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Comparison of Dirichlet–Neumann operator expansions for nonlinear surface gravity waves

Hemming A. Schäffer

SchafferWaves, Sortedam Dossering 59 H st., DK-2100 Copenhagen Ø, Denmark

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Abstract

The Dirichlet–Neumann operator for the water-wave problem was introduced and expanded by Craig and Sulem [Craig, W., Sulem, C., 1993. [CS] Numerical simulation of gravity waves. J. Comput. Phys. 108, 73–83] and in a slightly different form and for 3D waves by Bateman, Swan and Taylor [Bateman, W.J.D., Swan, C., Taylor, P.H., 2001. [BST] On the efficient numerical simulation of directionally spread surface water waves. J. Comput. Phys. 174, 277–305]. This approach is supposedly superior to techniques derived earlier by West et al. [West, B.J., Brueckner, K.A., Janda, R.S., Milder, D.M., Milton, R.L., 1987. [WW] A new numerical method for surface hydrodynamics. J. Geophys. Res. 92 (C11), 11803–11824] and Dommermuth and Yue [Dommermuth, D.G., Yue, D.K.P., 1987. [DY] A high-order spectral method for the study of nonlinear gravity waves. J. Fluid Mech. 184, 267–288] under seemingly more restrictive assumptions. This paper extracts the Dirichlet–Neumann operator expansions from West et al. and Dommermuth and Yue. Concerning the operator expansions alone it is found that Bateman et al. is identical to West et al. and Dommermuth and Yue while Craig and Sulem is slightly different due to minor differences in the operator definition. For application to the free-surface boundary conditions West et al. devised a consistent truncation at nonlinear order. This alters the equivalence of the different approaches when it comes to the evaluation of the temporal derivative of the free surface elevation, which is decisive for wave evolution. In this regard Craig and Sulem is found to be identical to West et al. while Bateman et al. is identical to Dommermuth and Yue. Pseudo code is provided for alternative computational schemes in Fourier-space and physical space, respectively, along with a discussion of efficiency and potential flexibility.

Keywords: Water waves; Dirichlet–Neuman operator; Higher-order spectral method; Perturbation expansion; Nonlinearity; Dispersion

1. Introduction

The Dirichlet–Neumann (DN) operator for water waves expresses the normal surface particle velocity in terms of the velocity potential at the surface. Given a procedure for estimating the DN operator (subject to lateral boundary conditions), the water-wave problem is reduced to the simultaneous time-integration of the kinematic and the dynamic free surface boundary conditions (KFSBC and DFSBC, respectively). Thus, the dependent variables are evaluated at the free surface only, also if the water depth is finite, and even if it is variable. A series expansion of the DN operator was derived by Craig and Sulem (CS, 1993) for 2D waves and in a slightly different form by Bateman, Swan and Taylor (BST, 2001) for 3D waves. Without using the terminology of a DN operator, West et al. (WW, 1987) and Dommermuth and Yue (DY, 1987) had already derived related expressions under seemingly more restrictive assumptions. Here West et al. (WW, 1987) is referred to as WW since this makes a nice superscript for symbols used below and at the same time it acknowledges Watson and West (1975), who West et al. quote for their methodology. Although the theoretical framework of the DN operator expansions of CS and BST appears quite different from the work of WW and DY, this paper brings all theories in a comparable form and shows that differences are either minor or non existent. This is in contradiction to the statements of CS, who claim that their developments are superior to the methods of WW and DY. In fact the recursive formulation of WW and DY is preferable for practical evaluation, since it helps to avoid multiples of identical sub-calculations. The same advantage was obtained a posteriori by BST in their "quick form".

The methods cited above are widely used in applications related to nonlinear, dispersive wave transformation. The WW/DY approach is typically referred to as Higher Order Spectral

E-mail address: Hemming@SchafferWaves.dk.

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(HOS) methods, while the DN-operator (DNO) is usually quoted for the CS/BST approach.

Especially the HOS method is widely cited and the following references just provide a small selection of work following WW and DY. Ölmez and Milgram (1995) compared the HOS method with a more accurate but far less efficient Boundary Integral Equation Method and e.g. found a good agreement for short waves riding on even very steep long waves. Tanaka (2001) employed the method for direct numerical verification of Hasselmann's energy transfer across the wave spectrum. Mori and Yasuda (2002) studied the evolution of 2D irregular waves e.g. identifying freak wave events in deep water and getting extreme wave height probabilities in deep/shallow water that were larger/smaller than those predicted by a Rayleigh distribution. Bingham and Agnon (2005) used the HOS procedure for including nonlinearity in their so-called Fourier Boussinesq model and e.g. looked at Bragg reflection as previously studied by Liu and Yue (1998), who extended the HOS method to account for bottom ripples. Simulations of a physical wave tank while modeling the actual wavemaker movements is one of several applications of the HOS method by the group in Ecole Centrale de Nantes, see e.g. the recent contribution by Ducrozet et al. (2006) and their references.

In addition to the work of DY and BST, examples of the use of the DNO method are Nicholls (2001), who computed steady hexagonal gravity waves, Gibbs and Taylor (2005) studying steep directionally spread focused wave groups in deep water and Craig et al. (2006) analyzing 2D solitary wave interactions.

Common to all the methods described in this paper is that they involve perturbation and Taylor series expansions which fail to converge for extremely steep waves. However, they perform well for waves of up to 80% of the limiting steepness (DY). Getting even closer to the limit, episodic waves with very high nonlinearity have also been modeled with success (BST), presumably due to the lack of time and space for inaccuracies to ruin the evolution. A detailed account for the type of wave forms that can be modeled is outside the scope of this paper, but the above references all include examples.

The difficulty encountered for very steep waves was analyzed by Nicholls and Reitich (2001a,b, 2003) who further resolved the problem by applying a sigma transformation of the vertical coordinate. However, the transformed system is more complicated and much more computationally demanding to solve. Global basis functions are no longer available and effectively a resolution of the vertical variation is required.

Considering only the traditional DN operator expansions (or equivalently, HOS methods), the present analysis was carried out in search of the preferred method for including nonlinearity in an ongoing development of a new wave model. This model is based on convolution integrals in physical space and involves only variables at the surface although the depth may vary by orders of magnitude provided the bottom slope is mild. Progress on this work was reported in Schäffer (2005, 2006).

This paper is closed by a short discussion of the DN operator computation with and without a toggle to Fourier space as suitable for a spectral and convolutional approach, respectively.

2. Governing equations

In the following $(\tilde{\mathbf{u}}, \tilde{w})$ is the particle velocity at the free surface, $z=\eta$, in a Cartesian coordinate system $(\mathbf{x}, z)$, where z is a vertical axis pointing upwards from e.g. the still water level. The velocity potential at the surface is $\tilde{\Phi} = \Phi(z=\eta)$, subscript t denotes the time derivative and ∇ is the horizontal gradient operator. Coordinates or components along the surface are not considered anywhere except for a secluded mention of the surface particle velocity normal to the surface (w' appearing in Eqs. (2.6) and (2.7)).

Including a small ordering parameter, ϵ, the KFSBC and DFSBC read

$$\eta_t = \left(1+(\epsilon\nabla\eta)^2\right)\tilde{w} - \epsilon\nabla\eta\cdot\nabla\tilde{\Phi} \tag{2.1}$$

and

$$\tilde{\Phi}_t = -g\eta - \frac{1}{2}\epsilon\left(\left(\nabla\tilde{\Phi}\right)^2 - \tilde{w}^2\left(1+(\epsilon\nabla\eta)^2\right)\right) \tag{2.2}$$

as used by WW, DY and essentially also by BST. To march η and $\tilde{\Phi}$ forward in time a closure is needed. An attractive explicit form is

$$\tilde{w} = G^{\mathrm{WW}}\left(\tilde{\Phi}\right), \tag{2.3}$$

where G^{WW} is the DN operator dependent on η and operating on $\tilde{\Phi}$. G^{WW} further depends on the constant or variable water depth, h, and it may also account for non-periodic lateral boundary conditions. A slightly different definition of the DN operator was introduced by CS as

$$\eta_t = G^{\mathrm{CS}}\left(\tilde{\Phi}\right). \tag{2.4}$$

This directly serves as the KFSBC and the CS form of the DFSBC condition is

$$\tilde{\Phi}_t = -g\eta - \frac{\epsilon}{2\left(1+(\epsilon\nabla\eta)^2\right)} \times\left(\left(\nabla\tilde{\Phi}\right)^2 - \eta_t^2 - 2\epsilon\nabla\eta\cdot\nabla\tilde{\Phi}\eta_t - \left(\varepsilon\nabla\eta\cdot\nabla\tilde{\Phi}\right)^2 + \left(\nabla\tilde{\Phi}\right)^2(\epsilon\nabla\eta)^2\right) \tag{2.5}$$

when generalized to two horizontal dimensions. Finally, we remark that the "genuine" but less practical DN operator, G^{DN} is defined by

$$w' = G^{\mathrm{DN}}\left(\tilde{\Phi}\right), \tag{2.6}$$

where w' is the particle velocity normal to the surface. This again is the KFSBC through the geometric relation

$$\eta_t = \sqrt{1+(\epsilon\nabla\eta)^2 w'}. \tag{2.7}$$

3. The Dirichlet–Neumann operator

In this section, WW's procedure for accounting for wave nonlinearity is detailed, and the equivalent expansion of the DN operator is explicitly extracted. Following DY yields identical results for the DN operator. However, as discussed subsequently, WW and DY differ in how the operator is used in the surface boundary conditions. The surface potential is expressed as the perturbation series,

$$\epsilon\Phi = \sum_{m=1}^{M} \epsilon^m \Phi^{(m)}. \tag{3.1}$$

With a vertical axis, z, typically originating at still water level, each term is expressed as a Taylor series about $z=0$ evaluated at $z=\eta$,

$$\tilde{\Phi}^{(m)} = \sum_{n=0}^{M-m} \epsilon^n \frac{\eta^n}{n!} \left[\frac{\partial^n \Phi^{(m)}}{\partial z^n}\right]_{z=0} \tag{3.2}$$

by which

$$\epsilon\tilde{\Phi} = \sum_{m=1}^{M} \sum_{n=0}^{M-m} \epsilon^{m+n} \frac{\eta^n}{n!} \left[\frac{\partial^n \Phi^{(m)}}{\partial z^n}\right]_{z=0}. \tag{3.3}$$

Collecting terms of order ϵ, this may be rewritten as

$$\epsilon\tilde{\Phi} = \sum_{m=1}^{M} \sum_{n=0}^{m-1} \epsilon^m \frac{\eta^n}{n!} \left[\frac{\partial^n \Phi^{(m-n)}}{\partial z^n}\right]_{z=0}. \tag{3.4}$$

Requiring this to be valid for all values of ϵ gives

$$\Phi_0^{(1)} = \tilde{\Phi} \tag{3.5}$$

at order ϵ, while the ϵ^m-terms result in

$$\Phi_0^{(m)} = -\sum_{n=1}^{m-1} \frac{\eta^n}{n!} \left[\frac{\partial^n \Phi^{(m-n)}}{\partial z^n}\right]_{z=0}. \tag{3.6}$$

This conveniently expresses $\Phi_0^{(m)} \equiv [\Phi^{(m)}]_{z=0}$ in terms of lower order expressions already known from the previous steps of this recursion. At each order the z-derivatives are computed from the recursion relation

$$\left[\frac{\partial^n \Phi^{(m)}}{\partial z^n}\right]_{z=0} = D^2 \left[\frac{\partial^{n-2} \Phi^{(m)}}{\partial z^{n-2}}\right]_{z=0}, \tag{3.7}$$

where $D^2 = -\nabla^2$, which holds due to $\Phi^{(m)}$ satisfying the Laplace equation. The two starting values (indices $n=0$ and $n=1$) needed for this recursion are $\Phi_0^{(m)}$ and

$$\left[\frac{\partial \Phi^{(m)}}{\partial z}\right]_{z=0} = G_0\left(\Phi_0^{(m)}\right). \tag{3.8}$$

Taking iD as the gradient or the divergence for scalar and vector operands, respectively,

$$G_0 = D \tanh hD \tag{3.9}$$

represents the linearized DN operator that for constant depth follows directly from small-amplitude wave theory for periodic waves. Variable depth and/or non-periodic lateral boundary conditions require that G_0 be adjusted accordingly. WW restricted their approach to the deep water case for which $G_0 = D$. Usually G_0 is considered as a Fourier multiplier, but it may also be expressed essentially as a convolution integral operator in physical space. This framework is suitable for generalization to variable depth, see Schäffer (2005, 2006).

The derivations (3.1)–(3.6) remain valid if the potential is replaced by the vertical velocity, w, and with

$$\epsilon\tilde{w} = \sum_{m=1}^{M} \epsilon^m \tilde{w}^{(m)} \tag{3.10}$$

we get

$$\tilde{w}^{(m)} = \sum_{n=0}^{m-1} \frac{\eta^n}{n!} \left[\frac{\partial^{n+1} \Phi^{(m-n)}}{\partial z^{n+1}}\right]_{z=0}. \tag{3.11}$$

The DN-operator formulation (2.3) is expanded as

$$\epsilon\tilde{w} = \sum_{m=1}^{M} \epsilon^m G_{m-1}^{\mathrm{WW}}\left(\tilde{\Phi}\right). \tag{3.12}$$

and the recipe for evaluating $\tilde{w}^{(m)}$ at any order and extracting the associated terms of the DN operator, G_m^{WW}, is now as follows: Evaluate Eq. (3.11) obtaining the z-derivatives from the recursion relation (3.7) with starting values (3.6) and (3.8), where Eq. (3.6) is a recursion relation initialized with Eq. (3.5). The first three terms are given by

$$G_0^{\mathrm{WW}} = G_0, \tag{3.13}$$

$$G_1^{\mathrm{WW}} = \eta D^2 - G_0 \eta G_0, \tag{3.14}$$

and

$$G_2^{\mathrm{WW}} = G_0\eta G_0\eta G_0 - \frac{1}{2}G_0^2\eta D^2 - \eta D^2 \eta G_0 + \frac{1}{2}\eta^2 D^2 G_0. \tag{3.15}$$

This exactly matches the DN operator derived by BST (their eq. 22a–c) as a slight variation of CS. For the special case of

Table 1
Summary of extracted DN operators and surface elevation derivatives

Reference	DN-operator	η_t
WW: West et al. (1987); Wastson and West (1975)	G^{WW}	η_t^{WW}
DY: Dommermuth and Yue (1987)	$G^{\mathrm{DY}} = G^{\mathrm{WW}}$	η_t^{DY}
CS: Craig and Sulem (1993)	G^{CS}	$\eta_t^{\mathrm{CS}} = \eta_t^{\mathrm{WW}}$
BST: Bateman, Swan and Taylor (2001)	$G^{\mathrm{BST}} = G^{\mathrm{WW}}$	$\eta_t^{\mathrm{BST}} = \eta_t^{\mathrm{DY}}$

deep water, where $G_0=D$, (3.13)–(3.15) in Eq. (3.12) reduces to the expression given already by Watson and West (1975) (eq. A5 in their Appendix).

To determine the slightly different DN operator of CS, Eqs. (2.4) and (2.3) are substituted into the KFSBC (2.1), to get

$$G^{\mathrm{CS}}\left(\tilde{\Phi}\right) = \left(1+(\epsilon\nabla\eta)^2\right)G^{\mathrm{WW}}\left(\tilde{\Phi}\right) - \epsilon\nabla\eta\cdot\nabla\tilde{\Phi}. \tag{3.16}$$

Expanding the DN operators and collecting the orders gives

$$G_0^{\mathrm{CS}} = G_0, \tag{3.17}$$

$$G_1^{\mathrm{CS}} = D\eta D - G_0\eta G_0, \tag{3.18}$$

and

$$G_2^{\mathrm{CS}} = G_0\eta G_0\eta G_0 - \frac{1}{2}G_{0\eta}^2D^2 - \frac{1}{2}D^2\eta^2G_0, \tag{3.19}$$

where the explicit appearance of the surface elevation gradient has been eliminated by incorporating these in the operators (The last term in G_2^{CS} accommodates the two last terms in G_2^{WW} as well as the term arising from G_0^{WW}). These results are identical to those of CS (their eq. 2.14). This contradicts the beliefs of CS, who describe their DN operator expansion as "uniformly valid in wavenumber", while stating that it "differs from the spectral methods of both West et al. and Dommermuth and Yue, where both Φ and η are assumed to be $O(\epsilon)$ quantities, and the expansion is not uniform in wavenumber".

4. Application in the surface boundary conditions

WW devised a consistent truncation with regard to nonlinear order in connection with the evaluation of η_t and $\tilde{\Phi}_t$ by which

$$\tilde{w}(\nabla\eta)^2 = (\nabla\eta)^2\sum_{m=1}^{M-2} w^{(m)}, \tag{4.1}$$

$$\tilde{w}^2 = \sum_{m=1}^{M}\sum_{n=1}^{M-m} w^{(n)}w^{(m)} \tag{4.2}$$

and

$$\tilde{w}^2(\nabla\eta)^2 = (\nabla\eta)^2\sum_{m=1}^{M-2}\sum_{n=1}^{M-2-m} w^{(n)}w^{(m)}. \tag{4.3}$$

Without this truncation, all upper limits of the summations would be M. According to the KFSBC Eq. (2.1) or Eq. (2.4), errors in the DN operator affect the determination of η_t already at a linear level. On the other hand $\tilde{\Phi}_t$ is less sensitive to errors, since the DN operator only appears in the nonlinear terms of the DFSBC, Eq. (2.2) or Eq. (2.5). Consequently, we skip the analysis of $\tilde{\Phi}_t$ and concentrate on η_t. It turns out that the WW method corresponds exactly to the procedure outlined above for evaluating G_m^{CS} from G_m^{WW}, see Eq. (3.16). This implies that $\eta_t^{\mathrm{CS}}=\eta_t^{\mathrm{WW}}\neq\eta_t^{\mathrm{BST}}=\eta_t^{\mathrm{DY}}$, although $G_m^{\mathrm{CS}}\neq G_m^{\mathrm{WW}}=G_m^{\mathrm{BST}}=G_m^{\mathrm{DY}}$ as summarized in Table 1. These results were backed up by numerical examples, some of which are given in Qureshi (2005). As a final remark, a DN operator is strictly one that

Dirichlet-Neumann operator expansions

$\gamma_0 = 1$

$\hat{\mu}_1 = \mathrm{FFT}(\tilde{\Phi})$

Do $m = 1, M$

  $\gamma_m = \frac{\eta}{m}\gamma_{m-1}$

  $w_m = 0$

  If $m \le M-1$ **then** $\mu_{m+1} = 0$

  Do $n = 1, m$

    $\hat{\rho} = E_n\hat{\mu}_{m+1-n}$

    $\rho = \mathrm{FFT}^{-1}(\hat{\rho})$

    $w_m = w_m + \gamma_{n-1}\rho$

    If $m \le M-1$ **then** $\mu_{m+1} = \mu_{m+1} - \gamma_n\rho$

  End Do

  If $m \le M-1$ **then** $\hat{\mu}_{m+1} = \mathrm{FFT}(\mu_{m+1})$

End Do

Fig. 1. Pseudo code for the Fourier-multiplier computation of $w_m \equiv \tilde{w}^{(m)}$, $m = 1, \ldots M$ from $\tilde{\Phi}$ and η using the WW-DY-BST-method. The indices are adjusted so that the subscript everywhere (except on E) indicates the order of a quantity in ϵ.

maps $\tilde{\Phi}$ to the normal surface particle velocity w', see Eqs. (2.6)–(2.7). However, with a consistent truncation involving an expansion of $1/\sqrt{1+(\epsilon\nabla\eta)^2}$ in ϵ, this choice turns out to produce the same $\eta_t = \eta_t^{CS} = \eta_t^{WW}$.

In retrospect one might say that to a specified order the methods investigated must all be identical provided no errors were made. For input $\tilde{\Phi}$ and fixed output $\tilde{\Phi}_t$ or $\tilde{w}$ the only allowable differences would be due to explicit or implicit inclusion of terms of higher order than those consistently retained. However, for all comparisons the expressions are polynomial in ϵ making the orders "pure" and the equivalence exact.

5. Evaluation procedures, FFT-count and G_0-count

While the explicit evaluation of G_m is interesting from a theoretical point of view, it is favourable to retain the iterative procedure for practical applications. This helps to avoid redundant computations. The WW-DY-BST-method for evaluating $\tilde{w}^{(1)}, \tilde{w}^{(2)}, \ldots \tilde{w}^{(M)}$ from $\tilde{\Phi}$ and η may be accomplished as illustrated by the pseudo code in Fig. 1, where

$$\mu_m = \Phi_0^{(m)}, \tag{5.1}$$

$$\rho = \left[\frac{\partial^n \Phi^{(m+1-n)}}{\partial z^n}\right]^{z=0} \tag{5.2}$$

and

$$E_n = \begin{cases} D^{n-1} G_0 & \text{for } n \text{ odd} \\ D_n & \text{for } n \text{ even} \end{cases} \tag{5.3}$$

Note that ρ needs no indexing and that E_n, $n=1,\ldots,M$ as time-independent Fourier multipliers can be computed prior to the time-stepping procedure. The pseudo code assumes a toggle to and from Fourier space by the Fast Fourier Transform (FFT). This

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γ0 = 1
v1=G0(Φ̃)
If M > 1 then λ1 = D²Φ̃
Do m = 1, M
    γm = (η/m) γm−1
    wm = 0
    If m ≤ M − 1 then μm+1 = 0
    Do n = 1, m
        If n is odd then do
            wm = wm + γn−1 vm+1−n
            If m ≤ M − 1 then μm+1 = μm+1 − γn vm+1−n
            If m ≤ M − 2 then vm+1−n = D² vm+1−n
        Else
            wm = wm + γn−1 λm+1−n
            If m ≤ M − 1 then μm+1 = μm+1 − γn λm+1−n
            If m ≤ M − 2 then λm+1−n = D² λm+1−n
        End If
    End Do
    If m ≤ M − 1 then vm+1 = G0 (μm+1)
    If m ≤ M − 2 then λm+1 = D² μm+1
End Do
```

Fig. 2. As the pseudo code of Fig. 1, but tailored for equivalent physical-space evaluation.

Table 2
Summary of operation counts for different computational spaces

Comput. Space	#FFT's	#(G_0-convolutions)	#(Physical-space ∇^2 s)
Fourier space	$\frac{1}{2}M(M+3)$	–	–
Physical space	–	M	$\frac{1}{2}M(M-1)$
Mixed space	2M	–	$\frac{1}{2}M(M-1)$

explicitly shows the FFT-count as M forward transforms applied to the μ's and $\frac{1}{2}M(M+1)$ back-ward transforms to get the ρ's. The total FFT-count is $\frac{1}{2}M(M+3)$ matching the "quick form" of BST (which they attribute to Vijfvinkel (1996)) and the findings of Le Touzé (2003) (our M equals their $M+1$). This is in contrast to the claim of DY that their computational effort is directly proportional to M. It seems DY might only be counting the M boundary-value problems for $\Phi^{(m)}$ neglecting the quadratic M-dependence arising from each boundary condition itself requiring $O(m)$ products with powers of η. Additional FFT's are needed if a procedure with purely alias-free products is pursued. This approach leads to a significant increase in the computational cost and the effort may be better spent by limiting aliasing through improved spatial resolution.

For complex physical situations like non-rectangular domains, appreciable depth variations or non-periodic lateral boundary conditions, the use of Fourier space is not feasible. All operations can then be made in physical space and the coupling of D^2 and G_0 in Eq. (5.3) is no longer beneficial. Fig. 2 shows pseudo code for the all-physical-space computation with M G_0-convolutions and $\frac{1}{2}M(M-1)$ Laplacian operations. Typically, the Laplacians would be evaluated using finite differences thus compromising spectral accuracy. Table 2 summarizes the operations count for various situations including a mixed procedure where G_0 is computed in Fourier space, while all Laplacians are evaluated in physical space. Although the effect of G_0 is in principle global, the associated impulse response function shows exponential decay and a practical width limited to the horizontal scale of a few water depths. This means that the computational effort of each physical-space convolution scales linearly (and not quadratically) with the number of computational points. The convolution approach allows for order-of-magnitude depth variations provided the bottom slope is mild, see Schäffer (2005, 2006). Generalization to complex-shaped domains is a major challenge currently pursued by the author.

Using the symbolic calculation software package Mathematica, both versions of pseudo code (see Figs. 1 and 2) were checked analytically to produce the desired terms of the DN-operator expansion.

6. Concluding remarks

Various formulations essentially coupling the nonlinear wave evolution problem to the linear one have been compared. Despite the different approaches taken and the diversity of the mathematical derivations, the approach of Craig and Sulem (CS, 1993) and the related one of Bateman, Swan and Taylor (BST, 2001) are found to match the earlier work of West et al. (WW, 1987) and Dommermuth and Yue (DY, 1987). One variation is due to the choice of definition for the Dirichlet–Neumann (DN) operator, where CS differs slightly.

The other variation is due to WW's consistent truncation at nonlinear order during application in the nonlinear free surface boundary conditions. With these variations, we find that given the surface potential, the vertical surface velocity is identically determined for WW, DY and BST with a slight variation for CS. With regard to the temporal derivative of the surface elevation, the result from CS matches that of WW exactly, while BST matches that of DY.

Using Fourier-space operations, practical applications take $\frac{1}{2}M(M+3)$ FFT's to retain $O(\epsilon^M)$ accuracy. Potentially more flexible applications operating in physical space take M convolution integrals with nearly compact support and $\frac{1}{2}M(M-1)$ Laplacian operations.

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