New Partial Differential Equations Governing the Joint, Response-Excitation, Probability Distributions of Nonlinear Systems, under General Stochastic Excitation. ([1])

by

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ABSTRACT: In the present work the problem of determining the probabilistic structure of the dynamical response of nonlinear systems subjected to general, external, stochastic excitation is considered. The starting point of our approach is a Hopf-type equation, governing the evolution of the joint, response-excitation, characteristic functional. Exploiting this equation, we derive new linear partial differential equations governing the joint, response-excitation, characteristic (or probability density) function, which can be considered as an extension of the well-known Fokker-Planck-Kolmogorov equation to the case of a general, correlated excitation and, thus, non-Markovian response character. These new equations are supplemented by initial conditions and a marginal compatibility condition (with respect to the known probability distribution of the forcing), which is of non-local character and, thus, difficult to implement. The validity of this new equation is also checked by showing its equivalence with the infinite system of moment equations. The method is applicable to any, state-space, differential system exhibiting polynomial nonlinearities, but in this paper it is illustrated through a detailed analysis of a simple, first-order, scalar equation, with a cubic nonlinearity. It is also shown that the same approach is also able to derive the Fokker-Planck-Kolmogorov equation for the case of independent-increment excitation.

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A numerical method for the solution of these new equations is introduced and illustrated through its application to the simple model problem. It is based on the representation of the joint probability density (or characteristic) function by means of a convex superposition of kernel functions, which permits to satisfy *a priori* the non-local marginal compatibility condition. On the basis of this representation, the partial differential equation is eventually transformed to a system of ordinary differential equations for the kernel parameters. Extension to general, multidimensional, dynamical systems exhibiting any polynomial nonlinearity will be presented in a forthcoming paper (Athanassoulis & Sapsis 2007).

KEYWORDS: Stochastic Dynamics, Numerical solution of Stochastic Differential Equations, Functional Differential Equations, Correlated Stochastic Excitation, Generalized Fokker-Planck-Kolmogorov Equation, Non-Markovian Responses, Characteristic Functional, Kernel Density Functions.

1 INTRODUCTION

Many problems occurring in applied sciences and engineering are successfully modelled as stochastic differential equations. A very important class of such problems are those modelled as stochastically excited, nonlinear, dynamical systems. Well-known examples include the dynamic responses of ships and other man-made structures and systems under the influence of wind-generated waves in the sea (Schlesinger & Swean 1998, Wilson 2002, Belenky & Sevastianov 2003, Arnold *et al.* 2004), the dynamic responses of buildings and bridges under the influence of earthquakes (Lin & Cai 1995, Deodatis 1996, Kafali & Grigoriu 2003), as well as the dynamic responses of structures and vehicles under the influence of wind forces (Simiu & Scanlan 1986, Kree & Soize 1986, Soong & Grigoriu 1993, Hemon & Santi 2006). In all these cases the excitation loads are assumed to be known stochastic processes, either Gaussian or non-Gaussian, as in the case of wind loads. Their probabilistic and correlation structure can be (and, usually, have been) inferred by means of statistical data analysis and, in most cases, have been conveniently parameterized for easy reference and use in calculations. Most of the foundational facts and aspects concerning the stochastic modelling philosophy in engineering and applied science, and the corresponding mathematical background can be found

nowadays in book form; see, e.g., Kree & Soize 1986, Sobczyk 1991, Soong & Grigoriu 1993, Roberts & Spanos 2003.

The ultimate objective in the analysis of such problems is to obtain a complete probabilistic description of the response process, permitting to answer any important questions about the response dynamics. Examples of such questions concerns the distributions of local extrema, of upcrossing rates at certain levels, of the first passage time associated with a critical level value, etc. To make this possible we need, in principle, to know the whole Kolmogorov hierarchy of the *n*-fold, joint, probability distributions $F_{x(t_1)x(t_2)...x(t_n)}(a_1,a_2,...,a_n)$ of the *n*-variate response random variables $(x(t_1),x(t_2),...,x(t_n))$ at any collection of time instances $(t_1,t_2,...,t_n)$ or, equivalently and more concisely, the Characteristic Functional (Ch.Fl) of the response process. Because of the obvious difficulties of this general concept of solution of the probabilistic dynamics problems, there is a constant tendency –at least in the applied and engineering literature– to avoid such an approach, resorting to simpler (partial) solution concepts.

An important, and extensively studied, context, permitting a relatively easy, complete characterization of the probabilistic responses of a dynamical system, occurs if we assume that the excitation is a process with independent increments (see, e.g., Pugachev & Shinitsyn 1987, Soize 1994, Grigoriu 2004). The key feature in this context is that the response vector, in the state-space formalism, is a Markovian process and, thus, its probability density function is governed by the Fokker-Planck-Kolmogorov (FPK) equation (in the Gaussian case) or by reasonable extensions of the FPK equation (in the non Gaussian case). Interestingly enough, there have been identified broad classes of problems in which analytic solutions of the classical FPK equation are available (see, e.g., Soize 1994), making this approach even more attractive.

An approximate method dealing with nonlinear systems under general stochastic excitation is the Statistical Linearization Method (see, e.g., Roberts & Spanos 2003), which is based on the approximation of the full system by a ‘statistically equivalent’ linear one. Some variations of the method, concerning local linearization in the phase space, have been recently presented (Pradlwarter 2001), giving promising results. It is also possible to develop approximate solution schemes by replacing the given dynamical system by a “statistically equivalent” nonlinear system provided that the

latter belongs to the class of problems which can be solved exactly. This method introduced by Caughey has been applied to various particular problems in the last three decades; see e.g . Lutes (1970), Caughey (1986) or Roberts & Spanos (2003).

Another well-known method that can be applied to any type of stochastic excitation and to any type of nonlinearity, is the method of moments, which reduces the initial stochastic dynamics problem to an infinite system of deterministic differential equations for the moment functions (Beran 1968, Pugachev & Shinitsyn 1987). This infinite system should be truncated and becomes closed (in the case of nonlinear problems) by means of appropriate closure schemes. Then, it is solved numerically, providing us with restricted information about the probabilistic characterization of the response process.

Another method, in principle well-known but in very little use for solving practical problems in stochastic dynamics, is the one based on the Ch.Fl of the full probability measure associated with the dynamic response process. The first step in this direction was made by Hopf (1952) who derived a Functional Differential Equation (FDE) for the Ch.Fl associated with the probabilistic solution of the Navier-Stokes equations. This approach, known as *the statistical approach to turbulence*, has been developed further by many authors (see, e.g., Lewis & Kraichnan 1962, Monin and Yaglom 1971, 1975, Foias 1974, Feller 1986), and, eventually, led to the derivation and exploitation of various transport-diffusion equations for pdfs of the velocities and composition in Turbulent reactive flows (Kollman 1990). In parallel, a simpler version of the same approach has been developed and applied to finite-dimensional dynamical systems, governed by Stochastic Ordinary Differential Equations (SODEs). See, e.g., Beran (1968). Such Hopf-type FDEs are always linear, and govern the Ch.Fl of the sought-for probability measure or –depending on the specific formulation– the Ch.Fl of the joint, response-excitation, probability measure. In recent years successful attempts have been reported towards the analytic determination of the response Ch.Fl for some classes of linear problems, even avoiding the explicit use of Hopf's FDE (Caseres & Budini 1997, Budini & Caseres 1999, 2004). For some non-linear problems, the Ch.Fl can be expressed as a formal infinite-dimensional (functional) integral (Monin & Yaglom 1975), which is of little (or no) practical use.

In this paper, Hopf's FDE is taken as the starting point of the probabilistic analysis of the considered stochastic dynamics problem. Because of the generality of Hopf's approach, the method is applicable to any (at least) polynomially non-linear system

and any kind of stochastic excitation. Nevertheless, for reasons of simplicity and clarity, our study will be carried out on a specific, first-order, dynamical system, with cubic nonlinearity. The excitation process will be assumed, in principle, completely known, with a given correlation structure and continuous (or smoother) sample functions. This implies a non-Markovian character of the response, making the approach based on the FPK equation inapplicable. Exploiting the Hopf FDE, new Partial Differential Equations (PDEs) governing the joint, response-excitation, characteristic functions (ch.f.), are derived. The corresponding equations for the joint pdfs are also obtained, by applying a Fourier transformation. These new PDEs, which are always linear, can be considered as a systematic and rigorous generalization of the FPK-type equations to the case of correlated excitation and non-Markovian responses. As an additional test of validity of these new PDEs, we show that they produce the correct infinite system of the moment equations. The same approach, i.e. starting from the Hopf FDE, is also applied to derive extended FPK equations, for the case of independent-increment excitation. In this connection, the results recently obtained by Grigoriu (Grigoriu 2004), concerning various cases of non-Gaussian, independent-increment forcing, are derived as special cases of our new extended FPK equations. We also show the consistency of our new PDE (for the joint, response-excitation, pdfs) with the usual (or extended) FPK equation, by deriving the latter as a limiting case of the former. A lack of rigor occurs here, when the sample functions of the response process are not continuous. It is conjectured that this derivation may be reformulated in a rigorous manner by invoking the dual of the space of cadlag (or regulated) functions, recently studied by Tvrdy (Tvrdy 2002).

Abbreviations

The following abbreviations –some of which have already been introduced above– will be consistently used in the sequel:

B-space	Banach space
ch.f(s)	characteristic function(s)
Ch.Fl(s)	characteristic functional(s)
F-derivative	Frechet derivative
FDE(s)	functional differential equation(s)
FPK	Fokker – Planck – Kolmogorov
ODE(s)	ordinary differential equation(s)

PDE(s)	partial differential equation(s)
pdf(s)	probability density function(s)
SODE(s)	stochastic ODE(s)

2 PRELIMINARIES AND NOTATION

In this work we consider ODEs (systems) of the form (in state space formulation):

$$\dot{x}(t) = G\big(x(t)\big) + y(t), \quad x(t_0) = x_0, \tag{2.1}$$

where x and y are scalar-valued or N-vector-valued, continuous (or smoother) functions, defined at least on an interval $I \equiv [t_0, T]$ (that is, $x, y : [t_0, T] \equiv I \to \mathbb{R}^N$), and $G : \mathbb{R}^N \to \mathbb{R}^N$, $N = 1$ or $N > 1$, is also a continuous (or smoother) function. Both the excitation $y(\bullet)$ and the initial conditions x_0 will be assumed known stochastic elements (function and variable, respectively). In contrast with the standard approach, followed in the case of an Ito SODE, the forcing $y(\bullet)$ is allowed to be smooth (e.g., k-times continuously differentiable), exhibiting any type of correlation structure in time. Thus, the sample functions $x(t)$ and $y(t)$ are considered as elements of smooth-function B-spaces, denoted by $\mathscr{X}$ and $\mathscr{Y}$, respectively. Our main results will refer to the case $N = 1$, $\mathscr{Y} = C^k(I)$, $I \subseteq \mathbb{R}$, $k = 0$ or $k > 0$, and $\mathscr{X}$ a similar space with smoother elements. The whole methodology can be extended to the vector case $N > 1$ with the usual trouble (see Athanassoulis & Sapsis 2007 for a detail analysis of a second-order system).

The topological dual spaces of $\mathscr{X}$ and $\mathscr{Y}$ are also B-spaces and will be denoted by $\mathscr{X}' = \mathscr{U}$ and $\mathscr{Y}' = \mathscr{V}$. The symbols $\langle u, x \rangle$ and $\langle v, y \rangle$ denote the standard duality pairings between $\mathscr{X}$ and $\mathscr{U}$, and $\mathscr{Y}$ and $\mathscr{V}$, respectively.

The underlying probability space is denoted by $\big(\Omega, \mathscr{B}(\Omega), \mathscr{P}_\Omega\big)$, where Ω is an abstract version of the sample (trial) space, $\mathscr{B}(\Omega)$ is the family of Borel sets of Ω, and $\mathscr{P}_\Omega$ is the corresponding probability measure over Ω. The stochastic processes x and y are measurable maps $x, y : \Omega \to \mathscr{X}, \mathscr{Y}$, which define the induced

probability spaces $\left(\mathscr{X},\mathscr{B}(\mathscr{X}),\mathscr{P}_{\mathscr{X}}\right)$ and $\left(\mathscr{Y},\mathscr{B}(\mathscr{Y}),\mathscr{P}_{\mathscr{Y}}\right)$, respectively. We shall also need and consider the joint process $x\times y:\Omega\to\mathscr{X}\times\mathscr{Y}$ with induced probability space $\left(\mathscr{X}\times\mathscr{Y},\mathscr{B}(\mathscr{X}\times\mathscr{Y}),\mathscr{P}_{\mathscr{X}\times\mathscr{Y}}\right)$. In the sequel we shall use the notation x or $x(\bullet)$ or $x(\bullet;\omega)$, and similarly for y, for the random element, and $x(t;\omega)$, $t\in[t_0,T]\equiv I\subseteq\mathbb{R}$, $\omega\in\Omega$, and similarly for y, for the sample functions, in accordance with the needs of the discussion.

The finite-dimensional distributions, densities and characteristic functions of the random element $x(\bullet;\omega)$ will be denoted by $F_{x(t_1)\ldots x(t_M)}(\alpha_1,\ldots,\alpha_M)$, $f_{x(t_1)\ldots x(t_M)}(\alpha_1,\ldots,\alpha_M)$ and $\phi_{x(t_1)\ldots x(t_M)}(\upsilon_1,\ldots,\upsilon_M)$, respectively. This implies a convenient notation for the joint random element $\left(x(\bullet;\omega),y(\bullet;\omega)\right)$; for example $f_{x(t_1)x(t_2)y(t_1)y(t_2)y(t_3)}(\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3)$ for the 2–x, $3-y$ density, and $\phi_{x(t_1)x(t_2)y(t_1)y(t_2)y(t_3)}(\upsilon_1,\upsilon_2,\nu_1,\nu_2,\nu_3)$ for the corresponding characteristic function. The usual (finite-dimensional) mean value operator (ensample average) will be denoted by $\mathbf{E}^{\omega}[\bullet]$. For example, the mean value function of the random element $x(\bullet;\omega)$ will be written as $m_{x(t)}=\mathbf{E}^{\omega}\left[x(t;\omega)\right]$. Slight variations (simplifications) of this notation will be introduced later, in accordance with the needs of the presentation.

Infinite-dimensional (global) moments, are defined by integrating over the whole sample space $\mathscr{X}$ with respect to the probability measure $\mathscr{P}_{\mathscr{X}}$ (See, e.g., Kree & Soize 1986, Vakhania *et al.* 1987, Egorov *et al.* 1993). For example, the mean (first moment) $m_{\mathscr{X}}$ is defined to be this element of $\mathscr{X}$, for which the following scalar equation holds true:

$$\langle u,m_{\mathscr{X}}\rangle=\int_{\mathscr{X}}\langle u,x\rangle\mathscr{P}(dx),\quad\forall u\in\mathscr{U},\tag{2.2a}$$

where $\mathscr{U}\equiv\mathscr{X}'$. Furthermore, the correlation operator (second moment) is defined to be this linear operator $R_{\mathscr{X}\mathscr{X}}:\mathscr{U}\to\mathscr{X}$, for which the following scalar equation is valid $\forall u,w\in\mathscr{U}$:

$$\langle w, R_{\mathscr{X}\mathscr{X}} u\rangle = \int_{\mathscr{X}} \langle w, x\rangle \langle u, x\rangle \mathscr{P}(dx). \tag{2.2b}$$

The integrals appearing in the right-hand side of eqs. (2.2) are infinite-dimensional (functional) integrals over *B*-spaces. (For detailed definitions and conditions ensuring existence of these integrals see references stated above or Dalecky & Fomin 1991). In general, the functional integral of any bounded, measurable, continuous functional $\mathscr{G}: \mathscr{X} \to \mathbb{C}$, with respect to a probability measure $\mathscr{P}$, is well defined, and will be denoted by $\int_{\mathscr{X}} \mathscr{G}(x)\, \mathscr{P}(dx)$.

Measures and integrals over infinite-dimensional vector spaces are related with the corresponding finite-dimensional ones through the concepts of cylinder sets, cylinder measures and cylinder functionals. Let $\mathscr{X}$ be a separable *B*-space, $\mathscr{U}$ be the dual of $\mathscr{X}$, and $u_1, \ldots, u_Q$, be Q linearly independent elements of $\mathscr{U}$. Then, to any element $x \in \mathscr{X}$ we associate the $Q-$dimensional projection $\prod_{u_1,\ldots,u_Q}: \mathscr{X} \to \mathbb{R}^Q$, defined by

$$\prod_{u_1,\ldots,u_Q}[x] = \left(\langle u_1, x\rangle, \ldots, \langle u_Q, x\rangle\right). \tag{2.3}$$

The inverse of $\prod_{u_1,\ldots,u_Q}[\bullet]$, applied to the Borel sets $\mathscr{B}\left(\mathbb{R}^Q\right)$, defines the cylinder sets of $\mathscr{X}$. The existence of a probability measure $\mathscr{P}_{\mathscr{X}}$ on $\mathscr{X}$ implies the existence of Q-dimensional (marginal) measures $P_{u_1,\ldots,u_Q}$ on $\mathbb{R}^Q$, associated with the random vectors $\left(\langle u_1, x(\bullet;\omega)\rangle, \ldots, \langle u_Q, x(\bullet;\omega)\rangle\right)$ by means of the relation

$$P_{u_1,\ldots,u_Q}\left(E_Q\right) = \mathscr{P}_{\mathscr{X}}\left(\prod\nolimits^{-1}_{u_1,\ldots,u_Q}\left[E_Q\right]\right). \tag{2.4}$$

for any $E_Q \in \mathscr{B}\left(\mathbb{R}^Q\right)$.

Consider now an arbitrary cylinder functional $\mathscr{G}: \mathscr{X} \to \mathbb{C}$, that is a functional of the form

$$\mathscr{G}(x) = g\left(\langle u_1, x\rangle, \ldots, \langle u_Q, x\rangle\right), \qquad x \in \mathscr{X}, \tag{2.5}$$

where $g : \mathbb{R}^Q \to \mathbb{C}$ is an arbitrary, measurable, integrable function. In this case, the infinite-dimensional integral of $\mathscr{G}(x)$ with respect to the probability measure $\mathscr{P}$ over the space $\mathscr{X}$, can be expressed as a *Q*-dimensional integral by means of the formula:

$$\int_{\mathscr{X}} \mathscr{G}(x)\, \mathscr{P}(dx) \;=\; \int_{\mathbb{R}^Q} g(\boldsymbol{a})\, P_{u_1,\ldots,u_Q}(d\boldsymbol{a})\,. \tag{2.6}$$

Eqs. (2.5) and (2.6) provide us with a powerful method for evaluating integrals over infinite-dimensional (function) spaces. They will be referred to as the (Q-dimensional) *Projection Theorem*.

3 A BRIEF REVIEW ON THE CHARACTERISTIC FUNCTIONAL AND ITS BASIC PROPERTIES

In this section we recall the definition and some basic properties of the Ch.Fl for probability measures defined on separable *B*-spaces.

3.1 *Definition of the Characteristic Functional*

Definition 3.1: Let $\mathscr{X}$ be a separable *B*-space and $\mathscr{P} = \mathscr{P}_{\mathscr{X}}$ be a probability measure defined on it. The Ch.Fl $\mathscr{F}$ of $\mathscr{P}$ is a cylinder functional defined on the dual space $\mathscr{X}' = \mathscr{U}$ by the formula

$$\mathscr{F}(u) \;=\; \int_{\mathscr{X}} e^{i\langle u,\, x\rangle}\, \mathscr{P}(dx), \qquad u \in \mathscr{U}\,. \tag{3.1}$$

This integral always exists provided that the corresponding probability measure is well defined.

3.2 *Infinite-Dimensional (Global) Moments*

Let the Ch.Fl be differentiable in the sense of Frechet. In order to calculate the *F*-derivative $D\mathscr{F}(u)$, we make use of the Gateaux derivative (which always exists for a *F*-differentiable map). Thus, we have

$$D\mathscr{F}(u)[z] = \left.\frac{d\mathscr{F}(u+\varepsilon z)}{d\varepsilon}\right|_{\varepsilon=0} = i\cdot\int_{\mathscr{X}} \langle z, x\rangle e^{i\langle u,\, x\rangle}\, \mathscr{P}(dx)\,, \quad u, z \in \mathscr{U}. \tag{3.2}$$

Setting $u = 0$, we obtain

$$D\mathscr{F}(0)[z] = i \cdot \int_{\mathscr{X}} \langle z, x \rangle \mathscr{P}(dx), \; z \in \mathscr{U}. \tag{3.3}$$

Since $D\mathscr{F}(0)[z]$ is a continuous, linear functional with respect to z, there should exists an element $m \in \mathscr{X}$, such that

$$\langle z, m \rangle = -i \cdot D\mathscr{F}(0)[z] = \int_{\mathscr{X}} \langle z, x \rangle \mathscr{P}(dx). \tag{3.3'}$$

Comparing the above equation with the eq. (2.2a), it easily seen that the element $m \in \mathscr{X}$ of eq. (3.3') coincides with the mean value $m_{\mathscr{X}}$ of the probability measure $\mathscr{P}$. The correlation operator $R_{\mathscr{P}}$ can be associated in a similar way with the second *F*-derivative of the Ch.Fl. In this case we have

$$\langle w, R_{\mathscr{P}} z \rangle = -D^2\mathscr{F}(0)[z, w] = \int_{\mathscr{X}} \langle w, x \rangle \langle z, x \rangle \mathscr{P}(dx), \quad z, w \in \mathscr{U}.$$

3.3 *Finite-Dimensional (Point) Moments*

In the case where the space $\mathscr{X}$ is a function space, apart from infinite-dimensional (global) moments, we are also interesting in finite-dimensional moments associated with finite-dimensional projections $\big(x(t_1;\omega), x(t_2;\omega), \ldots, x(t_n;\omega)\big)$, for any set of time instances $(t_1, t_2, \ldots, t_n)$. This kind of moments can be obtained also by differentiating the Ch.Fl, this time using Volterra functional derivatives. (See, e.g., Volterra 1927/1959/2002 or Beran 1968). Volterra derivatives, e.g. the first-order one $\delta\mathscr{F}(u)/\delta u(t)$, can be calculated either by applying the original definition to the functional, or by applying the Frechet derivative $D\mathscr{F}(u)[z]$ at $z(\bullet) = \delta(\bullet - t)$. Following the second approach, and using eqs. (3.3) and (2.6), we obtain

$$\frac{\delta\mathscr{F}(0)}{\delta u(t)} \overset{def}{=} D\mathscr{F}(0)\big[\delta(\bullet - t)\big] = i \int_{\mathscr{X}} x(t) \mathscr{P}(dx) = i \int_{\mathbb{R}} a \, dF_{x(t)}(a) = i \, \mathbf{E}^{\omega}\big[x(t;\omega)\big]$$

and thus

$$\mathbf{E}^{\omega}\left[x(t;\omega)\right] = \frac{1}{i}\frac{\delta\mathscr{F}(0)}{\delta x(t)}. \tag{3.4a}$$

Similarly we obtain

$$\mathbf{E}^{\omega}\left[x(t_1;\omega)x(t_2;\omega)\right] = \frac{1}{i^2}\cdot D^2\mathscr{F}(0)\left[\delta(\bullet - t_1),\, \delta(\bullet - t_2)\right] = \frac{1}{i^2}\frac{\delta^2\mathscr{F}(0)}{\delta u(t_1)\delta u(t_2)}, \tag{3.4b}$$

as well as analogous expressions for higher-order moments. Working similarly, and using appropriate generalized functions, we can derive equations for higher-order moments involving both the values of the random element at some time instances, as well as the values of its derivatives either at the same or at different time instances. As an example we give the formula:

$$\mathbf{E}^{\omega}\left[x'(t_1;\omega)x(t_2;\omega)\right] = \frac{1}{(-i)\,i}\cdot D^2\mathscr{F}(0)\left[\delta'(\bullet - t_1),\, \delta(\bullet - t_2)\right].$$

4 HOPF-TYPE EQUATION FOR THE CHARACTERISTIC FUNCTIONAL

In order to illustrate the derivation of Hopf-type FDEs for nonlinear dynamical systems, and pave the way to the next section, where these equations will be exploited to produce new PDEs for finite-dimensional ch.fs, we shall restrict ourselves here to a specific case of a simple (scalar, first-order) dynamical system having a cubic nonlinearity, which is described by the following SODE:

$$\dot{x}(t;\omega) + \mu x(t;\omega) + k x^3(t;\omega) = y(t;\omega), \tag{4.1a}$$

$$x(t_0;\omega) = x_0(\omega), \tag{4.1b}$$

where μ, k are deterministic constants, $x_0(\omega)$ is a random variable with known ch.f $\phi_0(v)$, $v \in \mathbb{R}$, and the forcing $y(\bullet,\omega)$ is a real-valued random function, with sample space $\mathscr{Y}$, probability measure $\mathscr{P}_y$, and Ch.Fl $\mathscr{F}_y(v)$, $v \in \mathscr{Y}' = \mathscr{V}$. The sample space $\mathscr{Y}$ can be taken to be a quite general, separable, *B*-space. In the present work, it will be taken as a space $\mathscr{Y} = C^k(I)$, $I \subseteq \mathbb{R}$, for some $k \in \mathbb{N} \cup \{0\}$.

Standard existence and uniqness theory (see, e.g., Bunke 1972, or Sobczyk 1991) assure that there is a stochastic process $x(\bullet;\omega)$, with sample space $\mathscr{X} = C^{k+1}(I)$ and probability measure $\mathscr{P}_x$, and a joint probability space $\left(\mathscr{X}\times\mathscr{Y}, \mathscr{B}(\mathscr{X}\times\mathscr{Y}), \mathscr{P}_{xy}\right)$, such that the joint process $\left(x(\bullet;\omega), y(\bullet;\omega)\right)$ verifies the SODE (4.1).

The joint, response-excitation, probability measure $\mathscr{P}_{xy}$ is equivalently described by the joint Ch.Fl

$$\mathscr{F}_{xy}(u,v) = \int_{\mathscr{Y}}\int_{\mathscr{X}} e^{i(\langle u,x\rangle+\langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy). \tag{4.2}$$

We shall now use the SODE (4.1) in order to obtain an FDE for $\mathscr{F}_{xy}(u,v)$. Let us consider the Volterra *u*-partial derivative of $\mathscr{F}_{xy}$ at time t:

$$\frac{\delta\mathscr{F}_{xy}(u,v)}{\delta u(t)} = \int_{\mathscr{Y}}\int_{\mathscr{X}} ix(t)e^{i(\langle u,x\rangle+\langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy). \tag{4.3}$$

Since the sample space $\mathscr{X}$ consists of smooth functions, we can differentiate (4.3) with respect to t, obtaining:

$$\frac{d}{dt}\frac{\delta\mathscr{F}_{xy}(u,v)}{\delta u(t)} = \int_{\mathscr{Y}}\int_{\mathscr{X}} i\,x'(t)e^{i(\langle u,x\rangle+\langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy). \tag{4.4}$$

Further, we compute the three-fold *u*-partial Volterra derivative of $\mathscr{F}_{xy}(u,v)$ at time instants $t_1,t_2,t_3\in I$:

$$\frac{\delta^3\mathscr{F}_{xy}(u,v)}{\delta u(t_1)\delta u(t_2)\delta u(t_3)} = \int_{\mathscr{Y}}\int_{\mathscr{X}} ix(t_1)ix(t_2)ix(t_3)e^{i(\langle u,x\rangle+\langle v,y\rangle)}\mathscr{P}_{xy}(dx,dy). \tag{4.5}$$

Setting $t_1 = t_2 = t_3 = t$ in the latter, and combining with equs. (4.3), (4.5) and (4.1a), we get

$$\frac{d}{dt}\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta u(t)}+\mu\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta u(t)}-k\frac{\delta^3 \mathscr{F}_{xy}(u,v)}{\delta u(t)^3}=$$

$$=i\int_{\mathscr{Y}}\int_{\mathscr{X}}\left[\,x'(t)+\mu x(t)+kx^3(t)\,\right]e^{i\left(\langle u,x\rangle+\langle v,y\rangle\right)}\mathscr{P}_{xy}(dx,dy)\overset{(4.1a)}{=}$$

$$=i\int_{\mathscr{Y}}\int_{\mathscr{X}}y(t)\,e^{i\left(\langle u,x\rangle+\langle v,y\rangle\right)}\mathscr{P}_{xy}(dx,dy). \tag{4.6}$$

Clearly, the last double functional integral can be expressed as a *v*-partial Volterra derivative:

$$\int_{\mathscr{Y}}\int_{\mathscr{X}}iy(t)e^{i\left(\langle u,x\rangle+\langle v,y\rangle\right)}\mathscr{P}_{xy}(dx,dy)=\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta v(t)}. \tag{4.7}$$

Combining (4.6) and (4.7) we derive the sought-for, Hopf-type, FDE that governs the joint Ch.Fl $\mathscr{F}_{xy}(u,v)$:

$$\frac{d}{dt}\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta u(t)}+\mu\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta u(t)}-k\frac{\delta^3 \mathscr{F}_{xy}(u,v)}{\delta u(t)^3}=\frac{\delta \mathscr{F}_{xy}(u,v)}{\delta v(t)}. \tag{4.8a}$$

Equ. (4.8a) is a <u>linear FDE</u> involving Volterra functional derivatives, as well as ordinary time derivatives. The cubic nonlinearity of the initial SODE corresponds to the 3-fold Volterra derivative $\delta^3 \mathscr{F}_{xy}/\delta u(t)^3$. From the above derivation it is clear that any n^{th}-order polynomial nonlinearity of the initial differential equation is transformed to an *n*-fold Volterra derivative in the corresponding Hopf-type FDE. Another important feature of equation (4.8a) is that it holds true for any continuous functionals $u\in\mathscr{U},\ v\in\mathscr{V}$.

Equ. (4.8a) has to be supplemented by an appropriate initial condition, expressing that the probability measure associated with the initial state $x(t_0,\omega)$ is given. This condition can be implemented by means of the joint Ch.Fl $\mathscr{F}_{xy}(u,v)$ as follows. Setting $v=0$ (to restrict ourselves to the response process only) and $u=\upsilon\cdot\delta(\bullet-t_0)$, $\upsilon\in\mathbb{R}$, (to concentrate only at the initial time instant), will result in

$$\mathscr{F}_{xy}\left(\upsilon\delta\left(\bullet-t_0\right),0\right)=\int_{\mathscr{Y}}\int_{\mathscr{X}}e^{i\left(\left\langle\upsilon\delta\left(\bullet-t_0\right),x\right\rangle+\left\langle 0,y\right\rangle\right)}\mathscr{P}_{xy}\left(dx,dy\right)=\int_{\mathscr{X}}e^{i\left\langle\upsilon\delta\left(\bullet-t_0\right),x\right\rangle}\mathscr{P}_{x}\left(dx\right)\;=\;\phi_0\left(\upsilon\right),$$

where $\phi_0\left(\upsilon\right)$ is the ch.f of $x\left(t_0,\omega\right)=x_0\left(\omega\right)$. Hence, the initial condition can be expressed as

$$\mathscr{F}_{xy}\left(\upsilon\delta\left(\bullet-t_0\right),0\right)=\phi_0\left(\upsilon\right),\qquad \upsilon\in\mathbb{R}\,. \tag{4.8b}$$

5 DERIVATION OF NEW PDEs FOR JOINT RESPONSE-EXCITATION CHARACTERISTIC FUNCTIONS

In this section we shall exploit the Hopf-type FDE (4.8), obtained above, to derive new PDEs for the joint, response-excitation, ch.f when the excitation is a known stochastic process either with a.e. continuous sample functions or smoother. In contrast with the case of an independent-increment excitation process, where the randomness of the excitation "regenerates" every time instant and allows us to write explicitly an equation involving only the response density (the well-known FPK equation), in the case of a stochastic excitation with smooth sample functions, the randomness evolves, in general, in a smoother way, as a result of the finite correlation time, making necessary to consider response and excitation jointly.

The causality principle dictates that the current value $x\left(t;\omega\right)$ of the response, depends only on the history of the excitation $y\left(t_0\le s<t;\omega\right)$. However, this does not prevent the stochastic dependence between $x\left(t;\omega\right)$ and $y\left(t+\varepsilon;\omega\right)$, $\varepsilon>0$, which is a natural result of the smoothness and the finite correlation time of the excitation, $C_{yy}\left(t+\varepsilon,t\right)\neq 0$.

We shall proceed to derive a PDE for the joint ch.f $\phi_{x(t)y(t)}\left(\upsilon,\nu\right)$ corresponding to the pair of random variables $\left(x\left(t;\omega\right),y\left(t;\omega\right)\right)$, $t=$ *fixed*. To this end we apply eq. (4.8a), above, to the pair

$$u\;=\;\upsilon\cdot\delta\left(\bullet-t\right),\quad v\;=\;\nu\cdot\delta\left(\bullet-s\right), \tag{5.1}$$

($\upsilon,\nu\in\mathbb{R}$) and take the limit $s\to t$, after some manipulations. For the first term of eq. (4.8a) (see also eq. (4.4)), we obtain

$$\frac{d}{dt}\left(\frac{\delta\mathscr{F}_{xy}\left(\upsilon\cdot\delta(\bullet-t),\,\nu\cdot\delta(\bullet-s)\right)}{\delta u(t)}\right) = \quad \left[\,\textit{Projection Theorem}\,\right]$$

$$= \iint\limits_{\mathscr{X}\times\mathscr{Y}} i\frac{dx(t)}{dt}\exp\left\{i\upsilon x(t)+i\nu y(s)\right\}\mathscr{P}_{xy}(dx,dy) =$$

$$= \frac{1}{\upsilon}\frac{\partial}{\partial t}\left[\iint\limits_{\mathscr{X}\times\mathscr{Y}}\exp\left\{i\upsilon x(t)+i\nu y(s)\right\}\mathscr{P}_{xy}(dx,dy)\right] =$$

$$= \frac{1}{\upsilon}\frac{\partial}{\partial t}\left[\iint\limits_{\mathbb{R}\times\mathbb{R}}\exp\left\{i\upsilon x+i\nu y\right\} f_{x(t)y(s)}(x,y)\,dxdy\right] = \frac{1}{\upsilon}\frac{\partial\phi_{x(t)y(s)}(\upsilon,\nu)}{\partial t}.$$

Taking now the limit $s\to t$, we get

$$\lim_{s\to t}\frac{d}{dt}\left(\frac{\delta\mathscr{F}_{xy}\left(\upsilon\cdot\delta(\bullet-t),\,\nu\cdot\delta(\bullet-s)\right)}{\delta u(t)}\right) = \frac{1}{\upsilon}\left.\frac{\partial\phi_{x(t)y(s)}(\upsilon,\nu)}{\partial t}\right|_{s=t}. \tag{5.2}$$

Working similarly, we readily obtain the following results concerning the remaining terms appearing in eq. (4.8a):

$$\frac{\delta\mathscr{F}\left(\upsilon\cdot\delta(\bullet-t),\nu\cdot\delta(\bullet-t)\right)}{\delta u(t)} = \frac{\partial\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon}, \tag{5.3}$$

$$\frac{\delta^3\mathscr{F}\left(\upsilon\cdot\delta(\bullet-t),\nu\cdot\delta(\bullet-t)\right)}{\delta u(t)^3} = \frac{\partial^3\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon^3}, \tag{5.4}$$

$$\frac{\delta\mathscr{F}\left(\upsilon\cdot\delta(\bullet-t),\nu\cdot\delta(\bullet-t)\right)}{\delta v(t)} = \frac{\partial\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\nu}. \tag{5.5}$$

Combining equs. (5.2)-(5.5) with the FDE (4.8a), we obtain the following PDE for the joint ch.f $\phi_{x(t)y(t)}(\upsilon,\nu)$, of the pair of random variables $\left(x(t;\omega),y(t;\omega)\right)$, for every $t>t_0$:

$$\left.\frac{1}{\upsilon}\frac{\partial\phi_{x(t)y(s)}(\upsilon,\nu)}{\partial t}\right|_{s=t} + \mu\frac{\partial\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon} - k\frac{\partial^3\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon^3} = \frac{\partial\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\nu}. \tag{5.6a}$$

Now, since the stochastic process $y(\bullet,\omega)$ is given, its ch.f $\phi_{y(t)}(\nu)$ is known. Hence, the *y*-marginal of the joint ch.f $\phi_{x(t)y(t)}(\upsilon,\nu)$ has to coincide with $\phi_{y(t)}(\nu)$, resulting in the following ***marginal compatibility condition*:**

$$\phi_{x(t)y(t)}(0,\nu) = \phi_{y(t)}(\nu), \qquad \nu \in \mathbb{R}, \quad t \geq t_0 . \tag{5.6b}$$

In addition, the initial condition (4.8b) implies the following ***initial condition*** to $\phi_{x(t)y(t)}(\upsilon,\nu)$:

$$\phi_{x(t_0)y(t_0)}(\upsilon,0) = \phi_{x(t_0)}(\upsilon) = \phi_0(\upsilon), \quad \upsilon \in \mathbb{R} . \tag{5.6c}$$

Finally, two obvious, yet essential, conditions that the sought-for function $\phi_{x(t)y(s)}(\upsilon,\nu)$ should obey are the following:

$$\phi_{x(t)y(s)}(0,0) = 1, \qquad t,s \geq t_0 , \tag{5.6d}$$

$$\phi_{x(t)y(s)}(\upsilon,\nu) \text{ is non-negative definite w.r.t. } \upsilon, \nu \text{, for any } \quad t, s \geq t_0 . \tag{5.6e}$$

which come directly from the fact that it is a characteristic function. The last two conditions will be referred to as ***constitutive conditions***.

The above problem (5.6a-e) can be equivalently reformulated in terms of the corresponding joint, response-excitation, pdf $f_{x(t)y(s)}(a,\beta)$. Recalling that $f_{x(t)y(s)}(a,\beta)$ and $\phi_{x(t)y(s)}(\upsilon,\nu)$ constitute a Fourier transform pair, i.e.

$$\phi_{x(t)y(s)}(\upsilon,\nu) = \mathcal{F}_{\substack{a\to\upsilon \\ \beta\to\nu}}\left\{ f_{x(t)y(s)}(a,\beta)\right\},$$

and applying the inverse Fourier transformation to (5.6a-e), we readily obtain the partial differential equation

$$\left.\frac{\partial f_{x(t)y(s)}(a,\beta)}{\partial t}\right|_{s=t} + \frac{\partial}{\partial a}\left[\left(\mu a + k a^3 \right) f_{x(t)y(t)}(a,\beta)\right] + \frac{\partial}{\partial a}\left[\beta\, f_{x(t)y(t)}(a,\beta)\right] = 0, \tag{5.7a}$$

the following alternative forms of the ***marginal compatibility condition*** and the ***initial condition***

$$\int_{\mathbb{R}} f_{x(t)y(t)}(a,\beta)da = f_{y(t)}(\beta),\ \ \beta \in \mathbb{R}\,, \qquad t \geq t_0\,, \tag{5.7b}$$

$$\int_{\mathbb{R}} f_{x(t_0)y(t_0)}(a,\beta)d\beta = f_{x(t_0)}(a) = f_0(a),\ \ a \in \mathbb{R}\,, \tag{5.7c}$$

as well as and the corresponding new forms of the ***constitutive conditions***

$$\int_{\mathbb{R}\times\mathbb{R}} f_{x(t)y(s)}(a,\beta)da d\beta = 1\,, \qquad t,s \geq t_0 \tag{5.7d}$$

$$f_{x(t)y(s)}(a,\beta) \geq 0\,, \qquad \textit{for any } a,\beta \in \mathbb{R} \quad \textit{any } \ t,\, s \geq t_0\,. \tag{5.7e}$$

To the best of our knowledge, equs. (5.6a-e) and (5.7a-e), governing the evolution of the joint, response-excitation, ch.f $\phi_{x(t)y(t)}(\upsilon,\nu)$ and pdf $f_{x(t)y(t)}(a,\beta)$, appear here for the first time. They can be considered as a new kind of mathematical models, providing us with the probabilistic characterization of the response $x(t,\omega)$, $\omega \in \Omega$, for each $t \in I$, obtained by taking the $y-$marginal of the joint ch.f or the joint pdf. This mathematical model is valid for any kind of stochastic excitation with a.e. continuous (or smoother) sample functions, having any (known) probabilistic structure.

Although the mathematical analysis (solvability theory) of problem (5.6a-e) or (5.7a-e) is an open problem, existing numerical evidence, presented in Section 8 (see also Sapsis & Athanassoulis 2006), suggests that it might be well-posed under reasonable assumptions.

In concluding this section we should emphasize that the above approach can be generalized in order to obtain similar, linear, PDEs for the joint, *N-x* and *M-y,* ch.f

$$\phi_{x(t_1)\ldots x(t_N)y(s_1)\ldots y(s_M)}(\upsilon_1,\ldots\upsilon_N,\nu_1,\ldots,\nu_M)$$

(or the corresponding joint pdf), along with appropriate (marginal compatibility and initial) conditions. This point will be further discussed in another work (Athanassoulis & Sapsis 2007). It seems that in this way it is possible to construct a closed (finitely-solvable) hierarchy of linear problems providing us with the full hierarchy of the finite-dimensional probabilities of the stochastic response $x(\bullet;\omega)$.

6 DERIVATION OF THE FPK EQUATION FOR THE CASE OF INDEPENDENT INCREMENT EXCITATION

Eqs. (5.6) and (5.7) hold true for any kind of stochastic excitation process, provided that the latter has at least a.e. continuous sample functions. We shall now turn to the most commonly studied case, those of an Ito SODE, where $y(t;\omega)$ represents the generalized derivative of an independent-increment process. In this case the response $x(t;\omega)$ is continuous but not differentiable. Thus, the treatment based on the Hopf equation, as developed in Section 5, is not valid, since the duality pairings (5.1) are not applicable. The question arises if it is possible to treat this case also by a similar method, starting from the Hopf equation and obtaining the usual FPK equation – which involves only the response ch.f (or pdf). In the present section we shall show how this is possible, by resorting back to the FDE for a finite-difference version of the SODE (4.1). The crucial property, to be exploited in this case, is the independence of the current value $x(t;\omega)$ of the response from the future increment $\Delta_\tau z(t;\omega) = z(t+\tau;\omega) - z(t;\omega)$, $\tau > 0$, of the excitation. Everything presented in this Section can be generalized to multidimensional nonlinear dynamical systems.

Let us rewrite the SODE (4.1a,b) in a finite-difference form:

$$\frac{\Delta_\tau x(t;\omega)}{\tau} + \mu x(t;\omega) + k x^3(t;\omega) = \frac{\Delta_\tau z(t;\omega)}{\tau}, \tag{6.1a}$$

$$x(t_0;\omega) = x_0(\omega), \tag{6.1b}$$

where $z(\bullet;\omega)$ is a known, real-valued process with independent increments, and $x_0(\omega)$ is a known random variable. The time increment τ is assumed to be positive, $\tau > 0$, and this is essential in what follows.

The sample functions of the stochastic process $z(\bullet;\omega)$ may be either continuous functions (as in the case of normally-distributed, independent-increment processes) or non-decreasing, piecewise-constant functions (as in the case of Poisson distributed independent-increment processes). In the first case (continuous sample functions), it is clear that the previously developed approach can be applied to equ. (6.1). In the second case (*cadlag* sample functions) the applicability of the same arguments is not directly justifiable. Nevertheless, we shall take the liberty not to be completely

rigorous, and apply the same approach to the general case as well. It seems to us quite remarkable and fascinating that the obtained PDE for the ch.f of the response $x(\bullet;\omega)$ coincides with the known one in all examined cases. Thus, the results of the present section can be considered as a rigorous rederivation of the classical FPK equation from the Hopf FDE, in the case of Gaussian forcing, and as a heuristic method to derive analogous equations in the case of a Poissonian or an $\alpha-$stable or a general Levy process forcing.

Working similarly as in Section 4, we obtain the following Hopf-type FDE that governs the evolution of the Ch.Fl $\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)$, parametrically dependent on $\tau>0$:

$$\tau^{-1}\Delta_\tau\left(\frac{\delta\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)}{\delta u(t)}\right)+\mu\,\frac{\delta\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)}{\delta u(t)}-k\,\frac{\delta^3\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)}{\delta u(t)^3}$$

$$=\frac{\delta\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)}{\delta v(t)}\quad, \tag{6.2a}$$

$$\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}\big(\upsilon\delta(\bullet-t_0),0\big)=\phi_0(\upsilon)\,,\qquad \upsilon\in\mathbb{R}\,. \tag{6.2b}$$

Note that $\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)$ is the finite-difference version of $\mathscr{F}_{xy}(u,v)=\lim_{\tau\to 0}\mathscr{F}_{x(\tau^{-1}\Delta_\tau z)}(u,v)$.

Using again the arguments $u(\bullet)$, $v(\bullet)$, given by (5.1), and applying the same treatment as in Section 5, we obtain the following PDE that governs the joint ch.f $\phi_{x(\tau^{-1}\Delta_\tau z)}$:

$$\frac{1}{\upsilon}\tau^{-1}\Delta_\tau\left(\phi_{x(t)(\tau^{-1}\Delta_\tau z(s))}(\upsilon,\nu)\right)\Big|_{s=t}+\mu\frac{\partial\phi_{x(t)(\tau^{-1}\Delta_\tau z(t))}(\upsilon,\nu)}{\partial\upsilon}-k\,\frac{\partial^3\phi_{x(t)(\tau^{-1}\Delta_\tau z(t))}(\upsilon,\nu)}{\partial\upsilon^3}=$$

$$=\frac{\partial\phi_{x(t)(\tau^{-1}\Delta_\tau z(t))}(\upsilon,\nu)}{\partial\nu}. \tag{6.3}$$

Setting $\nu=0$ in the equ. (6.3) and taking the limit as $\tau\to 0^+$, we obtain:

$$\frac{1}{\upsilon}\frac{\partial\phi_{x(t)}(\upsilon)}{\partial t}+\mu\frac{\partial\phi_{x(t)}(\upsilon)}{\partial\upsilon}-k\frac{\partial^3\phi_{x(t)}(\upsilon)}{\partial\upsilon^3}=\lim_{\tau\to0^+}\left.\frac{\partial\phi_{x(t)\left(\tau^{-1}\Delta_\tau z(t)\right)}(\upsilon,\nu)}{\partial\nu}\right|_{\nu=0}. \qquad (6.4)$$

In the left-hand side of eq. (6.4) we can already recognize the sought-for result. In the right-hand side, because of the $\nu-$derivative, the situation is more complicated and should be studied further. In analogy with eqs (4.8a) and (4.7), the "source" term $\partial\phi_{x(t)\left(\tau^{-1}\Delta_\tau z(t)\right)}(\upsilon,\nu)/\partial\nu$, appearing in the right-hand side of eqs. (6.3) and (6.4), comes from the following functional integral

$$\frac{\partial\phi_{x(t)\left(\tau^{-1}\Delta_\tau z(t)\right)}(\upsilon,\nu)}{\partial\nu}\equiv I_\tau(\upsilon,\nu)=i\int_{\mathscr{Y}}\int_{\mathscr{X}}\left(\tau^{-1}\Delta_\tau z(t)\right)e^{i\left(\langle u,x\rangle+\left\langle\nu,\tau^{-1}\Delta_\tau z\right\rangle\right)}\mathscr{P}_{x\left(\tau^{-1}\Delta_\tau z(t)\right)}\left(dx,d\left(\tau^{-1}\Delta_\tau z\right)\right)$$

where $\mathscr{Y}$ is an appropriate space of continuous functions. Now, using the identity

$$\mathscr{P}_{x,\left(\tau^{-1}\Delta_\tau z\right)}\left(dx,d\left(\tau^{-1}\Delta_\tau z\right)\right)=\mathscr{P}_{x,\Delta_\tau z}\left(dx,d\left(\Delta_\tau z\right)\right),$$

we obtain

$$I_\tau=i\int_{\mathscr{Y}}\int_{\mathscr{X}}\left(\tau^{-1}\Delta_\tau z(t)\right)e^{i\left(\langle u,x\rangle+\left\langle\nu,\tau^{-1}\Delta_\tau z\right\rangle\right)}\mathscr{P}_{x,\Delta_\tau z}\left(dx,d\left(\Delta_\tau z\right)\right).$$

Let us now evaluate the above functional integral (for $\tau=\mathit{fixed}>0$), under the specific choice of arguments $u(\bullet)=\upsilon\cdot\delta(\bullet-t)$ and $v(\bullet)=\nu\cdot\delta(\bullet-t)$:

$$I_\tau\left(\upsilon\delta(\bullet-t),\ \nu\cdot\delta(\bullet-t)\right)=i\int_{\mathscr{Y}}\int_{\mathscr{X}}\left(\frac{\Delta_\tau z(t)}{\tau}\right)e^{i\left(\upsilon x(t)+i\nu\frac{\Delta_\tau z(t)}{\tau}\right)}\mathscr{P}_{x,\Delta_\tau z}\left(dx,d\left(\Delta_\tau z\right)\right). \qquad (6.5)$$

Because of the specific form of the excitation (independent-increment process), the response $x(t;\omega)$ is stochastically independent from the future increment of the forcing $\Delta_\tau z(t;\omega)=z(t+\tau;\omega)-z(t;\omega)$. (at this point we make use of the assumption $\tau>0$). As a consequence, the joint probability measure $\mathscr{P}_{x,\Delta_\tau z}\left(dx,d\left(\Delta_\tau z\right)\right)$ can be written in multiplicative form

$$\mathscr{P}_{x,\Delta_\tau z}\left(dx,d\left(\Delta_\tau z\right)\right)=\mathscr{P}_x(dx)\cdot\mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right). \qquad (6.6)$$

Taking this into account, and making the substitution $\nu = \upsilon\tau$ (note that we are interested in the double limit $\nu \to 0$ and $\tau \to 0^+$), the double functional integral in the right-hand side of (6.5) can be factored out as follows:

$$I_s\left(\upsilon\delta(\bullet - t),\ \upsilon\tau\cdot\delta(\bullet - t)\right) = \int_{\mathscr{X}} \exp\{i\upsilon x(t)\}\mathscr{P}_x(dx)\times$$

$$\times \int_{\mathscr{Z}} i\left(\frac{\Delta_\tau z(t)}{\tau}\right)\exp\{i\upsilon\Delta_\tau z(t)\}\ \mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right). \qquad (6.7)$$

On the basis of the Projection Theorem (eq. (2.6)), the first functional integral of the right-hand side of the above equation is simply the ch.f of $x(t;\omega)$:

$$\int_{\mathscr{X}} \exp\{i\upsilon x(t)\}\mathscr{P}_x(dx) = \phi_{x(t)}(\upsilon). \qquad (6.8)$$

To calculate the second functional integral in the right-hand side of eq. (6.7), we start by considering the functional integral:

$$J_\tau(\upsilon) = \int_{\mathscr{Z}} \exp\{i\upsilon(\Delta_\tau z(t))\}\ \mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right).$$

By the same token as above, $J_\tau(\upsilon)$ is the ch.f of the increment $\Delta_\tau z(t;\omega)$: $J_\tau(\upsilon) = \phi_{\Delta_\tau z(t)}(\upsilon)$. Assuming the latter is τ–differentiable in the vicinity of $\tau = 0^+$, we get

$$\left.\frac{\partial\phi_{\Delta_\tau z(t)}(\upsilon)}{\partial\tau}\right|_{\tau=0^+} = \lim_{\tau\to 0^+}\int_{\mathscr{Z}} i\upsilon\frac{\partial\left(\Delta_\tau z(t)\right)}{\partial\tau}e^{i\upsilon(\Delta_\tau z(t))}\mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right) =$$

$$= i\upsilon\lim_{\tau\to 0^+}\int_{\mathscr{Z}}\left(\frac{\Delta_\tau z(t)}{\tau}\right)e^{i\upsilon(\Delta_\tau z(t))}\mathscr{P}_{\Delta_\tau z}\left(d\left(\Delta_\tau z\right)\right). \qquad (6.9)$$

The last term in (6.9) coincides –apart from the factor υ– with the second integral in the right-hand side of equ. (6.7). Thus, on the basis of (6.8) and (6.9), we can rewrite eq. (6.7) as follows:

$$\lim_{\tau \to 0^+} I_\tau \left(\upsilon \delta(\bullet - t),\ \upsilon\tau \cdot \delta(\bullet - t) \right) \ = \ \frac{1}{\upsilon}\, \phi_{x(t)}(\upsilon) \cdot \lim_{\tau \to 0^+} \frac{\partial \phi_{\Delta_\tau z(t)}(\upsilon)}{\partial \tau} \ . \tag{6.10}$$

Combining now (6.4) and (6.10), we obtain

$$\frac{\partial \phi_{x(t)}(\upsilon)}{\partial t} \ + \ \mu\upsilon \frac{\partial \phi_{x(t)}(\upsilon)}{\partial \upsilon} \ - k\upsilon \frac{\partial^3 \phi_{x(t)}(\upsilon)}{\partial \upsilon^3} = \phi_{x(t)}(\upsilon) \lim_{\tau \to 0^+} \frac{\partial \phi_{\Delta_\tau z(t)}(\upsilon)}{\partial \tau}\ ,\ \ \upsilon \in \mathbb{R}\ . \tag{6.11}$$

This is in fact the generalized FPK equation for general, independent-increment, excitation, written in terms of the ch.f of the response process. The corresponding FPK equation, in terms of the pdf, is easily derived by applying a Fourier transformation

$$\frac{\partial f_{x(t)}(a)}{\partial t} + \frac{\partial}{\partial a}\left[\left(\mu a + k a^3 \right) f_{x(t)}(a) \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\upsilon(y-a)} f_{x(t)}(y) \lim_{\tau \to 0^+} \frac{\partial \phi_{\Delta_\tau z(t)}(\upsilon)}{\partial \tau} dy d\upsilon \tag{6.12}$$

It can be shown (Athanassoulis & Sapsis 2007) that the above equation includes as special cases various generalized FPK equations, recently obtained by Grigoriu (2004).

7 MOMENT EQUATIONS FROM THE NEW PDE (5.6)

It is worth noticing that the PDE (5.6a), derived at Section 5, can reproduce the infinite set of moment equations corresponding to the dynamical system equation (4.1a). This is a very important consistency result that can be interpreted twofold. From the point of view of the new PDE (5.6a), it provides an independent check of validity. From the point of view of the infinite system of moment equations, it provides an "integrating scheme" permitting the replacement of the infinite system of ODEs by a single linear PDE. The remaining of this section is devoted to the proof of the above mentioned consistency result.

Let us denote by $M_{nm}(t,s) = \mathbf{E}^\omega \left[x^n(t;\omega) \cdot y^m(s;\omega) \right]$, $n,m = 0,1,\ldots$, the joint $(n,m)^{th}-$order moment of $x(t;\omega)$ and $y(s;\omega)$. Then, by direct integration of eq. (4.1a), it is easily seen that infinite system of moment equations has the form

$$\frac{1}{n+1} \cdot \left. \frac{dM_{n+1,m}(t,s)}{dt} \right|_{s=t} \ + \ \mu M_{n+1,m}(t,t) \ = -k M_{n+3,m}(t,s) + M_{n,m+1}(t,s). \tag{7.1}$$

We shall now derive the same equations (7.1) using the PDE (5.6a). Recall first that

$$\left.\frac{\partial^{n+m}\phi_{x(t)y(s)}(\upsilon,\nu)}{\partial\upsilon^n\partial\nu^m}\right|_{\substack{\upsilon=0\\ \nu=0}} = i^{n+m}\mathbf{E}^{\omega}\left[x^n(t;\omega)\cdot y^m(s;\omega)\right] = i^{n+m}M_{n,m}(t,s), \quad n,m=0,1,\ldots . \tag{7.2}$$

By direct differentiation of eq. (5.6a) we obtain

$$\left.\frac{\partial^{n+m+2}\phi_{x(t)y(s)}(\upsilon,\nu)}{\partial\upsilon^{n+1}\partial\nu^m\partial t}\right|_{s=t} + \mu\frac{\partial^{n+m+1}}{\partial\upsilon^{n+1}\partial\nu^m}\left[\upsilon\frac{\partial\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon}\right] + k\frac{\partial^{n+m+1}}{\partial\upsilon^{n+1}\partial\nu^m}\left[\upsilon\frac{\partial^3\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial^3\upsilon}\right] =$$
$$= \frac{\partial^{n+m+1}}{\partial\upsilon^{n+1}\partial\nu^m}\left[\upsilon\frac{\partial\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\nu}\right]. \qquad \upsilon,\nu\in\mathbb{R}\,. \tag{7.3}$$

In accordance with (7.2), the first term in the left-hand side of eq (7.3) can be written as

$$\frac{\partial}{\partial t}\left.\frac{\partial^{n+m+1}\phi_{x(t)y(s)}(\upsilon,\nu)}{\partial\upsilon^{n+1}\partial\nu^m}\right|_{\substack{\upsilon=0\\ \nu=0}} = i^{n+m+1}\frac{\partial M_{n+1,m}(t,s)}{\partial t}\,. \tag{7.4a}$$

To proceed with the remaining three terms in eq. (7.3), use will be made of the following Lemma:

Lemma 7.1**:** For every C^n-differentiable function $f:\mathbb{R}\to\mathbb{R}$, we have

$$\left.\frac{d^n}{dx^n}\left[xf(x)\right]\right|_{x=0} = n\left.\frac{d^{n-1}f(x)}{dx^{n-1}}\right|_{x=0}. \quad \blacksquare$$

Hence,

$$\left.\frac{\partial^{n+m+1}}{\partial\upsilon^{n+1}\partial\nu^m}\left[\upsilon\frac{\partial\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon}\right]\right|_{\substack{\upsilon=0\\ \nu=0}} = (n+1)\left.\frac{\partial^{n+m+1}\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon^{n+1}\partial\nu^m}\right|_{\substack{\upsilon=0\\ \nu=0}}, \tag{7.4b}$$

$$\left.\frac{\partial^{n+m+1}}{\partial\upsilon^{n+1}\partial\nu^m}\left[\upsilon\frac{\partial^3\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial^3\upsilon}\right]\right|_{\substack{\upsilon=0\\ \nu=0}} = (n+1)\left.\frac{\partial^{n+m+3}\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon^{n+3}\partial\nu^m}\right|_{\substack{\upsilon=0\\ \nu=0}}, \tag{7.4c}$$

and

$$\left.\frac{\partial^{n+m+1}}{\partial\upsilon^{n+1}\partial\nu^m}\left[\upsilon\frac{\partial\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\nu}\right]\right|_{\substack{\upsilon=0\\ \nu=0}} = (n+1)\left.\frac{\partial^{n+m+1}\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon^{n}\partial\nu^{m+1}}\right|_{\substack{\upsilon=0\\ \nu=0}}. \tag{7.4d}$$

Substituting eq. (7.4b,c,d) in eq. (7.3) we obtain

$$\left(\frac{1}{(n+1)}\frac{\partial}{\partial t}\frac{\partial^{n+m+1}\phi_{x(t)y(s)}(\upsilon,\nu)}{\partial\upsilon^{n+1}\partial\nu^{m}}+\mu\frac{\partial^{n+m+1}\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon^{n+1}\partial\nu^{m}}+k\frac{\partial^{n+m+3}\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon^{n+3}\partial\nu^{m}}\right)\Bigg|_{\substack{\upsilon=0\\ \nu=0}}=$$

$$=\frac{\partial^{n+m+1}\phi_{x(t)y(t)}(\upsilon,\nu)}{\partial\upsilon^{n}\partial\nu^{m+1}}\Bigg|_{\substack{\upsilon=0\\ \nu=0}}. \quad (7.5)$$

By applying eq. (7.2) and (7.4a), the above equation reduces to the infinite system of moment equations (7.1). This completes the proof of the consistency result announced at the beginning of this section.

8 KERNEL DENSITY REPRESENTATION OF JOINT pdfs

Clearly, problem (5.6) –either in the form (5.6a-e) or in the form (5.7a-e)– exhibits some peculiarities making it distinctly different from the usual initial-boundary value problems for PDEs, coming from problems of Mathematical Physics. These peculiarities reflect the probabilistic origin of the present problem.

In the remaining part of this paper, an original (particle-type) method for the numerical solution of problem (5.6) (or (5.7)) is developed, and some first, illustrative, numerical results are presented. The main tool, on which the formulation of the numerical scheme relies, is the representation of the sought-for pdf and ch.f by means of convex superpositions of kernel density functions (kdfs) and their Fourier transformation, the kernel characteristic functions (kch.fs), respectively. A short presentation of the basic facts about kdfs is given below.

Kernel density functions constitute a key notion/tool within the framework of nonparametric statistical estimation. See, e.g., Scott 1992. In our approach, a kdf $K(x;x_*,h)$ is mainly thought of as a generalized (non-symmetric) summability kernel, appropriate to represent pdfs (Gavriliadis 2005). The defining properties of an $M-$ variate kdf are the following:

Pr.1) $K(x;x_*,h)$ is a continuous, real-valued function defined on a domain of the form $\mathscr{D}_K = A\times A\times\mathscr{M}_{M\times M}^{NonNegDef}$, where $A\subseteq\mathbb{R}^M$ is taken to be contained in (or to be equal to) the support of the target pdf, say $f(x)$, which is to be represented (see Lemma 8.1 and Theorem 8.2, below), and $\mathscr{M}_{M\times M}^{NonNegDef}$ is the set

of non-negative definite, $M\times M-$ matrices, which can serve as covariance matrices.

Pr.2) $K(x;x_*,h)\geq 0$, for $(x;x_*,h)\in \mathscr{D}_K$.

Pr.3) $\int_A K(x;x_*,h)\,dx_* = 1$, for $(x,h)\in A\times \mathscr{M}_{M\times M}^{NonNegDef}$.

Pr.4) $\lim_{\|h\|\to 0} \int_{\|x-x_*\|>\delta} K(x;x_*,h)\,dx_* = 0$, *for any* $x_*\in A$ *and* $\delta>0$.

A kernel characteristic function is defined as the Fourier transformation of a kdf. Clearly, properties Pr.2), Pr.3) ensure that each kdf is a pdf on its own. The ***shape*** of the kernel function $K(x;x_*,h)$ is controlled by its covariance matrix h, also called *bandwidth* (or *shape*) *parameter*. h quantifies the spreading of the kernel probability mass around its "***center***" x_*. Another –simpler and in many cases adequate– choice of the shape parameter is the *M*-variate vector of the eigenvalues of the covariance matrix. In this sense, the domain $\mathscr{D}_K = A\times A\times \mathscr{M}_{M\times M}^{NonNegDef}$ can be (and will be) simplified as $A\times A\times [0,\infty)^M$.

Using the defining properties Pr.1) – Pr.4), and only these, it is not difficult to prove the following

Lemma 8.1: If $f(x)$ is a continuous pdf and $K(\cdot;\cdot,\cdot)$ is any kernel function satisfying Pr.1) – Pr.4), then, for any x,

$$\lim_{\|h\|\to 0}\int_A K(x;x_*,h)\,f(x_*)\,dx_* = f(x). \qquad \blacksquare \tag{8.1}$$

That is, as the bandwidth decreases, the kernel function shrinks around its "center" x_*, having the weak asymptotic limit

$$K(x;x_*,h) \xrightarrow{\|h\|\to 0} \delta(x-x_*). \tag{8.2}$$

On the other hand, as the bandwidth increases the kernel function spreads out.

Theorem 8.2: The set of all convex finite superpositions of the form $\sum_{n=1}^{N} p_n K(x; x_n, h_n)$, where $p_1 + p_2 + \ldots + p_N = 1,\ p_n \geq 0$ for all n, and $K(\cdot;\cdot,\cdot)$ is any kernel function satisfying Pr.1) – Pr.4), is dense within the set of all continuous pdfs supported in A. That is, given any continuous pdf $f(x)$, a specific kernel function $K(x; x_*, h)$, and an arbitrary (small) number $\varepsilon > 0$, there exist a bandwidth parameter h_*, a finite set of centers $\{x_n\}_{n=1}^{N}$ in A, and a vector $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ lying in the positive cone of $\mathbb{R}^N$, such that

$$\max_{x \in A} \left| f(x) - f^N(x) \right| < \varepsilon, \tag{8.3a}$$

where $$f^N(x) = \sum_{n=1}^{N} p_n K(x; x_n, h_n). \quad \blacksquare \tag{8.3b}$$

The clue of the proof of this theorem is Lemma 8.1, in conjunction with the properties of the Riemann sum approximation of the integral $\int_A K(x; x_*, h)\, f(x_*)\, dx_*$ (Athanassoulis and Gavriliadis 2002). The technical details are omitted. The above theorem makes clear that any (continuous) pdf can be approximated, as closely as it is required, by a representation of the form (8.3b).

9 REFORMULATION OF THE PROBLEM BY USING KERNEL DENSITY REPRESENTATIONS

We shall now apply the pdf representation (8.3b) (or the corresponding ch.f representation, obtained by means of a Fourier transformation) in order to reformulate problem (5.7) (or (5.6a)) in a way facilitating its numerical solution. Again here and subsequently, as in the Introduction, $f_{\mathbf{xy}} = f_{x(t)y(s)}(a, \beta)$, $\phi_{xy} = \phi_{x(t)y(s)}(\upsilon, \nu)$ are four-argument, two-variate, joint, response-excitation pdf and ch.f, respectively. For clarity, in the present and the subsequent sections, vector or matrix quantities will be explicitly denoted by using bold letters.

Applying the representation (8.3b) for the pdf, and the corresponding one for the ch.f we define the approximants

$$f^N_{x(t)y(s)}(a, \beta) = \sum_{k=1}^{N} p_k(t,s)\, K\left(a, \beta; \mathbf{m}^k(t,s), \mathbf{h}^k(t,s)\right), \tag{9.1}$$

$$\phi^N_{x(t)y(s)}(\upsilon,\nu)=\mathcal{F}_{\substack{a\to\upsilon\\ \beta\to\nu}}\left\{f^N_{x(t)y(s)}(a,\beta)\right\}=\sum_{k=1}^{N}p_k(t,s)\tilde{K}\left(\upsilon,\nu;\mathbf{m}^k(t,s),\mathbf{h}^k(t,s)\right). \qquad (9.2)$$

Here $\mathbf{m}^k=\left(m^k_x,m^k_y\right)$ is the location parameter, namely the position of the most probable (highest) value of the kdf, and $\mathbf{h}^k$ is the shape parameter, represented either by the $2\times 2-$ covariance matrix of the kdf or by the two eigenvalues of the latter (both pictures will be applied to the numerical treatment). For the numerical computations, $K\left(a,\beta;\mathbf{m}^k,\mathbf{h}^k\right)$ is taken to be a Gaussian pdf. (See, e.g., Härdle 1990, Sec. 2.9).

Our main goal now, is to exploit the representations (9.1), (9.2) in order to solve the system (5.6) or the equivalent (5.7). Conditions (5.6d,e), or the equivalent (5.7d,e), are automatically satisfied since the approximants (9.1), (9.2) are by construction pdfs and ch.fs, respectively.

To facilitate the discussion, we define the linear differential operators

$$\mathcal{L}\bullet=\left.\frac{\partial\bullet}{\partial t}\right|_{s=t}+\mu\frac{\partial[a\bullet]}{\partial a}+k\frac{\partial\left[a^3\bullet\right]}{\partial a}-\frac{\partial[\beta\bullet]}{\partial a}. \qquad (9.3a)$$

$$\tilde{\mathcal{L}}\bullet=\left.\frac{\partial\bullet}{\partial t}\right|_{s=t}+\mu\upsilon\frac{\partial\bullet}{\partial\upsilon}-k\upsilon\frac{\partial^3\bullet}{\partial\upsilon^3}-\upsilon\frac{\partial\bullet}{\partial\nu}, \qquad (9.4a)$$

And rewrite equs. (5.6a), (5.7a) in the following concise form:

$$\mathcal{L}\left[f_{xy}\right](a,\beta,t)=0\ ,\qquad (a,\beta)\in\mathbb{R}^2,\quad t\ge t_0\,, \qquad (9.3b)$$

$$\tilde{\mathcal{L}}\left[\phi_{\mathbf{xy}}\right](\upsilon,\nu,t)=0\ ,\qquad (\upsilon,\nu)\in\mathbb{R}^2,\quad t\ge t_0\,, \qquad (9.4b)$$

It is interesting to note here that the two equivalent formulations –(5.7a) or (9.3) in terms of the pdf, and (5.6a) or (9.4) in terms of the ch.f– are both useful and they will be considered in parallel, since the conceptual arguments are better stated using the pdf formulation, while the numerical analysis is better developed using the ch.f formulation.

Substituting the approximation (9.1) into eq. (9.3), we obtain

$$\sum_{k=1}^{N}\mathcal{L}\left[p_kK\left(a,\beta;\mathbf{m}^k,\mathbf{h}^k\right)\right]=0\ ,\qquad (a,\beta)\in\mathbb{R}^2 \qquad (9.5)$$

Let us denote by $\varepsilon(\mathbf{h}^j)$ the radius of the effective support of $K(a,\beta;\mathbf{m}^j,\mathbf{h}^j)$. $\varepsilon(\mathbf{h}^j)$ will be taken and always kept to be small. Since each kernel function $K(a,\beta;\mathbf{m}^j,\mathbf{h}^j)$ is taken to be concentrated around its center $\mathbf{m}^j=(m_x^j,m_y^j)$ and it is positive there, eq. (3.5), restricted in a neighborhood $\mathcal{N}(\mathbf{m}^j,\varepsilon(\mathbf{h}^j))$, is locally equivalent with the equation

$$\sum_{k=1}^{N}\mathcal{L}\left[p_k K(a,\beta;\mathbf{m}^k,\mathbf{h}^k)\right]\cdot K(a,\beta;\mathbf{m}^j,\mathbf{h}^j)=0,\quad (a,\beta)\in\mathcal{N}(\mathbf{m}^j,\varepsilon(\mathbf{h}^j)) \tag{9.6}$$

Assuming that the system of neighborhoods $\left\{\mathcal{N}(\mathbf{m}^j,\varepsilon(\mathbf{h}^j)),\ j=1,\ldots,N\right\}$ covers the essential support of the sought-for density function f_{xy}, we can assert that the global equation (3.5) is equivalent with the system of local equations

$$\sum_{k=1}^{N}\mathcal{L}\left[p_k K(a,\beta;\mathbf{m}^k,\mathbf{h}^k)\right]\cdot K(a,\beta;\mathbf{m}^j,\mathbf{h}^j)=0,\forall\ j\in\{1,\ldots,N\},\ \text{and}$$

$$\forall\ (a,\beta)\in\bigcup_{j=1,\ldots,N}\mathcal{N}(\mathbf{m}^j,\varepsilon(\mathbf{h}^j)). \tag{9.7}$$

By taking a Fourier transformation, eq. (9.7) is equivalently rewritten as

$$\sum_{k=1}^{N}\tilde{\mathcal{L}}\left[p_k \tilde{K}(\upsilon,\nu;m^k,h^k)\right]*\tilde{K}(\upsilon,\nu;m^j,h^j)=0,\quad \forall\ j\in\{1,\ldots,N\},\ \text{and}$$

$$\forall\ (\upsilon,\nu)\in\mathbb{R}^2, \tag{9.8}$$

where $*$ denotes the convolution operator. Although the latter equation could be considered as being more complicated than eq. (9.7), an efficient numerical solution scheme will be based on it.

10 A TWO-LEVEL NUMERICAL SOLUTION SCHEME FOR THE SET OF EQs (9.8)

To proceed to the numerical solution, use will be made of a specific choice of the kdf. Assuming a Gaussian density as the kdf, we have

$$K\left(a,\beta;\mathbf{m}^k,\mathbf{C}^k\right)=\frac{1}{2\pi\sqrt{\left|\det\left[\boldsymbol{C}\right]\right|}}\exp\left[-\frac{1}{2}\begin{pmatrix}a-m_x^k\\ \beta-m_y^k\end{pmatrix}^T\left[\boldsymbol{C}^k\right]^{-1}\begin{pmatrix}a-m_x^k\\ \beta-m_y^k\end{pmatrix}\right], \tag{10.1a}$$

with corresponding kernel characteristic function

$$\tilde{K}\left(\upsilon,\nu;\mathbf{m}^k,\mathbf{C}^k\right)=\exp\left(i\begin{bmatrix}m_x^k\\ m_y^k\end{bmatrix}^T\cdot\begin{bmatrix}\upsilon\\ \nu\end{bmatrix}-\frac{1}{2}\begin{bmatrix}\upsilon\\ \nu\end{bmatrix}^T\cdot\left[\boldsymbol{C}^k\right]\cdot\begin{bmatrix}\upsilon\\ \nu\end{bmatrix}\right), \tag{10.1b}$$

where

$$\mathbf{m}^k=\mathbf{m}^k\left(t,s\right)=\left(m_x^k\left(t\right),\,m_y^k\left(s\right)\right) \tag{10.2a}$$

is the mean vector, and

$$\boldsymbol{C}^k=\boldsymbol{C}^k\left(t,s\right)=\begin{pmatrix}C_{xx}^k\left(t,t\right) & C_{xy}^k\left(t,s\right)\\ C_{yx}^k\left(s,t\right) & C_{yy}^k\left(s,s\right)\end{pmatrix} \tag{10.2b}$$

is the covariance matrix of our Gaussian kdf. As we have already mentioned above, a dual realization of the shape parameter will be considered herewith. Apart from the covariance matrix $\boldsymbol{C}^k$, the vector $\mathbf{h}^k=\left(h_x^k,h_y^k\right)$ having as elements the two eigenvalues of the matrix $\boldsymbol{C}^k$, will also be used in this case.

Our numerical solution scheme will be implemented by restricting the kdf to be highly concentrated, so that the effective supports of any pair of two different kernels to be practically ***non overlapping***. This permits us to neglect the interaction between any pair of Gaussian kernels, i.e. to disregard the summation in the left-hand side of eq. (9.7) and its equivalent eq. (9.8). Thus, under the above assumption, which is equivalent with the condition $\left\|\mathbf{h}^k\right\|<\varepsilon_1$, for all $k\in\left\{1,\ldots,N\right\}$, where ε_1 is an appropriate (small) constant, eq. (9.8) simplifies to

$$\tilde{\mathcal{L}}\left[p_j\tilde{K}\left(\upsilon,\nu;\,\mathbf{m}^j,\mathbf{C}^j\right)\right]=0,\qquad t\geq s\geq t_0,\quad\text{and}\quad\forall\ j\in\left\{1,\ldots,N\right\},\ \text{and}\ \forall\left(\upsilon,\nu\right)\in\mathbb{R}^2. \tag{10.3a}$$

Furthermore, assuming the amplitudes p_j are positive and piecewise constant, the above equation is further simplified to

$$\tilde{\mathcal{L}}\left[\tilde{K}\left(\upsilon,\nu;\mathbf{m}^j,\mathbf{C}^j\right)\right]=0,\ \text{within each time interval}\quad \tau^{(\ell)}\le s\le t\le\tau^{(\ell+1)},$$

$$\forall\ j\in\{1,\ldots,N\},\ \text{and}\quad \forall\ (\upsilon,\nu)\in\mathbb{R}^2. \tag{10.3b}$$

On the basis of the above discussion, a two-level (two-time scale) approach comes into the scene:

a. Solve the set of independent equations (10.3b) within each interval $\tau^{(\ell)}\le s\le t\le\tau^{(\ell+1)}$ (this is the ***short-time phase*** or ***inner-cycle phase***), and then

b. Come back to the complete representation and update the values of the amplitudes p_j, passing from the interval $\left[\tau^{(\ell)},\tau^{(\ell+1)}\right]$ to the interval $\left[\tau^{(\ell+1)},\tau^{(\ell+2)}\right]$ (this is the ***coarse-time phase*** or ***the outer-cycle phase***).

The ***criterion*** for defining the sequence of coarse updating times $\tau^{(\ell)}$, $\ell=1,2,3,\ldots$, is formulated as a sufficient condition for the validity of the assumptions underlying the derivation of the set of independent equations (10.3b). It turns out that the most critical assumption is the restriction of each kdf to be highly concentrated around its center. As expected, because of the diffusive character of the problem, it has been found that, during the short-time phase solution, kernel parameters evolve in a way leading to a continuous increase of the variance parameter $\left\|\mathbf{h}^k\right\|$. (See, for example, Figures 2c, 3c, in Section 11, below, and the discussion therein). The growth of the quantity $\left\|\mathbf{h}^k\right\|$ leads to the spreading of the mass of the corresponding kdf, which results in the violation of the assumption of negligible interaction between the kernels.

Thus, the set of kernel parameters $\mathbf{m}^k(t,s)$ and $\boldsymbol{C}^k(t,s)$ evolve in accordance with the simplified dynamical equations (10.3b) from time $\tau^{(\ell)}$, until the spreading index $\left\|\mathbf{h}^k(t)\right\|$, of some kernel, exceeds a certain critical value, say $\varepsilon_1>0$. This value of *t* is taken to be the next updating time $\tau^{(\ell+1)}$. At that time instant, the inner-cycle (short-time) solution phase is interrupted, and an approximation of the total joint pdf $f^N_{x\left(\tau^{(\ell+1)}\right)y\left(\tau^{(\ell+1)}\right)}(a,\beta)$ is calculated by means of eq. (9.1), in the specific form:

$$f^N_{x\left(\tau^{(\ell+1)}\right)y\left(\tau^{(\ell+1)}\right)}(a,\beta)=\sum_{k=1}^{N}p_k\left(\tau^{(\ell)}\right)K\left(a,\beta\,;\mathbf{m}^k\left(\tau^{(\ell+1)},\tau^{(\ell+1)}\right),\mathbf{h}^k\left(\tau^{(\ell+1)},\tau^{(\ell+1)}\right)\right). \tag{10.4}$$

Then, the calculated pdf (10.4) is re-approximated, by using a new set of kdfs, satisfying the concentration condition $\left\|\mathbf{h}^k\right\| = \varepsilon_2 < \varepsilon_1$, with different amplitudes $p_k\left(\tau^{(\ell+1)}\right)$. The latter are calculated by means of an optimization algorithm (used also for the set up of the initial conditions), which is described in the Appendix. After the updating of the amplitudes, the next inner-cycle begins, and the procedure continues as described above.

During each time interval $\left[\tau^{(\ell)}, \tau^{(\ell+1)}\right]$ the amplitudes are considered constant and, thus, globally, p_j are piecewise constant functions of time. In fact, the evolution of the amplitudes p_j is much slower than the evolution of the kernel parameters $\mathbf{m}^k$ and $\mathbf{h}^k$, and this is what justifies the piece-wise constant assumption for p_j in our numerical scheme. An improved numerical solution, taking also into account the evolution of p_j in a continuous fashion, can be constructed and will be published elsewhere.

It should be stressed that the accuracy of the method proposed and developed herewith is critically dependant on the threshold value ε_1 for the variance parameter (spreading index) $\left\|\mathbf{h}^k(t)\right\|$.

10.1 *A local-moment method for the numerical solution of Eqs (10.3b)*

We are now focusing on the numerical treatment of equations (10.3b). For each value of $j \in \{1,\ldots,N\}$, eq. (10.3b) contains three unknown functions, namely the response mean value $m_x^j(t)$, and covariances $C_{xx}^j(t,t)$ and $C_{xy}^j(t,s)$, which should be determined, and two known functions, namely the excitation mean value $m_y^j(s)$ and the autocovariance $C_{yy}^j(s,s)$, introducing the appropriate, inner-cycle, forcing. Thus, any solution scheme of eq. (10.3b) should provide us with a number of equations (hopefully three) governing the evolution of the three unknown functions, along with the evidence that introducing the obtained solution in the operator $\tilde{\mathcal{L}}\left[\tilde{K}\left(\upsilon,\nu;\mathbf{m}^j,\mathbf{C}^j\right)\right]$ will result in 0 (at least approximately) for all values of $(\upsilon,\nu) \in \mathbb{R}^2$.

Since the (Gaussian) kernel $\tilde{K}\left(\upsilon,\nu;\mathbf{m}^j,\mathbf{C}^j\right)$ is $C^\infty\left(\mathbb{R}^2\right)$ in (υ,ν) and dies out as $\left\|(\upsilon,\nu)\right\|\to\infty$, eq. (10.3b) is equivalent to the following system of localized moment equations:

$$\frac{\partial^{p+q}}{\partial^p\upsilon\,\partial^q\nu}\tilde{\mathcal{L}}\left[\tilde{K}\left(\upsilon,\nu;\mathbf{m}^j,\mathbf{C}^j\right)\right]\Bigg|_{\substack{\upsilon=0\\ \nu=0}} = 0\,,\ \ \forall(p,q)\in\mathbb{N}_0\times\mathbb{N}_0\,,\ \mathbb{N}_0=\{0,1,2,3,...\}\,. \quad (10.5)$$

Exploiting the specific (Gaussian) form of the kernel, and considering the cases $(p,q)=(1,0)$, $(2,0)$ and $(1,1)$, the following three (nonlinear) ODEs are obtained from (10.5):

$$m^j_{x,t}(t)+\mu\,m^j_x(t)+3km^j_x(t)C^j_{xx}(t,t)+k\left[m^j_x(t)\right]^3=m^j_y(t)\,, \quad (10.6a)$$

$$C^j_{xx,t}(t,s)+\mu C^j_{xx}(t,s)+3km^j_x(t)C^j_{xx}(t,s)m^j_x(t)+3kC^j_{xx}(t,t)C^j_{xx}(t,s)=C^j_{xy}(t,s)\,, \quad (10.6b)$$

$$C^j_{xy,t}(t,s)+\mu C^j_{xy}(t,s)+3kC^j_{xx}(t,t)C^j_{xy}(t,s)+3km^j_x(t)C^j_{xy}(t,s)m^j_x(t)=C^j_{yy}(t,s)\,, \quad (10.6c)$$

These equations involve the three unknown functions $m^j_x(t)$, $C^j_{xx}(t,t)$ and $C^j_{xy}(t,s)$, and they are differential equations with respect to t, parametrically dependent on s. (No derivatives with respect to s appear.) They should be satisfied for all values of (t,s) such that $\tau^{(\ell)}\le s\le t\le\tau^{(\ell+1)}$. We are especially interesting in the solution of system (10.6) on the diagonal $s=t$.

It has been found that if the three moment equations (10.6a,b,c) holds true, then various other –but not all– moments, corresponding to other values of (p,q), are also zero. Besides, there are also values of (p,q), corresponding to higher-order moments, for which eqs. (10.5) are not satisfied. In any case, the system (10.6a,b,c) is closed and can be efficiently solved, providing us with a reasonable approximation of the evolution of the kernel parameters $m^j_x(t)$, $C^j_{xx}(t,\,s=t)$ and $C^j_{xy}(t,\,s=t)$. When the value of $\left\|\mathbf{h}^j(t)\right\|$ exceeds the threshold value ε_1, the current inner-cycle phase is finished and the procedure switches to outer-cycle phase.

The numerical solution of the set of nonlinear ODEs (10.6) is implemented by using the method of the ***quasilinearization*** (Bellman 1973, Lakshmikantham and Malek 1994). Taking advantage of the symmetry properties of the correlation matrix,

the equations can be solved on the 'diagonal', that is around $s = t$. The sequence of time instants for the numerical scheme has the form

$$(t,s):\ (t_i,t_i) \rightarrow (t_{i+1},t_i) \rightarrow (t_{i+1},t_{i+1}).$$

An important aspect of the present method is its suitability for parallel computation. Parallelization techniques can be applied both to the dynamical evolution of the kernels and to the optimization algorithm. In the first case the algorithm can take advantage of the independent evolution of each kernel. For the parallelization of the optimization algorithm we can split the group of kdfs into subgroups and then independently approximate each subgroup by new kernels with small variance. Hence, we can probably succeed fast computations for systems of higher dimensions, subjected to general (smooth) excitation.

11 NUMERICAL EXAMPLES

We shall now apply the above described numerical scheme to the numerical determination of the response pdf of a dynamical system (4.1), excited by a known stochastic process (see below), with system parameters μ and k having the values given in Table 1, under Cases I and II.

Table 1: System Parameters

System parameters	Case I	Case II
μ	1	1
k	1	-1

By performing a stability analysis to problem (4.1) we found that for $\mu/k > 0$ (Case I in Table 1), the nonlinear system has one stable fixed point located at zero. A pitchfork bifurcation occurs at $\mu/k = 0$, and the fixed point at zero becomes unstable in the semi-axis $\mu/k < 0$ (Case II). In the same region ($\mu/k < 0$) two symmetric stable points appear at $\pm\sqrt{|\mu/k|}$. Hence, we have the bifurcation diagram shown in Figure 1.

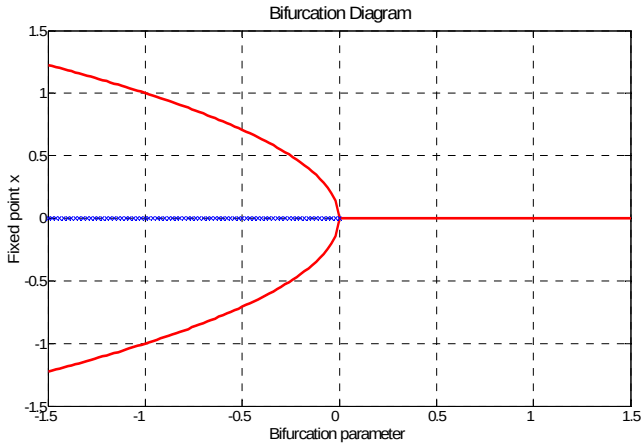

Figure 1: Bifurcation diagram for system (4.1) with respect to the bif. parameter μ / k .

On the basis of the above described dynamical features of the studied problem, it is natural to expect that, in Case I, the evolved pdf will become eventually a unimodal distribution centered at zero, while in Case II, the probability will concentrate around the pair of the two symmetric stable fixed points $\pm\sqrt{\mu/|k|}$, hence ultimately a bimodal distribution will appear. Since the stable fixed points are global attractors we expect to attain these results after some time, independently of the initial density. The numerical results to be presented and discussed below clearly comply with this behavior, dictated by the qualitative analysis of the studied system.

Consider first Case I, with a bimodal initial pdf, defined as a convex superposition of two Gaussians with parameter values $m_1 = 0,\ \ m_2 = 0.6$, $\sigma_1 = 0.1$, $\sigma_2 = 0.6$, and amplitudes $p_1 = 0.4$ and $p_2 = 0.6$, respectively. This initial pdf is shown in Figure 2b, at the section $t = 0$. The excitation process is taken to be, in this case, a Gaussian stationary random function with zero mean and covariance function given by

$$C_{YY}(\tau) = \frac{1}{2}\cos^2(2\tau). \qquad (11.1)$$

Numerical results are presented in Figure 2. More specifically, in the two upper plots of this figure (Figures 2a and 2b), the evolution of the probability density $f_{x(t)}(a)$ is shown, for the time interval $0 < t < 1.4\text{sec}$, large enough to get the steady state response pdf. Also, in the same figure (Figure 2a) the orbits of $m_x^k(t)$ are plotted by using thick black lines. The apparent discontinuities every 0.2sec are due to the re-approximation of the calculated density by means of a new convex superposition of kdfs with smaller variance every time the concentration parameters h^k exceeds the

critical value ε_1 (which in this example was taken to be $\varepsilon_1 = 0.3$). In Figure 2c the evolution of the variance for some kdfs of the response density is shown. The diffusive character of the evolution (strictly increasing variances with respect to time) is clearly seen in the numerical results. Again, the apparent discontinuities are due to the re-approximation of the response pdf by kdfs with smaller variances.

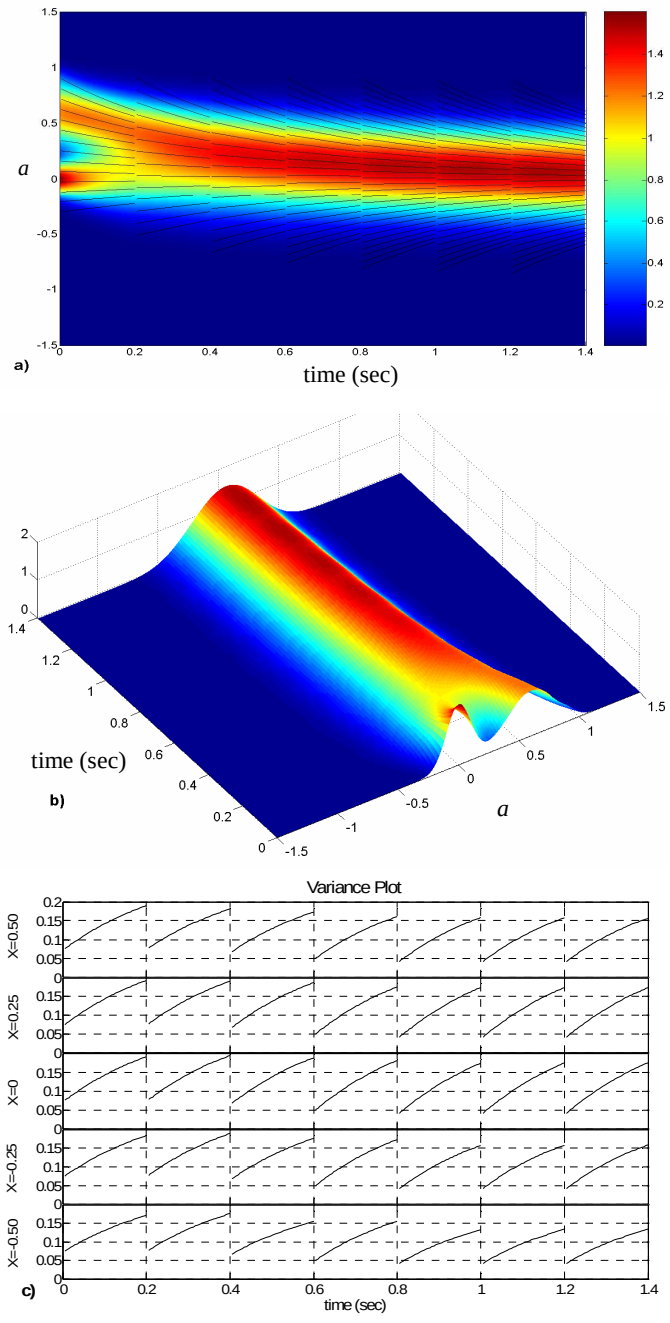

Figure 2: a) Response pdf $f_{x(t)}(a)$ and $m_x^j(t)$ curves for Case I with stationary excitation. b) 3D plot of the response pdf. c) Variance plots for some kdfs.

Let us now consider our system (4.1) with parameter values as in Case II. Two cases of stochastic excitation will be considered. First we will study the same stationary Gaussian excitation as before, having zero mean and covariance function given by eq. (11.1). The initial distribution is taken to be bimodal (strongly asymmetric for this case), and is defined as a convex superposition of two Gaussian pdfs with parameters $m_1 = -0.4$, $m_2 = 0.6$, $\sigma_1 = 0.1$, $\sigma_2 = 0.6$, and amplitudes $p_1 = 0.4$, $p_2 = 0.6$, respectively.

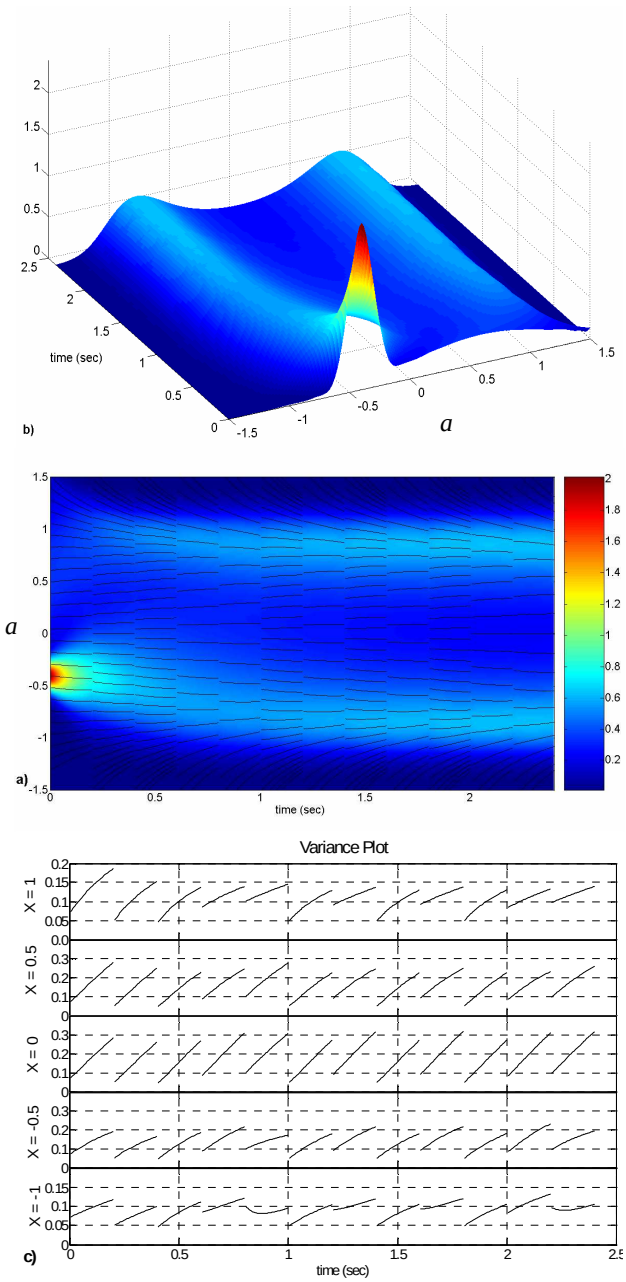

Figure 3: a) Response pdf $f_{X(t)}(a)$ and $m_x^j(t)$ curves for Case II with stationary excitation. b) 3D plot of the response pdf. c) Variance plots for some kdfs.

Numerical results concerning the evolution of the response pdf, for the time interval $0 < t < 2.4\,\text{sec}$, are presented in Figure 3. Although the initial pdf has taken to be a strongly asymmetric bimodal one, the eventually resulting response density turns to be a symmetric bimodal pdf, with modes exactly at the stable fixed points, located at $\pm\sqrt{|\mu/k|} = \pm 1$, as expected. The interchange of probability between the kernels (implemented by means of the re-approximation of the response pdf in terms of a new convex superposition of kdfs with smaller variances) takes place approximately every 0.2 seconds. This is shown in the figure as an apparent discontinuity of the mean-value and variance curves.

From both Figures 2a and 3a (see also Figure 4a, below), we can easily observe a permanent tendency of $m_x^k(t)$-orbits to be attracted by the stable fixed points. This means that there is a continuous inflow of probability mass from the outer region of the phase space ($|a| > 1$) to a strip around the locus of the stable fixed points, which is not stopping even after the response pdf has been reached its stationary form. This apparently paradoxical behaviour should be addressed to the discrepancy between the tail form of the response pdf $f_{x(t)}(a)$, and the tail form of the Gaussian kernels which are used to represent $f_{x(t)}(a)$. This fact reveals the necessity for an asymptotic study of the tail behaviour directly from the differential equation (5.7a), which will permit the construction and use of the kdfs suitably adapted to the specific system, i.e., exhibiting the correct tail behaviour. Such a construction will also facilitate and accelerate the convergence of the numerical solution procedure.

Finally, in Figure 4 we present numerical results for the Case II, with a non-stationary (cyclostationary) Gaussian excitation, with zero mean and covariance function given by

$$C_{YY}(t,s) = \frac{1}{2}\left(1 + 0.2\cos\left(\frac{\pi t}{2}\right)\right)\cos^2(t-s). \tag{11.2}$$

Again the initial distribution is constructed as a superposition of two Gaussian pdfs with parameters $m_1 = -0.4$, $m_2 = 0.6$, $\sigma_1 = 0.3$, $\sigma_2 = 0.7$, and amplitudes $p_1 = 0.4$ and $p_2 = 0.6$, respectively. The evolution of the response probability density function is plotted for the time interval $0 < t < 8.0\,\text{sec}$, long enough so that the periodic character of the response to become clear.

From Figures 4a, 4b we are able to observe that, after a transient state ($0 < t < 1.5\text{sec}$), the response density function exhibits a periodic behavior with a period of approximately 4sec, which is the period of the excitation, i.e., the period of the correlation function $C_{YY}(t,s)$ (eq. (11.2)) with respect to its first argument. Furthermore, it is easily seen that, in this case, a greater amount of kernels is necessary in order to approximate satisfactorily the sought-for pdf, due to the fact that the non-stationary excitation produces a more complicated response.

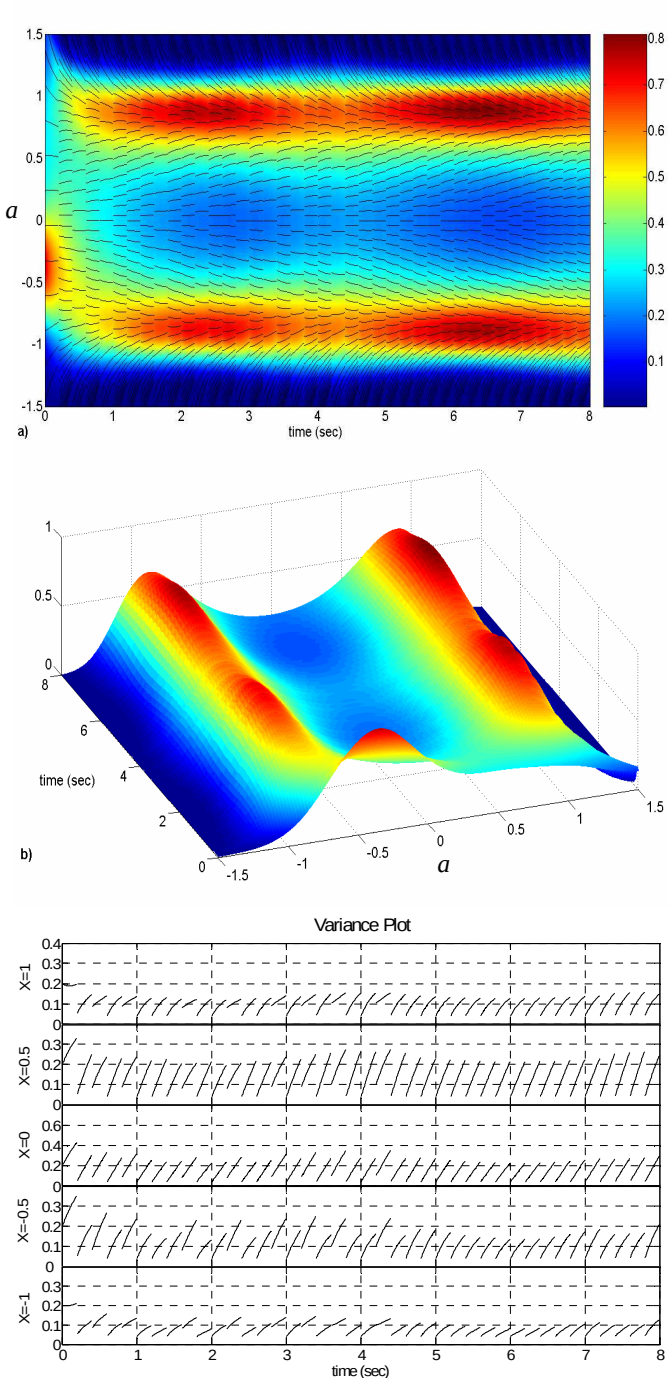

Figure 4: a) Response pdf $f_{X(t)}(a)$ and $m_x^j(t)$ curves for Case II with non-stationary excitation. b) 3D plot of the response pdf. c) Variance plots for some kdfs.

12 DISCUSSION AND CONCLUSIONS

In this paper new PDEs governing the evolution of the joint, response-excitation, ch.f and pdf of nonlinear dynamical systems under general stochastic excitation have been derived. The starting point of our approach is a Hopf-type equation, that governs the joint, response-excitation, characteristic functional, providing a probabilistically complete reformulation, equivalent with the underlying (nonlinear) stochastic differential equation. This 'infinite-dimensional' equation is appropriately reduced (by projection) to *linear* partial differential equations that governs the response-excitation, characteristic (or probability density) function (see, e.g. eq. (5.6a) or (5.7a)). The latter equations are supplemented with (non-local) marginal compatibility conditions (see, e.g. eq. (5.6b) or (5.7b)) and initial conditions (see, e.g. eq. (5.6c) or (5.7c)), and they can provide us with the evolution of the joint ch.f. (or pdf).

For the numerical solution of these novel PDEs (e.g., either in the form (5.6) or (5.7)) an original, particle-type, method is developed and illustrated through its application to a specific, simple, nonlinear system. The key point of the numerical method is the representation of the joint, response-excitation, pdfs and ch.fs by means of appropriate convex superpositions of kernel density or kernel characteristic functions, respectively. In this way, the non-local marginal compatibility conditions are satisfied *a priori*, and the PDEs governing the evolution of the sought-for pdf and ch.f are eventually transformed to systems of nonlinear ODEs for the kernel parameters.

From the results presented in this work we conclude that the proposed method is able to produce quite satisfactory results for systems subjected to general stochastic excitation. Important aspects of the method are (**i**) It is a *two-level*, *particle-type* method, separating the fast, inner-cycle (short-term) phase, which describes the particle dynamics separately for each particle, from the slow, outer-cycle (long-term) phase, which accounts for the interchange of probability mass between the particles and the evolution of the particles' amplitudes. (**ii**) It can be improved, keeping its two-level, particle-type character, so that to avoid the piece-wise smoothness assumption for the amplitudes p_j, and to ensure the "exact" satisfaction of the PDE, by solving a linear evolution problem in the outer-cycle phase. (**iii**) It can be generalized to higher dimensional systems. And (**iv**) It is plainly suitable for parallelized computations, since the nonlinear ODEs describing the evolution of each particle in the inner-cycle

phase can be solved independently. In addition, the computationally demanding optimization algorithm (see Appendix) is easily parallelizable.

APPENDIX

In this Appendix a brief outline is given of the optimization algorithm used for the construction of appropriate approximants of the sought-for pdf in terms of kdfs exhibiting a small variance (either given or under specific control). This algorithm is used quite often throughout the numerical solution, i.e. each time the solution procedure switches from the inner-cycle to the outer-cycle and the calculated density is re-approximated by means of kdfs of small variance. It is also used for implementing the initialization, by representing the given initial pdf as a convex superposition of appropriate kdfs. The basic optimization problem is formulated as follows:

Given $f(x)$ and σ_0, find M and $\{p_k, m_k\}_{k=1}^{M}$ such that

$$\int_{-\infty}^{+\infty}\left[f(x)-\sum_{k=1}^{M}\frac{p_k}{\sqrt{2\pi}\sigma_0}\exp\left\{-\frac{1}{2}\left(\frac{x-m_k}{\sigma_0}\right)^2\right\}\right]^2 dx \;=\; \min. \qquad \text{(A.1)}$$

under the constraints:

$$p_1 + p_2 + \ldots + p_M = 1 \quad \text{and} \quad p_k \geq 0, \quad \text{for all } k\,.$$

For the inner-cycle/outer-cycle re-approximation of the sought-for pdf, the integrations can be carried out analytically (since $f(x)$ is already represented as a superposition of Gaussian kernels, with different parameters of course) leading to an explicit linear optimization problem, if M is given. M is obtained by using a variant of an iterative, adaptive procedure, developed by Gavriliadis (2005).

For the initial data representation, the optimization procedure is performed quite similarly. However, in this case, the integrations in (A.1) are performed numerically, since, in general, the initial probability distribution may not be analytically described. A detailed description of the solution algorithm will be presented elsewhere.

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