneley wave equation by numerical computation; notwithstanding this elaborate method their results are in complete agreement with equations (1) and (2). The reader who is interested in the derivation of these equations may be referred to a paper by the present author † published in 1942. (Figs. 4 and 5 of that paper show some obvious mistakes and are to be replaced by Figs. 2 and 3 of this paper.)

3. *The existence of surface waves in a superficial layer.*—In this section the method to determine this range of existence will be described; the mathematical analysis of the period equation on which this method is based will not be given here as it was published in an earlier paper.‡

It is advisable to have some general knowledge about this range before starting with a more detailed investigation. Roughly stated there exists a surface wave system (SWS) for every value of μ/μ' and T/L if $\beta' > S$, S being the velocity of Rayleigh waves in a semi-infinite body consisting of the same material as the surface layer. For not too small values of T/L and μ/μ' a SWS is possible, which changes with increasing value of T/L gradually into the Stoneley wave system. If $\beta' < S$ a SWS can only exist for every value of T/L if μ/μ' is small enough and also for any value of μ/μ' if T/L is sufficiently small.

* K. Sezawa and K. Kanai, *Bull. Earthq. Res. Inst. Tokyo*, **17**, 1, 1939.
† J. G. Scholte, *Proc. Acad. Sci. Amst.*, **45**, 20 and 159, 1942.
‡ J. G. Scholte, *Ibid.*, **45**, 380, 449 and 516, 1942.

In a more detailed description of the region where a SWS is possible the determination of the maximum and minimum values of μ/μ' and T/L just mentioned, must be given. For this purpose we need the Stoneley "boundary-equations" (1) and (2), and also the period equation of waves in an isolated layer * :

$$\frac{4\sqrt{(1-\nu\omega)(1-\omega)}}{(2-\omega)^2} = \left\{\frac{\tanh \pi \, T/L \, \sqrt{(1-\nu\omega)/\omega}}{\tanh \pi \, T/L \, \sqrt{(1-\omega)/\omega}}\right\}^{\pm 1}. \tag{3}$$

In Figs. 3 and 4 the graphic representations of these equations have been drawn.

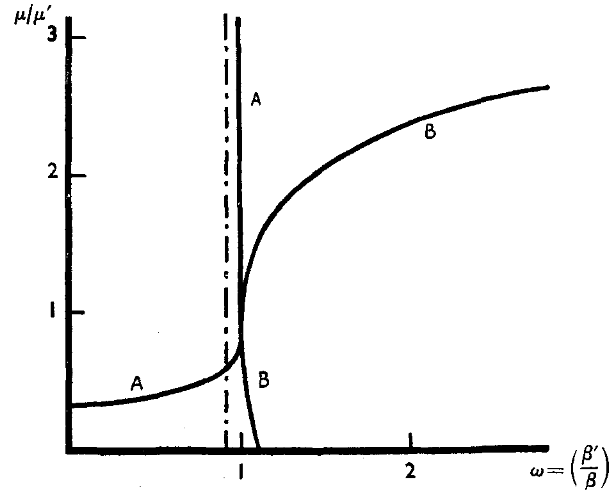

FIG. 3.—*Stoneley boundary curves ; broken line* $\omega = 0{\cdot}9126$.

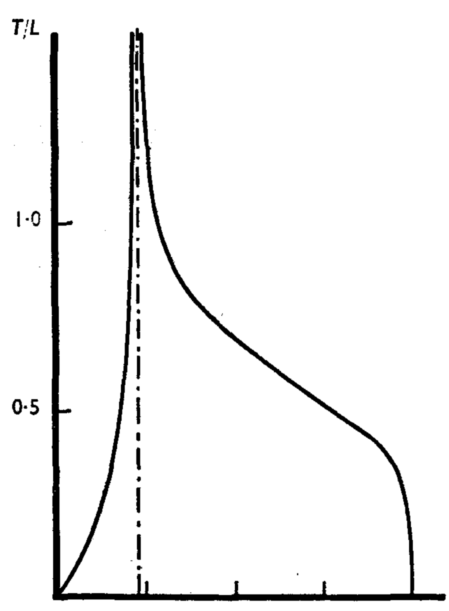

FIG. 4.—*Equation* 3, $\omega = 0{\cdot}9126$.

* H. Lamb, *Proc. Roy. Soc.* A, **93**, 114, 1917.

The use of these curves will be demonstrated by some examples; taking first a case where $\beta' > S$ we suppose $\beta' = 1{\cdot}4\beta$. In Fig. 3 we see that the solution of (2) is $\mu/\mu' \approx 2$ and in Fig. 4 that the root of (3) is $T/L \approx 4{\cdot}4$. A SWS is possible for every value of μ/μ' and T/L, and a second SWS can exist if $T/L > 4{\cdot}4$ and $\mu/\mu' > 2$.

In the second example ($\beta' < S$) we assume $\beta = 1\frac{1}{4}\beta'$. The two roots are $\mu/\mu' \approx 0{\cdot}5$ and $T/L \approx 0{\cdot}4$; a SWS is possible for every value of T/L if $\mu/\mu' < 0{\cdot}5$, and for every value of μ/μ' if $T/L < 0{\cdot}4$.

As can be seen in Figs. 3 and 4 there are three important values of ω: $(S/\beta)^2$, 1 and $(\beta'/S')^2$; we therefore proceed to describe the region where a SWS is possible in every one of the four intervals of ω separated by these values.

(*a*) $\omega < (S/\beta)^2$ or $\beta' < S$. With increasing value of ω the roots of (1) and (3) are also increasing; the region where a SWS can exist extends more and more, till at $\omega = (S/\beta)^2$ or $\beta' = S$ both roots are infinite and a SWS is possible for every value of μ/μ' and T/L (Fig. 5): in the horizontally shaded area no SWS can exist.

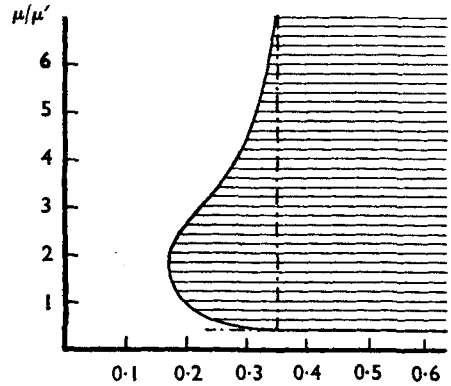

FIG. 5.—$\beta < S'$ ($\omega = 0{\cdot}64$).

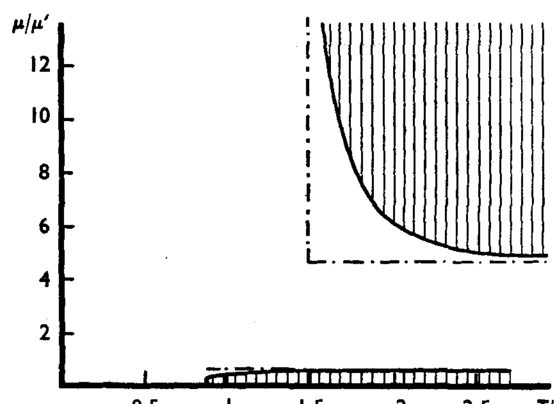

FIG. 6.—$S < \beta' < \beta$ ($\omega = 0{\cdot}96$).

(*b*) $(S/\beta)^2 < \omega < 1$ or $S < \beta' < \beta$. Equation (1) yields two positive roots, equation (3) only one root. In the vertically shaded area of Fig. 6 two SWS's are possible; with growing value of ω the roots of (1) approach each other, while the root of (3) decreases, so that the Stoneley wave area also increases. At $\omega = 1$ the two roots of (1) coincide at $\mu/\mu' = 1$ and the root of (3) is determined by the limiting form of this equation at $\omega = 1$:

$$\pi T/L \tanh \pi T/L \sqrt{1-\nu} = 4\sqrt{1-\nu}. \qquad (4)$$

(*c*) $1 < \omega < (\beta'/S')^2$ or $\beta' > \beta > S'$. Equation (3) has for values of $\omega > 1$ to be replaced by (4) and equation (1) by (2). With increasing ω the smallest root of (2) rapidly approaches zero and the largest one slowly increases; the root of equation (4) is independent of ω (its value is about 1 if $\lambda/\mu = \lambda'/\mu' = 1$ and 1·12 if $\lambda = \lambda' = \infty$). We see therefore that the area where a second SWS can occur slowly diminishes (Fig. 7).

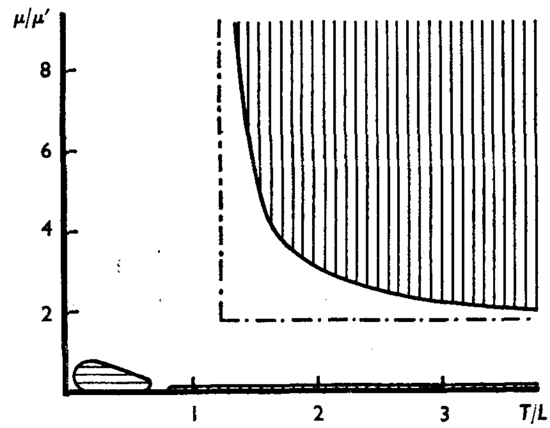

FIG. 7.—$\beta' > \beta > S'(\omega = 1{\cdot}10)$.

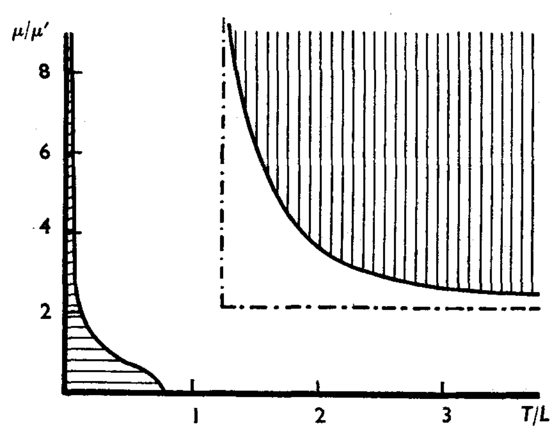

FIG. 8.—$\beta < S'(\omega = 1{\cdot}33)$.

(*d*) $\omega > (\beta'/S')^2$ or $\beta < S'$. The single positive root of (2) increases with growing value of ω to a limiting value, which is equal to 2·91 if $\lambda/\mu = \lambda'/\mu' = 1$ and to 3·27 if $\lambda = \lambda' = \infty$; as the root of (4) remains the same the Stoneley wave area decreases very slowly (Fig. 8).

The method, just described, to determine the range of existence of surface waves is not exact, but will generally be accurate enough for practical purposes.

The exact boundary curves of the regions where one or two SWS's are possible are given by the equations

$$\begin{aligned}(\mu/\mu')^2.[\{16(1-\nu\omega)(1-\omega)+(2-\omega)^4\}\tanh a\tanh b-8(2-\omega)^2\sqrt{(1-\nu\omega)(1-\omega)}\,.\,\phi]\\ -(\mu/\mu').[\{16(1-\nu\omega)(1-\omega)+2(2-\omega)^3\}\tanh a\tanh b\\ -8(2-\omega)(2-\tfrac{1}{2}\omega)\sqrt{(1-\nu\omega)(1-\omega)}\,.\,\phi-4\omega\sqrt{(1-\nu')(1-\nu\omega)}\tanh b\\ +\omega(2-\omega)^2\sqrt{(1-\nu')(1-\omega)}\tanh a]\\ +[\{4(1-\nu\omega)(1-\omega)+(2-\omega)^2\}\tanh a\tanh b\\ -4(2-\omega)\sqrt{(1-\nu\omega)(1-\omega)}\phi-\omega^2\sqrt{(1-\nu\omega)(1-\omega)}]=0, \quad \text{if} \quad \omega<1;\end{aligned}$$

here $a=2\pi T/L.\sqrt{1-\nu\omega}$, $b=2\pi T/L\sqrt{1-\omega}$ and $\phi=1-\operatorname{sech} a \operatorname{sech} b$.

$$\begin{aligned}(\mu/\mu')^2.\{1-\sqrt{(1-\nu'/\omega)(1-1/\omega)}\}(8\phi\sqrt{1-\nu}-N\tanh a)\\ -(\mu/\mu').[2\{2-1/\omega-2\sqrt{(1-\nu'/\omega)(1-1/\omega)}\}(6\phi\sqrt{1-\nu}-N\tanh a)\\ +1/\omega\{4(1-\nu)\sqrt{1-1/\omega}-\sqrt{1-\nu'/\omega}\}\tanh a-1/\omega\sqrt{(1-\nu)(1-1/\omega)}N]\\ +\{(2-1/\omega)^2-4\sqrt{(1-\nu'/\omega)(1-1/\omega)}\}(4\phi\sqrt{1-\nu}-N\tanh a+\sqrt{1-\nu})=0,\\ \text{if } \omega>1;\end{aligned}$$

where $N=2\pi T/L$, $a=2\pi T/L\sqrt{1-\nu}$ and $\phi=1-\operatorname{sech} a$.

In Figs. 5 to 8 the curves have been drawn; the values of μ/μ' and T/L calculated from equations (1) to (4) determine the asymptotes to these curves. The arithmetic involved in computing these curves is not heavy, as μ/μ' can be calculated in an elementary way if the values of T/L and ω are known. These calculations show that there exists if $\beta<\beta'$ a small region where no SWS is possible; this region has the form of a loop if $S'<\beta<\beta'$ and changes into a narrow strip along the μ/μ' axis if $S'>\beta$. As the value of T/L in this area is always small the analysis of Bromwich, Lee and others * can be used.

In Figs. 5 to 8 the non-shaded area is the region where one SWS is possible; in the vertically (horizontally) shaded area two (no) SWS's can exist.

4. *Conclusion.*—The range of existence of simple Stoneley waves has been determined by the evaluation of the boundary curves of these waves; the equations of these curves are also given. The range of existence of the generalized Rayleigh and Stoneley wave systems has also been determined. Small values of T/L excepted (where another approximation can be used) a fairly good approximation of this range can be obtained by the calculation of the asymptotes of the boundary curves; these asymptotes are determined by the equation of the simple Stoneley boundary curves and by the period equation of the waves in an isolated layer. This approximation will generally be accurate enough for practical purposes.

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1946 *February* 3.

* H. Jeffreys, *Geophys. Suppl.*, 3, 253, 1935