ANOTHER STEP TOWARDS A POST-BOUSSINESQ WAVE MODEL

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Progress toward a new non-linear and fully dispersive irregular-wave model is presented. The model relies on time stepping the two surface boundary conditions with a closure between the horizontal and vertical velocity expressed in terms of convolution integrals. The formulation is fully explicit in space and thus no equations need to be inverted. The convolution integrals involve impulse response functions exhibiting exponential decay. In practice, this reduces the infinite limits of the integrals to a horizontal distance of only several water depths. The model is derived for linear waves over a mildly sloping one-dimensional bottom, and the shoaling property is verified for a linear regular wave. It is further discussed how an existing perturbation method can be used for the inclusion of wave nonlinearity of arbitrary order.

1. Introduction

Advanced deterministic models for the transformation of water waves have evolved significantly over recent years. This is indeed the case for Boussinesq-type models, see Kirby (2003) for a recent review. The latest developments practically eliminate the usual water depth to wavelength restriction and treat highly nonlinear waves with great accuracy (Madsen et al., 2003, 2002). However, the increase in accuracy and range of application has been accompanied by a significant increase in computational effort. The present paper suggests an alternative approach by which the high accuracy and large application range is attainable at lesser computational cost. The model requires no restriction on the water depth. A similar approach has previously been used for computing the kinematics field from the kinematics at the still water level as assumed to be provided by a numerical wave model (Schäffer, 2003a).

2. Linear Waves over a Horizontal Bottom

To illustrate the basic idea, we first consider linear waves propagating over a horizontal bottom in one horizontal dimension. In line with Agnon et al. (1999), the governing equations may be written as the two surface conditions

$$\eta_t - w_0 = 0 \tag{1}$$

and

$$u_{0,t} + g\eta_x = 0 \tag{2}$$

with a closure

$$f_0(u_0, w_0; h) = 0 \tag{3}$$

obtained from the continuity equation and the bottom boundary condition. Here u and w are the horizontal and vertical component of the particle velocity vector and subscript 0 denotes variables at the still water level, SWL. Furthermore, h is depth, η is surface elevation, g is gravitational acceleration, and subscripts t and x refer to differentiation with respect to time, t, and horizontal coordinate, x. Finally, f_0 is an operator. The two surface conditions can be stepped forward in time by standard-time integration techniques provided that Eq. (3) is solved for w_0 at every time step. For the linear shallow water equations, Eq. (3) becomes $w_0 = -hu_{0,x}$, which provides w_0 explicitly. Boussinesq formulations include at least third order differential operators and except for formulations with very poor dispersion characteristics, the closure is an implicit relation that requires the solution of an algebraic system of equations involving the variables at all computational points. In traditional Boussinesq formulations for which the vertical velocity has been eliminated from the mass and momentum equations, an equivalent algebraic system appears as a consequence of mixed spatial and temporal derivatives.

Now linear Stokes theory provides a simple relation that can be regarded as a connection between the wave-number-space counterparts (U_0, W_0) of (u_0, w_0),

$$W_0 = -\frac{\tanh kh}{kh} hU_{0,x} \tag{4}$$

Using the convolution theorem, the inverse Fourier transform yields an expression which has the explicit form

$$w_0 = f_0(u_0; h) \tag{5}$$

namely

$$w_0(x) = -h \int_{-\infty}^{\infty} u_{0,x}(x - x')r(x')dx' \tag{6}$$

where $r(x)$ is the impulse response function corresponding to the transfer function $(\tanh kh)/(kh)$. This impulse response function may be computed analytically to get

$$r(x) = -\frac{1}{\pi}\log_e\left(\tanh\left(\frac{\pi}{4}\left|\frac{x}{h}\right|\right)\right) \tag{7}$$

The full line in Figure 1 shows r versus x/h. The figure shows that in practice it suffices to take the infinite integration limits in Eq. (6) as plus minus a horizontal distance of just a few water depths. The exponential far field decay is

$$r \approx \frac{2}{\pi}\exp\left(-\frac{\pi}{2}\left|\frac{x}{h}\right|\right), \quad \left|\frac{x}{h}\right| >> 1 \tag{8}$$

The convolution integral in Eq. (6) provides an exact closure valid for irregular waves over any finite, constant water depth. The method is explicit and thus, the discretized formulation does not involve the inversion of large algebraic systems of equations as for Boussinesq formulations.

3. Linear Waves Over a Variable Depth

The advantage of the above approach to the derivation lies in the striking simplicity. However, when it comes to generalizing the theory to variable depth, another line of attack is required. Infinite series operators as applied in recent Boussinesq-type formulations may be used for an alternative derivation of Eq. (4). What more is, this approach also provides a way to handle variable depth. Schäffer (2003b) used this procedure to derive the following expression

$$\begin{aligned} W_0 = &-\frac{\tanh kh}{kh} hU_{0,x} - \\ &h_x\left(1 - kh\tanh(kh) - \tanh^2(kh) + kh\tanh^3(kh)\right)U_0 \end{aligned} \tag{9}$$

which is the mild-slope generalization of Eq. (4). Proceeding as before by using the convolution theorem, the inverse Fourier transform yields

$$w_0(x) = -h(x)\int_{-\infty}^{\infty} u_{0,x}(x-x')r(x')dx' - h_x(x)\int_{-\infty}^{\infty} u_0(x-x')\tilde{r}(x')dx' \tag{10}$$

where

$$\tilde{r}(x) = \frac{1}{8}\pi\left(\frac{x}{h}\right)^2 \coth\left(\frac{\pi}{2}\left|\frac{x}{h}\right|\right)\operatorname{csch}\left(\frac{\pi}{2}\left|\frac{x}{h}\right|\right) \tag{11}$$

is shown by the dashed line in Figure 1.

The far field expression for the impulse response function, $\tilde{r}(x)$ is

$$\tilde{r} \approx \frac{\pi}{4}\left(\frac{x}{h}\right)^2 \exp\left(-\frac{\pi}{2}\left|\frac{x}{h}\right|\right), \quad \left|\frac{x}{h}\right| >> 1 \tag{12}$$

which shows exponential decay as for $r(x)$, but falls off more slowly due to the quadratic factor. Unfortunately, $\tilde{r}(x)$ is much wider than $r(x)$. However, for a typical horizontal to vertical aspect ratio of near-shore regions subject to wave propagation studies, the function is still very local.

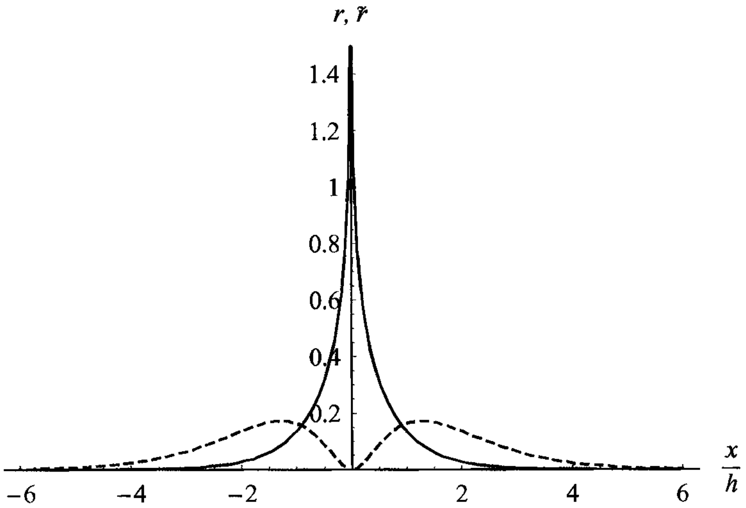

Figure 1. Impulse response functions, r (full line) for constant-depth terms and $\tilde{r}$ (dashed line) for mild-slope terms

In summary, the linear model is given by the surface conditions Eq. (1) and Eq. (2) with a closure on w_0 given by the explicit convolution-integral formulation in Eq. (10), that uses the impulse response functions given by Eq. (7) and Eq. (11). The model is valid for irregular waves travelling over a mildly sloping bed. Notice that the final model is wavenumber free.

To verify the model, a simple shoaling test was made. Copying the test used by Bingham and Agnon (2004) for convenience, a linear monochromatic wave of length 10m was generated at 5m depth and propagated over a mild and smooth slope to a water depth of 0.05m. Figure 2 shows a snapshot of the surface elevation after the wave has reached the shallow depth. For reference, the theoretically determined shoaling curve for the envelope is also shown. The good match verifies the theory.

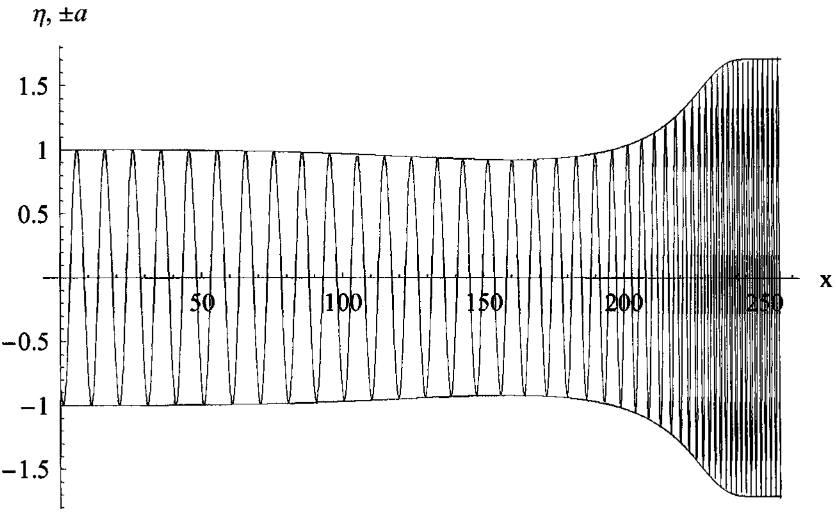

Figure 2. Snapshot of the surface elevation as obtained from the convolution wave model compared to the theoretically determined shoaling envelope. From Schäffer (2003b)

4. Nonlinearity

The fully nonlinear free surface boundary conditions may be expressed as

$$\eta_t = \tilde{w}(1+\eta_x^2) - \tilde{V}\eta_x \tag{13}$$

$$\tilde{V}_t = -g\eta_x - \tfrac{1}{2}\left(\tilde{V}^2 - \tilde{w}^2(1+\eta_x^2)\right)_x \tag{14}$$

see e.g. Agnon et al. (1999). Here tilde denotes variables at the free surface and $\tilde{V} \equiv \tilde{u} + \eta_x \tilde{w} = \tilde{\Phi}_x$, where Φ is the velocity potential.

To advance these equations in time, a relation between the dependent variables is needed by which $\tilde{w}$ can be determined at every time step given $\tilde{V}$ and η. Like for the linear model, an explicit form is used,

$$\tilde{w} = \tilde{f}(\tilde{V}, \eta; h) \tag{15}$$

where $\tilde{f}$ is a nonlinear operator. For the case of linear waves, this reduces to Eq. (5). To obtain a link with the linear expression, we follow Dommermuth and Yue (1987), who developed a perturbation method connecting variables at the SWL with variables at the surface. A similar technique was derived simultaneously by West et al. (1987), only it appears that they were slightly more consistent in that they truncated the expressions according to their place in the free surface boundary conditions. The procedure was derived for the velocity potential as follows. Let ε be a small parameter of the order of the wave steepness, then the surface potential is expressed as the perturbation series

$$\varepsilon\tilde{\Phi} = \sum_{m=1}^{M} \varepsilon^m \tilde{\Phi}^{(m)} \tag{16}$$

With a vertical axis, z, originating at SWL, each term is expressed as a Taylor series

$$\tilde{\Phi}^{(m)} = \sum_{n=0}^{M-m} \varepsilon^n \frac{\eta^n}{n!} \frac{\partial^n \Phi_0^{(m)}}{\partial z^n} \tag{17}$$

by which

$$\varepsilon\tilde{\Phi} = \sum_{m=1}^{M} \sum_{n=0}^{M-m} \varepsilon^{m+n} \frac{\eta^n}{n!} \frac{\partial^n \Phi_0^{(m)}}{\partial z^n} \tag{18}$$

Collecting terms of order ε yields

$$\Phi_0^{(1)} = \tilde{\Phi} \tag{19}$$

while a collection of order ε^m -terms results in

$$\Phi_0^{(m)} = -\sum_{n=1}^{m-1} \frac{\eta^n}{n!} \frac{\partial^n \Phi_0^{(m-n)}}{\partial z^n} \tag{20}$$

which conveniently expresses $\Phi_0^{(m)}$ in terms of lower order expressions already known from the previous steps of this recursion relation.

To adapt this method to the velocity formulation, Eq. (19) and Eq. (20) are differentiated with respect to x to get

$$u_0^{(1)} = \tilde{V} \tag{21}$$

and

$$u_0^{(m)} = -\left(\sum_{n=1}^{m-1} \frac{\eta^n}{n!} \frac{\partial^{n-1} w_0^{(m-n)}}{\partial z^{n-1}} \right)_x \tag{22}$$

Given $u_0^{(m)}$ at each order, $w_0^{(m)}$ and its z -derivatives are computed as

$$w_0^{(m)} = f_0(u_0^{(m)};h) \tag{23}$$

$$\frac{\partial w_0^{(m)}}{\partial z} = -\left(u_0^{(m)}\right)_x \tag{24}$$

$$\frac{\partial^n w_0^{(m)}}{\partial z^n} = -\left(\frac{\partial^{n-2} w_0^{(m)}}{\partial z^{n-2}}\right)_{xx} \tag{25}$$

by virtue of the convolution integral formulation, local continuity and the Laplace equation, respectively. The ingredients for the determination of $\tilde{w}$ are now in place and we can evaluate the vertical surface velocity as

$$\tilde{w} = \sum_{m=1}^{M} \sum_{n=0}^{M-m} \frac{\eta^n}{n!} \frac{\partial^n w_0^{(m)}}{\partial z^n} \tag{26}$$

analogous to Eq. (18) but omitting the ordering parameter.

Dommermuth and Yue (1987) tested the perturbation procedure for fully nonlinear regular waves in deep water and concluded that the method was applicable for waves of up to 80 percent of the maximum wave steepness. Their target wave was computed from the analytical continuation of the Stokes-type expansion given by Schwartz (1974). Here, we test the perturbation procedure using Stream Function waves (Dean, 1965, Fenton, 1988). This allows for a test of the shallow water wave as well.

With H and L denoting wave height and wavelength, respectively, we chose $H/L = 0.11$ and $h/L = 0.5$ for the deepwater case and $H/h = 0.6$ and $h/L = 0.05$ for the shallow water case. For both cases, the wave height is about 80 percent of the maximum value.

For the deepwater case, Figures 3, 4 and 5 show the respective profiles of η/L, $\tilde{V}/\sqrt{gL}$ and $\tilde{w}/\sqrt{gL}$. Figure 6 shows the relative error on $\tilde{w}$ as obtained from Eq.(26) for orders $M = 1$ to 6. The visual impression of convergence is in line with the more thorough analysis of Dommermuth and Yue.

For the shallow water case, Figures 7, 8 and 9 show the respective profiles of η/h, $\tilde{V}/\sqrt{gh}$ and $\tilde{w}/\sqrt{gh}$, while Figure 10 shows the error for the estimated $\tilde{w}$. Again, we omit a detailed analysis and note that a visual inspection indicates convergence.

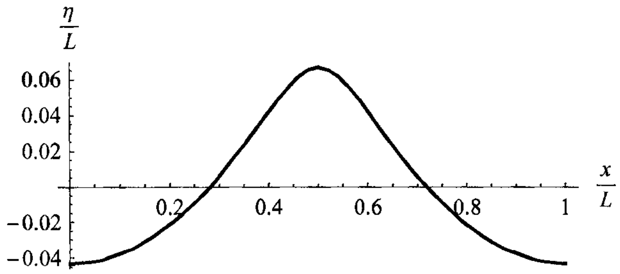

Figure 3. Profile of surface elevation for the deepwater case

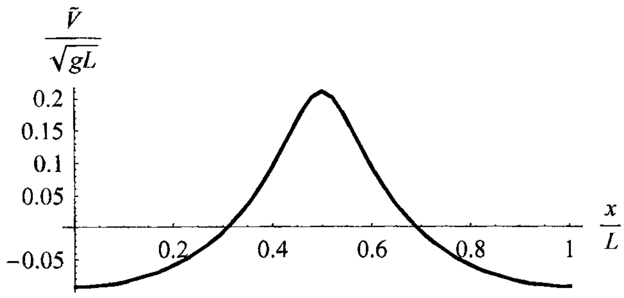

Figure 4. Profile of the gradient of the surface velocity potential for the deepwater case

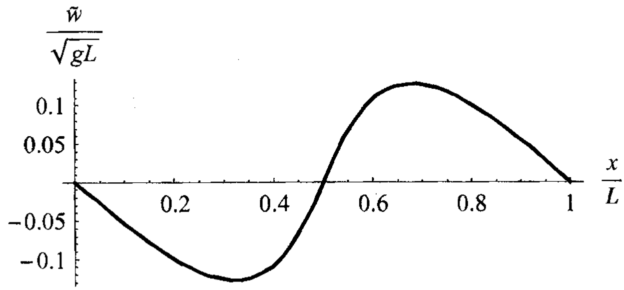

Figure 5. Profile of the vertical surface velocity for the deepwater case

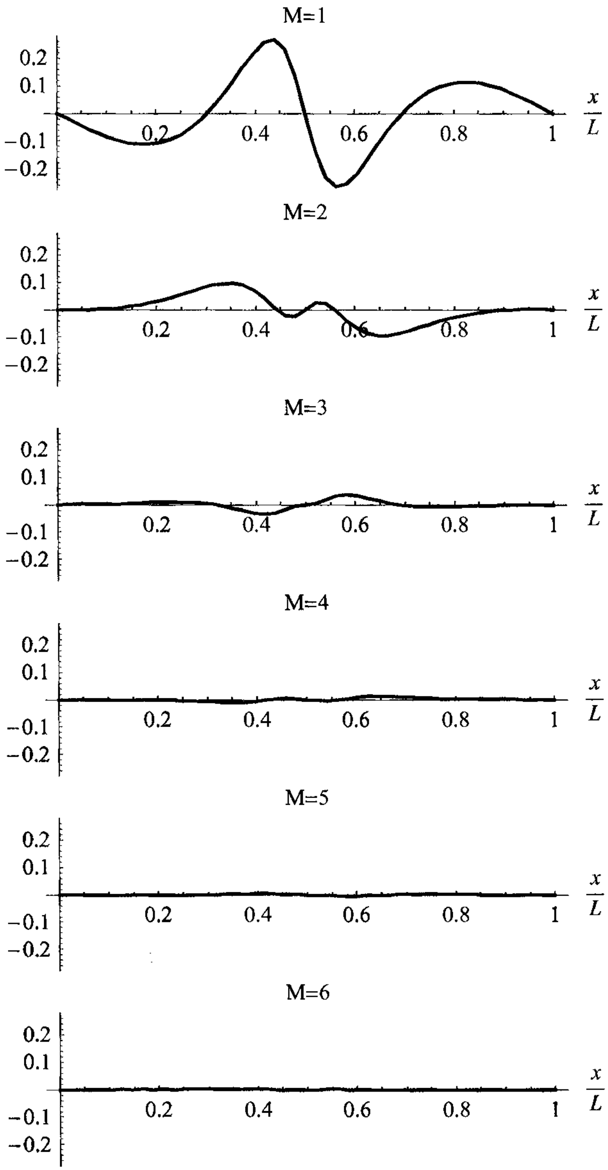

Figure 6. Profiles of the vertical surface velocity error $(\tilde{w}-\tilde{w}^{(M)})/\mathrm{Max}(\tilde{w})$ for the deepwater case as estimated to order M

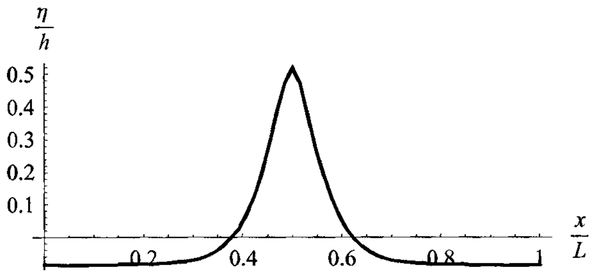

Figure 7. Profile of surface elevation for the shallow water case

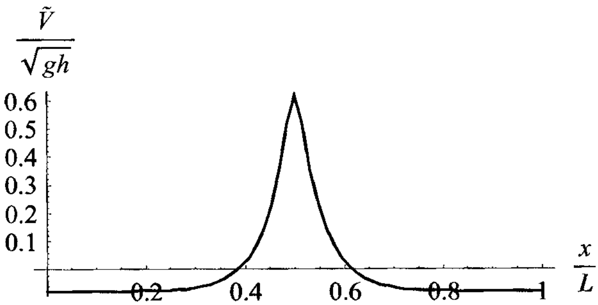

Figure 8. Profile of the gradient of the surface velocity potential for the shallow water case

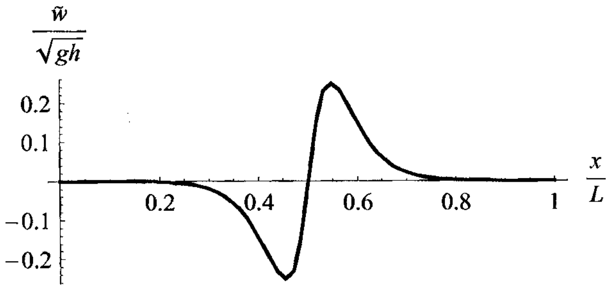

Figure 9. Profile of the vertical surface velocity for the shallow water case

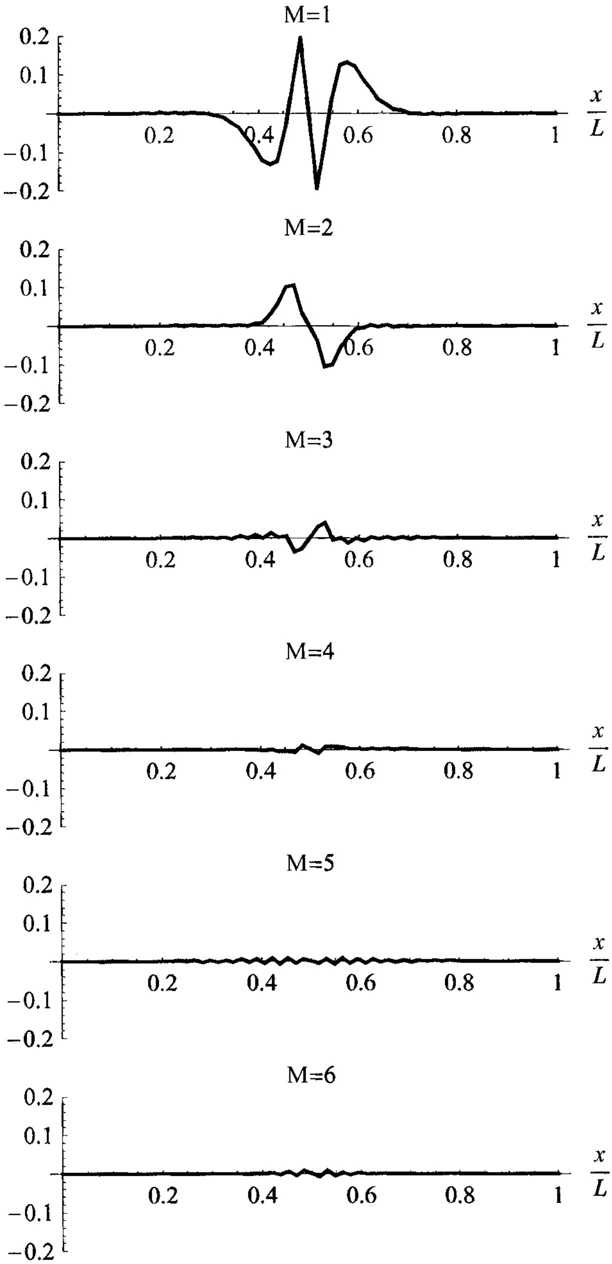

Figure 10. Profiles of the vertical surface velocity error $(\tilde{w}-\tilde{w}^{(M)})/\mathrm{Max}(\tilde{w})$ for the shallow water case as estimated to order M

5. Conclusions and Future Work

The prospects for a new and very desirable kind of wave transformation model have been shown to be good. So far, the model has been developed for linear waves over a one-dimensional mildly sloping bottom. The method involves a fully explicit closure of the kinematic relation needed to step the free-surface boundary conditions forward in time. This means that the model does not involve the solution of a large system of algebraic equations, as is the case for most wave propagation models for irregular waves.

The next challenges will be to implement nonlinearity in the model and to generalize it to two horizontal dimensions.

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